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Numerical Radius and p-Schatten Norm Inequalities for Analytic Functions of Operators in Hilbert Spaces

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Abstract

Let H be a complex Hilbert space, $f : G \subset \mathbb{C} \to \mathbb{C}$ an analytic function on the domain G and $A \in \mathscr{B}(H)$ with $\mathrm{Sp}(A) \subset G$ and γ a closed rectifiable path in G and such that $\mathrm{Sp}(A) \subset \mathrm{ins}(\gamma)$. If we denote

$$B(f,\gamma;A) := \frac{1}{2\pi}\int_\gamma |f(\xi)|\left(|\xi| - \|A\|\right)^{-1}|d\xi|,$$

then for B, $C \in \mathscr{B}(H)$ we have

$$|\langle C^* A f(A) Bx, y\rangle| \le B(f,\gamma;A)\left\langle \left||A|^\alpha B\right|^2 x, x\right\rangle^{1/2}\left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle^{1/2}$$

for $\alpha \in [0,1]$ and x, $y \in H$. Some natural applications for *numerical radius* and *p-Schatten norm* are also provided.

Keywords: *Schwarz inequality, Vector inequality, Bounded operators, Numerical radius, Operator trace, p-Schatten norm.*
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1. Introduction

In 1988, F. Kittaneh obtained the following generalization of Schwarz inequality [9]:

Theorem 1.1. *Assume that h and g are non-negative functions on* $[0,\infty)$ *which are continuous and satisfying the relation* $h(t)g(t) = t$ *for all* $t \in [0,\infty)$. *For any* $T \in \mathscr{B}(H)$

$$|\langle Tx, y\rangle| \le \|h(|T|)x\| \, \|g(|T^*|)y\| \tag{1.1}$$

for all x, $y \in H$.

If we take $h(t) = t^\alpha$, $g(t) = t^{1-\alpha}$ with $\alpha \in [0,1]$, then we obtain *Kato's inequality*

$$|\langle Tx, y\rangle| \le \left\||T|^\alpha x\right\| \left\||T^*|^{1-\alpha} y\right\| \text{ for all } x, y \in H. \tag{1.2}$$

The *numerical radius* $w(T)$ of an operator T on H is given by

$$\omega(T) = \sup\{|\langle Tx, x\rangle|, \|x\| = 1\}. \tag{1.3}$$

Obviously, by (1.3), for any $x \in H$ one has

$$|\langle Tx, x\rangle| \le w(T)\|x\|^2. \tag{1.4}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $T : H \to H$, i.e.,

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(i) $\omega(T) \geq 0$ for any $T \in B(H)$ and $\omega(T) = 0$ if and only if $T = 0$;
(ii) $\omega(\lambda T) = |\lambda| \omega(T)$ for any $\lambda \in \mathbb{C}$ and $T \in B(H)$;
(iii) $\omega(T+V) \leq \omega(T) + \omega(V)$ for any $T, V \in B(H)$.

This norm is equivalent with the operator norm. In fact, the following more precise result holds:

$$\omega(T) \leq \|T\| \leq 2\omega(T) \tag{1.5}$$

for any $T \in B(H)$.
F. Kittaneh, in 2003 [10], showed that for any operator $T \in B(H)$ we have the following refinement of the first inequality in (1.5):

$$\omega(T) \leq \frac{1}{2}\left(\|T\| + \left\|T^2\right\|^{1/2}\right). \tag{1.6}$$

Utilizing the Cartesian decomposition for operators, F. Kittaneh in [11] improved the inequality (1.5) as follows:

$$\frac{1}{4}\|T^*T + TT^*\| \leq \omega^2(T) \leq \frac{1}{2}\|T^*T + TT^*\| \tag{1.7}$$

for any operator $T \in B(H)$.
For powers of the absolute value of operators, one can state the following results obtained by El-Haddad & Kittaneh in 2007, [8]:
If for an operator $T \in B(H)$ we denote $|T| := (T^*T)^{1/2}$, then

$$\omega^r(T) \leq \frac{1}{2}\left\||T|^{2\alpha r} + |T^*|^{2(1-\alpha)r}\right\| \tag{1.8}$$

and

$$\omega^{2r}(T) \leq \left\|\alpha|T|^{2r} + (1-\alpha)|T^*|^{2r}\right\|, \tag{1.9}$$

where $\alpha \in (0,1)$ and $r \geq 1$.
If we take $\alpha = \frac{1}{2}$ and $r = 1$ we get from (1.8) that

$$\omega(T) \leq \frac{1}{2}\||T| + |T^*|\| \tag{1.10}$$

and from (1.9) that

$$\omega^2(T) \leq \frac{1}{2}\left\||T|^2 + |T^*|^2\right\|. \tag{1.11}$$

For more related results, see the recent books on inequalities for numerical radii [2] and [6] .
Let $(H;\langle .,. \rangle)$ be a complex Hilbert space and $\mathscr{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathscr{B}(H)$ is of *trace class* if

$$\|A\|_1 := \sum_{i \in I} \langle |A| e_i, e_i \rangle < \infty. \tag{1.12}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathscr{B}_1(H)$ the set of trace class operators in $\mathscr{B}(H)$.
We define the *trace* of a trace class operator $A \in \mathscr{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i \in I} \langle Ae_i, e_i \rangle, \tag{1.13}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.13) converges absolutely and it is independent from the choice of basis.
The following result collects some properties of the trace:

Theorem 1.2. *We have:*
(i) If $A \in \mathscr{B}_1(H)$ then $A^* \in \mathscr{B}_1(H)$ and

$$tr(A^*) = \overline{tr(A)}; \tag{1.14}$$

(ii) If $A \in \mathscr{B}_1(H)$ and $T \in \mathscr{B}(H)$, then AT, $TA \in \mathscr{B}_1(H)$ and

$$tr(AT) = tr(TA) \text{ and } |tr(AT)| \leq \|A\|_1 \|T\|; \tag{1.15}$$

(iii) $tr(\cdot)$ is a bounded linear functional on $\mathscr{B}_1(H)$ with $\|tr\| = 1$;
(iv) If A, $B \in \mathscr{B}_2(H)$ then AB, $BA \in \mathscr{B}_1(H)$ and $tr(AB) = tr(BA)$;
(v) $\mathscr{B}_{fin}(H)$, the space of operators of finite rank, is a dense subspace of $\mathscr{B}_1(H)$.

For a large number of results concerning trace inequalities, see the recent survey paper [7].
An operator $A\in\mathscr{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathscr{B}_p(H)$, $1\le p<\infty$ if the *p-Schatten norm* is finite [16, p. 60-64]

$$\|A\|_p:=[\operatorname{tr}(|A|^p)]^{1/p}=\left(\sum_{i\in I}\langle |A|^p e_i,e_i\rangle\right)^{1/p}<\infty.$$

For $1<p<q<\infty$ we have that

$$\mathscr{B}_1(H)\subset\mathscr{B}_p(H)\subset\mathscr{B}_q(H)\subset\mathscr{B}(H) \tag{1.16}$$

and

$$\|A\|_1\ge\|A\|_p\ge\|A\|_q\ge\|A\|. \tag{1.17}$$

For $p\ge1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathscr{B}_p(H)$ and $\left(\mathscr{B}_p(H),\|\cdot\|_p\right)$ is a Banach space.
Also, see for instance [16, p. 60-64],

$$\|A\|_p=\|A^*\|_p,\ A\in\mathscr{B}_p(H) \tag{1.18}$$

$$\|AB\|_p\le\|A\|_p\|B\|_p,\ A,B\in\mathscr{B}_p(H) \tag{1.19}$$

and

$$\|AB\|_p\le\|A\|_p\|B\|,\ \ \|BA\|_p\le\|B\|\,\|A\|_p,\ A\in\mathscr{B}_p(H),\ B\in\mathscr{B}(H). \tag{1.20}$$

This implies that

$$\|CAB\|_p\le\|C\|\,\|A\|_p\|B\|,\ A\in\mathscr{B}_p(H),\ B,\ C\in\mathscr{B}(H). \tag{1.21}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for $p,\,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$

$$(|\operatorname{tr}(AB)|\le)\,\|AB\|_1\le\|A\|_p\|B\|_q,\ A\in\mathscr{B}_p(H),\ B\in\mathscr{B}_q(H). \tag{1.22}$$

For the theory of trace functionals and their applications the reader is referred to [15] and [16].
For $\mathscr{E}:=\{e_i\}_{i\in I}$ an orthonormal basis of H we define for $A\in\mathscr{B}_p(H)$, $p\ge1$

$$\|A\|_{\mathscr{E},p}:=\left(\sum_{i\in I}|\langle Ae_i,e_i\rangle|^p\right)^{1/p}.$$

We observe that $\|\cdot\|_{\mathscr{E},p}$ is a norm on $\mathscr{B}_p(H)$ and

$$\|A\|_{\mathscr{E},p}\le\|A\|_p\ \text{ for } A\in\mathscr{B}_p(H).$$

Further, we can take the supremum over all orthonormal basis in H we can also define, for $A\in\mathscr{B}_p(H)$, that

$$\omega_p(A):=\sup_{\mathscr{E}}\|A\|_{\mathscr{E},p}\le\|A\|_p,$$

which is a *norm* on $\mathscr{B}_p(H)$.
It is also known that, if $\mathscr{E}=\{e_i\}_{i\in I}$ and $\mathscr{F}=\{f_i\}_{i\in I}$ are orthonormal basis, then [13]

$$\sup_{\mathscr{E},\mathscr{F}}\sum_{i\in I}|\langle Te_i,f_i\rangle|^s=\|T\|_s^s\ \text{ for } s\ge1. \tag{1.23}$$

Let $\mathscr{B}$ be a unital Banach algebra, $A\in\mathscr{B}$ and G be a domain of $\mathbb{C}$ with $\operatorname{Sp}(A)\subset G$. If $f:G\to\mathbb{C}$ is analytic on G, we define an element $f(A)$ in $\mathscr{B}$ by

$$f(A):=\frac{1}{2\pi i}\int_\delta f(\xi)(\xi-A)^{-1}d\xi, \tag{1.24}$$

where $\delta\subset G$ is taken to be closed rectifiable curve in G and such that $\operatorname{Sp}(A)\subset\operatorname{ins}(\delta)$, the inside of δ.
It is well known (see for instance [4, pp. 201-204]) that $f(A)$ does not depend on the choice of δ and the *Spectral Mapping Theorem* (SMT)

$$\operatorname{Sp}(f(A))=f(\operatorname{Sp}(A)) \tag{1.25}$$

holds.
Concerning other basic definitions and facts in the theory of Banach algebras, the reader can consult the classical books [5] and [14].

2. Vector Inequalities

In 1988, F. Kittaneh [9, Corollary 7] obtained the following Schwarz type inequality for natural powers of operators:

Lemma 2.1. *Let $A\in\mathscr{B}(H)$ and $\alpha\in[0,1]$. Then for natural number $n\geq 1$ we have*

$$|\langle A^n x,y\rangle|^2\leq\|A\|^{2n-2}\left\langle|A|^{2\alpha}x,x\right\rangle\left\langle|A^*|^{2(1-\alpha)}y,y\right\rangle \tag{2.1}$$

for all $x,\,y\in H$.

We can state the following result as well:

Corollary 2.2. *Let $A,\,B,\,C\in\mathscr{B}(H)$ and $\alpha\in[0,1]$. Then for $n\geq 1$ we have*

$$|\langle C^*A^nBx,y\rangle|^2\leq\|A\|^{2n-2}\left\langle\left||A|^{\alpha}B\right|^2x,x\right\rangle\left\langle\left||A^*|^{1-\alpha}C\right|^2y,y\right\rangle \tag{2.2}$$

for all $x,\,y\in H$.

Proof. If we replace x by Bx and y by Cy in (2.1), then we get

$$|\langle C^*A^nBx,y\rangle|^2\leq\|A\|^{2n-2}\left\langle B^*|A|^{2\alpha}Bx,x\right\rangle\left\langle C^*|A^*|^{2(1-\alpha)}Cy,y\right\rangle \tag{2.3}$$

for all $x,\,y\in H$.

Observe that $B^*|A|^{2\alpha}B=\left||A|^{\alpha}B\right|^2$ and $C^*|A^*|^{2(1-\alpha)}C=\left||A^*|^{1-\alpha}C\right|^2$, then by (2.3) we get (2.2). □

We also have:

Lemma 2.3. *Assume that $A,\,B,\,C\in\mathscr{B}(H)$ with $\|A\|<1$, then*

$$\left|\left\langle C^*A(I-A)^{-1}Bx,y\right\rangle\right|\leq(1-\|A\|)^{-1}\times\left\langle\left||A|^{\alpha}B\right|^2x,x\right\rangle^{1/2}\left\langle\left||A^*|^{1-\alpha}C\right|^2y,y\right\rangle^{1/2} \tag{2.4}$$

for $\alpha\in[0,1]$ and $x,\,y\in H$. In particular,

$$\left|\left\langle C^*A(I-A)^{-1}Bx,y\right\rangle\right|\leq(1-\|A\|)^{-1}\times\left\langle\left||A|^{1/2}B\right|^2x,x\right\rangle^{1/2}\left\langle\left||A^*|^{1/2}C\right|^2y,y\right\rangle^{1/2} \tag{2.5}$$

for $x,\,y\in H$.

Proof. If we put $n=k+1$, $k\in\mathbb{N}$ in (2.2) and take the square root, then we get

$$\left|\left\langle C^*AA^kBx,y\right\rangle\right|\leq\|A\|^k\left\langle\left||A|^{\alpha}B\right|^2x,x\right\rangle^{1/2}\left\langle\left||A^*|^{1-\alpha}C\right|^2y,y\right\rangle^{1/2}$$

for all $x,\,y\in H$.

Further, if we sum over k from 0 to m, then we obtain

$$\left|\left\langle C^*A\sum_{k=0}^{m}A^kBx,y\right\rangle\right|=\left|\sum_{k=0}^{m}\left\langle C^*AA^kBx,y\right\rangle\right|\leq\sum_{k=0}^{m}\left|\left\langle C^*AA^kBx,y\right\rangle\right|\leq\sum_{k=0}^{m}\|A\|^k\left\langle\left||A|^{\alpha}B\right|^2x,x\right\rangle^{1/2}\left\langle\left||A^*|^{1-\alpha}C\right|^2y,y\right\rangle^{1/2} \tag{2.6}$$

for all $x,\,y\in H$.

Since $\|A\|<1$, then series $\sum_{k=0}^{\infty}A^k$ and $\sum_{k=0}^{\infty}\|A\|^k$ are convergent and

$$\sum_{k=0}^{\infty}A^k=(I-A)^{-1}\text{ and }\sum_{k=0}^{\infty}\|A\|^k=(1-\|A\|)^{-1}.$$

By taking now the limit over $m\to\infty$ in (2.6) we deduce the desired result (2.4). □

Our first main result is as follows:

Theorem 2.4. *Let $f:G\subset\mathbb{C}\to\mathbb{C}$ be an analytic function on the domain G and $A\in\mathscr{B}(H)$ with $Sp(A)\subset G$ and γ a closed rectifiable path in G and such that $Sp(A)\subset ins(\gamma)$. If we denote*

$$B(f,\gamma;A):=\frac{1}{2\pi}\int_{\gamma}|f(\xi)|(|\xi|-\|A\|)^{-1}|d\xi|,$$

then for $B,\,C\in\mathscr{B}(H)$ we have

$$|\langle C^*Af(A)Bx,y\rangle|\leq B(f,\gamma;A)\left\langle\left||A|^{\alpha}B\right|^2x,x\right\rangle^{1/2}\left\langle\left||A^*|^{1-\alpha}C\right|^2y,y\right\rangle^{1/2} \tag{2.7}$$

for $\alpha\in[0,1]$ and $x,\,y\in H$. In particular,

$$\begin{aligned}&|\langle C^*Af(A)Bx,y\rangle|\\&\leq B(f,\gamma;A)\left\langle\left||A|^{1/2}B\right|^2x,x\right\rangle^{1/2}\left\langle\left||A^*|^{1/2}C\right|^2y,y\right\rangle^{1/2}\end{aligned} \tag{2.8}$$

for $x,\,y\in H$.

Proof. We have, by the representation

$$f(A) := \frac{1}{2\pi i}\int_{\gamma} f(\xi)(\xi I - A)^{-1} d\xi,$$

that

$$\langle C^*Af(A)Bx,y\rangle = \left\langle C^*A\frac{1}{2\pi i}\int_{\gamma} f(\xi)(\xi I - A)^{-1} d\xi Bx,y\right\rangle = \frac{1}{2\pi i}\int_{\gamma} f(\xi)\left\langle C^*A(\xi I - A)^{-1}Bx,y\right\rangle d\xi$$

for $x, y \in H$.
By taking the modulus and using the complex integral properties, we get

$$|\langle C^*Af(A)Bx,y\rangle| \le \frac{1}{2\pi}\int_{\gamma} |f(\xi)|\left|\left\langle C^*A(\xi I - A)^{-1}Bx,y\right\rangle\right| |d\xi| = \frac{1}{2\pi}\int_{\gamma} |f(\xi)||\xi|^{-1}\left|\left\langle C^*A\left(I - \frac{A}{\xi}\right)^{-1}Bx,y\right\rangle\right| |d\xi| \tag{2.9}$$

for $x, y \in H$.
Since $\left\|\frac{A}{\xi}\right\| < 1$ for $\xi \in \gamma$, then by Lemma 2.3 for $\frac{A}{\xi}$ we have

$$\begin{aligned}|\xi|^{-1}\left|\left\langle C^*A\left(I - \frac{A}{\xi}\right)^{-1}Bx,y\right\rangle\right| &= \left|\left\langle C^*\frac{A}{\xi}\left(I - \frac{A}{\xi}\right)^{-1}Bx,y\right\rangle\right| \le \left(1 - \left\|\frac{A}{\xi}\right\|\right)^{-1}\left\langle\left|\left|\frac{A}{\xi}\right|^{\alpha}B\right|^2 x,x\right\rangle^{1/2}\left\langle\left|\left|\frac{A^*}{\overline{\xi}}\right|^{1-\alpha}C\right|^2 y,y\right\rangle^{1/2} \\ &= \left(\frac{|\xi| - \|A\|}{|\xi|}\right)^{-1}\left\langle\left|\left|\frac{A}{\xi}\right|^{\alpha}B\right|^2 x,x\right\rangle^{1/2}\left\langle\left|\left|\frac{A^*}{\overline{\xi}}\right|^{1-\alpha}C\right|^2 y,y\right\rangle^{1/2} \\ &= \frac{|\xi|}{|\xi|^{\alpha}\left|\overline{\xi}\right|^{1-\alpha}}(|\xi| - \|A\|)^{-1}\left\langle\left||A|^{\alpha}B\right|^2 x,x\right\rangle^{1/2}\left\langle\left||A^*|^{1-\alpha}C\right|^2 y,y\right\rangle^{1/2} \\ &= (|\xi| - \|A\|)^{-1}\left\langle\left||A|^{\alpha}B\right|^2 x,x\right\rangle^{1/2}\left\langle\left||A^*|^{1-\alpha}C\right|^2 y,y\right\rangle^{1/2}\end{aligned} \tag{2.10}$$

for $x, y \in H$.
By utilizing (2.10) we derive

$$\frac{1}{2\pi}\int_{\gamma} |f(\xi)||\xi|^{-1}\left|\left\langle C^*A\left(I - \frac{A}{\xi}\right)^{-1}Bx,y\right\rangle\right| |d\xi| \le \left(\frac{1}{2\pi}\int_{\gamma} |f(\xi)|(|\xi| - \|A\|)^{-1}|d\xi|\right) \times \left\langle\left||A|^{\alpha}B\right|^2 x,x\right\rangle^{1/2}\left\langle\left||A^*|^{1-\alpha}C\right|^2 y,y\right\rangle^{1/2} \tag{2.11}$$

for $x, y \in H$.
By making use of (2.9) and (2.11) we obtain (2.7). □

Remark 2.5. *For $B = C = I$ in (2.7) we get the one operator inequalities*

$$|\langle Af(A)x,y\rangle| \le B(f,\gamma;A)\left\langle|A|^{2\alpha}x,x\right\rangle^{1/2}\left\langle|A^*|^{2(1-\alpha)}y,y\right\rangle^{1/2} \tag{2.12}$$

for $\alpha \in [0,1]$ and $x, y \in H$. In particular,

$$|\langle Af(A)x,y\rangle| \le B(f,\gamma;A)\langle|A|x,x\rangle^{1/2}\langle|A^*|y,y\rangle^{1/2} \tag{2.13}$$

for all $x, y \in H$.
If A is invertible and take $C = I$, $B = A^{-1}$ in (2.7), then we get

$$|\langle f(A)x,y\rangle| \le B(f,\gamma;A)\left\langle|A|^{-2(1-\alpha)}x,x\right\rangle^{1/2}\left\langle|A^*|^{2(1-\alpha)}y,y\right\rangle^{1/2} \tag{2.14}$$

for $\alpha \in [0,1]$ and $x, y \in H$. In particular,

$$|\langle f(A)x,y\rangle| \le B(f,\gamma;A)\left\langle|A|^{-1}x,x\right\rangle^{1/2}\langle|A^*|y,y\rangle^{1/2} \tag{2.15}$$

for $x, y \in H$.
If $A > 0$ and we take $B = A^{-\beta}$, $C = A^{-1+\beta}$, $\beta \in [0,1]$, then we derive

$$|\langle f(A)x,y\rangle| \le B(f,\gamma;A)\left\langle A^{2(\alpha-\beta)}x,x\right\rangle^{1/2}\left\langle A^{2(\beta-\alpha)}y,y\right\rangle^{1/2} \tag{2.16}$$

for $\alpha \in [0,1]$ and $x, y \in H$. In particular, for $\alpha = \beta$ we obtain

$$|\langle f(A)x,y\rangle| \le B(f,\gamma;A)\|x\|\|y\| \tag{2.17}$$

for $x, y \in H$.

Corollary 2.6. *With the assumptions of Theorem 2.4 and if*

$$\|f\|_{\gamma,\infty} := \sup_{\xi\in\gamma} |f(\xi)| < \infty,$$

then, by denoting

$$B_\infty(f,\gamma;A) := \frac{1}{2\pi}\|f\|_{\gamma,\infty}\int_\gamma (|\xi| - \|A\|)^{-1}|d\xi|,$$

we have

$$|\langle C^*Af(A)Bx,y\rangle| \le B_\infty(f,\gamma;A)\left\langle \left||A|^\alpha B\right|^2 x,x\right\rangle^{1/2}\left\langle \left||A^*|^{1-\alpha}C\right|^2 y,y\right\rangle^{1/2} \tag{2.18}$$

for $\alpha\in[0,1]$ and x, $y\in H$. In particular,

$$|\langle C^*Af(A)Bx,y\rangle| \le B_\infty(f,\gamma;A)\left\langle \left||A|^{1/2} B\right|^2 x,x\right\rangle^{1/2}\left\langle \left||A^*|^{1/2}C\right|^2 y,y\right\rangle^{1/2} \tag{2.19}$$

for x, $y\in H$.

Remark 2.7. *If we assume that $f: G\subset\mathbb{C}\to\mathbb{C}$ is an analytic function on the domain G and $A\in\mathscr{B}(H)$ with $Sp(A)\subset D(0,R)\subset D$ where $D(0,R)$ is an open disk centered in 0 and of radius R, then by taking γ parametrized by $\xi(t)=Re^{2\pi it}$ where $t\in[0,1]$, then $d\xi(t)=2\pi iRe^{2\pi it}dt$, $|d\xi(t)|=2\pi Rdt$, $|\xi|=R$ and by (2.18) we get for A, $B\in\mathscr{B}(H)$ that*

$$|\langle C^*Af(A)Bx,y\rangle| \le \frac{R}{R-\|A\|}\int_0^1\left|f\left(Re^{2\pi it}\right)\right|dt\times\left\langle \left||A|^\alpha B\right|^2 x,x\right\rangle^{1/2}\left\langle \left||A^*|^{1-\alpha}C\right|^2 y,y\right\rangle^{1/2} \tag{2.20}$$

where $\alpha\in[0,1]$ and x, $y\in H$. In particular,

$$|\langle C^*Af(A)Bx,y\rangle| \le \frac{R}{R-\|A\|}\int_0^1\left|f\left(Re^{2\pi it}\right)\right|dt\times\left\langle \left||A|^{1/2} B\right|^2 x,x\right\rangle^{1/2}\left\langle \left||A^*|^{1/2}C\right|^2 y,y\right\rangle^{1/2} \tag{2.21}$$

for x, $y\in H$.

Moreover, if $\|f\|_{R,\infty} := \sup_{t\in[0,1]}\left|f\left(Re^{2\pi it}\right)\right| < \infty$, then we have the simpler inequalities

$$|\langle C^*Af(A)Bx,y\rangle| \le \frac{R\|f\|_{R,\infty}}{R-\|A\|}\left\langle \left||A|^\alpha B\right|^2 x,x\right\rangle^{1/2}\left\langle \left||A^*|^{1-\alpha}C\right|^2 y,y\right\rangle^{1/2} \tag{2.22}$$

for x, $y\in H$. In particular,

$$|\langle C^*Af(A)Bx,y\rangle| \le \frac{R\|f\|_{R,\infty}}{R-\|A\|}\times\left\langle \left||A|^{1/2} B\right|^2 x,x\right\rangle^{1/2}\left\langle \left||A^*|^{1/2}C\right|^2 y,y\right\rangle^{1/2} \tag{2.23}$$

for x, $y\in H$.

3. Norm and Numerical Radius Inequalities

The following vector inequality for positive operators $A\ge 0$, obtained by C. A. McCarthy in [12] is well known,

$$\langle Ax,x\rangle^p \le \langle A^p x,x\rangle,\ p\ge 1$$

for $x\in H$, $\|x\|=1$.

Buzano's inequality [3],

$$\frac{1}{2}[\|x\|\|y\| + |\langle x,y\rangle|] \ge |\langle x,e\rangle\langle e,y\rangle| \tag{3.1}$$

that holds for any x, y, $e\in H$ with $\|e\|=1$ will also be used in the sequel.

We also have the following norm and numerical radius inequalities:

Theorem 3.1. *Let $f: G\subset\mathbb{C}\to\mathbb{C}$ be an analytic function on the domain G and $A\in\mathscr{B}(H)$ with $Sp(A)\subset G$ and γ a closed rectifiable path in G and such that $Sp(A)\subset ins(\gamma)$. If $B, C\in\mathscr{B}(H)$, then we have the norm inequality*

$$\|C^*Af(A)B\| \le B(f,\gamma;A)\left\||A|^\alpha B\right\|\left\||A^*|^{1-\alpha}C\right\|. \tag{3.2}$$

We also have the numerical radius inequalities

$$\omega(C^*Af(A)B) \le \frac{1}{2}B(f,\gamma;A)\left\|\left||A|^\alpha B\right|^2 + \left||A^*|^{1-\alpha}C\right|^2\right\| \tag{3.3}$$

and

$$\omega^2(C^*Af(A)B) \le \frac{1}{2}B^2(f,\gamma;A)\left[\left\||A|^\alpha B\right\|^2\left\||A^*|^{1-\alpha}C\right\|^2 + \omega\left(\left||A^*|^{1-\alpha}C\right|^2\left||A|^\alpha B\right|^2\right)\right]. \tag{3.4}$$

Proof. We have from (2.7), by taking the supremum over $\|x\| = \|y\| = 1$, that

$$\begin{aligned}\|C^*Af(A)B\|^2 &= \sup_{\|x\|=\|y\|=1} |\langle C^*Af(A)Bx,y\rangle|^2 \\ &\le B^2(f,\gamma;A) \sup_{\|x\|=1}\left\langle \left||A|^\alpha B\right|^2 x,x\right\rangle \sup_{\|y\|=1}\left\langle \left||A^*|^{1-\alpha}C\right|^2 y,y\right\rangle \\ &= B^2(f,\gamma;A)\left\|\left||A|^\alpha B\right|^2\right\| \left\|\left||A^*|^{1-\alpha}C\right|^2\right\| \\ &= B^2(f,\gamma;A)\left\||A|^\alpha B\right\|^2 \left\||A^*|^{1-\alpha}C\right\|^2,\end{aligned}$$

which gives (3.2).
From (2.7) we get, by taking $y = x$, the square root and using the *A-G-mean inequality*, that

$$\begin{aligned}|\langle C^*Af(A)Bx,x\rangle| &\le B(f,\gamma;A)\left\langle \left||A|^\alpha B\right|^2 x,x\right\rangle^{1/2}\left\langle \left||A^*|^{1-\alpha}C\right|^2 x,x\right\rangle^{1/2} \\ &\le \frac{1}{2}B(f,\gamma;A)\left(\left\langle \left||A|^\alpha B\right|^2 x,x\right\rangle + \left\langle \left||A^*|^{1-\alpha}C\right|^2 x,x\right\rangle\right) \\ &= \frac{1}{2}B(f,\gamma;A)\left\langle\left(\left||A|^\alpha B\right|^2 + \left||A^*|^{1-\alpha}C\right|^2\right)x,x\right\rangle\end{aligned} \tag{3.5}$$

for all $x \in H$.
By taking the supremum over $\|x\| = 1$ in (3.5) we get that

$$\begin{aligned}\omega(C^*Af(A)B) &= \sup_{\|x\|=1}|\langle C^*Af(A)Bx,x\rangle| \\ &\le \frac{1}{2}B(f,\gamma;A)\sup_{\|x\|=1}\left\langle\left(\left||A|^\alpha B\right|^2 + \left||A^*|^{1-\alpha}C\right|^2\right)x,x\right\rangle \\ &= \frac{1}{2}B(f,\gamma;A)\left\|\left||A|^\alpha B\right|^2 + \left||A^*|^{1-\alpha}C\right|^2\right\|,\end{aligned}$$

which proves (3.2).
From (2.7) for $y = x$ and Buzano's inequality we derive that

$$\begin{aligned}|\langle C^*Af(A)Bx,x\rangle|^2 &\le B^2(f,\gamma;A)\left\langle \left||A|^\alpha B\right|^2 x,x\right\rangle\left\langle x,\left||A^*|^{1-\alpha}C\right|^2 x\right\rangle \\ &\le \frac{1}{2}B^2(f,\gamma;A) \\ &\times\left[\left\|\left||A|^\alpha B\right|^2 x\right\|\left\|\left||A^*|^{1-\alpha}C\right|^2 x\right\| + \left|\left\langle \left||A|^\alpha B\right|^2 x,\left||A^*|^{1-\alpha}C\right|^2 x\right\rangle\right|\right] \\ &= \frac{1}{2}B^2(f,\gamma;A) \\ &\times\left[\left\|\left||A|^\alpha B\right|^2 x\right\|\left\|\left||A^*|^{1-\alpha}C\right|^2 x\right\| + \left|\left\langle \left||A^*|^{1-\alpha}C\right|^2\left||A|^\alpha B\right|^2 x,x\right\rangle\right|\right]\end{aligned} \tag{3.6}$$

for all $x \in H$.
By taking the supremum over $\|x\| = 1$ in (3.6) we get that

$$\begin{aligned}\omega^2(C^*Af(A)B) &= \sup_{\|x\|=1}|\langle C^*Af(A)Bx,x\rangle|^2 \\ &\le \frac{1}{2}B^2(f,\gamma;A)\times\sup_{\|x\|=1}\left[\left\|\left||A|^\alpha B\right|^2 x\right\|\left\|\left||A^*|^{1-\alpha}C\right|^2 x\right\| + \left|\left\langle \left||A^*|^{1-\alpha}C\right|^2\left||A|^\alpha B\right|^2 x,x\right\rangle\right|\right] \\ &\le \frac{1}{2}B^2(f,\gamma;A)\times\left[\sup_{\|x\|=1}\left\{\left\|\left||A|^\alpha B\right|^2 x\right\|\left\|\left||A^*|^{1-\alpha}C\right|^2 x\right\|\right\} + \sup_{\|x\|=1}\left|\left\langle \left||A^*|^{1-\alpha}C\right|^2\left||A|^\alpha B\right|^2 x,x\right\rangle\right|\right] \\ &\le \frac{1}{2}B^2(f,\gamma;A)\times\left[\sup_{\|x\|=1}\left\|\left||A|^\alpha B\right|^2 x\right\|\sup_{\|x\|=1}\left\|\left||A^*|^{1-\alpha}C\right|^2 x\right\| + \sup_{\|x\|=1}\left|\left\langle \left||A^*|^{1-\alpha}C\right|^2\left||A|^\alpha B\right|^2 x,x\right\rangle\right|\right] \\ &= \frac{1}{2}B^2(f,\gamma;A)\left[\left\|\left||A|^\alpha B\right|^2\right\|\left\|\left||A^*|^{1-\alpha}C\right|^2\right\| + \omega\left(\left||A^*|^{1-\alpha}C\right|^2\left||A|^\alpha B\right|^2\right)\right] \\ &= \frac{1}{2}B^2(f,\gamma;A)\left[\left\||A|^\alpha B\right\|^2\left\||A^*|^{1-\alpha}C\right\|^2 + \omega\left(\left||A^*|^{1-\alpha}C\right|^2\left||A|^\alpha B\right|^2\right)\right],\end{aligned}$$

which proves (3.4). □

Remark 3.2. *If we take $\alpha = 1/2$ in Theorem 3.1, then we get the norm inequality*

$$\|C^*Af(A)B\| \le B(f,\gamma;A)\left\||A|^{1/2}B\right\|\left\||A^*|^{1/2}C\right\| \tag{3.7}$$

and the numerical radius inequalities

$$\omega(C^*Af(A)B) \le \frac{1}{2}B(f,\gamma;A)\left\|\left||A|^{1/2}B\right|^2+\left||A^*|^{1/2}C\right|^2\right\| \tag{3.8}$$

and

$$\omega^2(C^*Af(A)B) \tag{3.9}$$
$$\le \frac{1}{2}B^2(f,\gamma;A)\left[\left\||A|^{1/2}B\right\|^2\left\||A^*|^{1/2}C\right\|^2+\omega\left(\left||A^*|^{1/2}C\right|^2\left||A|^{1/2}B\right|^2\right)\right].$$

The second main result is as follows:

Theorem 3.3. *Assume that the conditions of Theorem 3.1 are satisfied. If $\alpha \in [0,1]$, $r>0$, $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $pr, qr \ge 1$, then*

$$\omega^{2r}(C^*Af(A)B) \le B^{2r}(f,\gamma;A)\left\|\frac{1}{p}\left||A|^{\alpha}B\right|^{2rp}+\frac{1}{q}\left||A^*|^{1-\alpha}C\right|^{2rq}\right\|. \tag{3.10}$$

If $r \ge 1$, then

$$\omega^{2r}(C^*Af(A)B) \le \frac{1}{2}B^{2r}(f,\gamma;A)\left[\left\||A|^{\alpha}B\right\|^{2r}\left\||A^*|^{1-\alpha}C\right\|^{2r}+\omega^r\left(\left||A^*|^{1-\alpha}C\right|^2\left||A|^{\alpha}B\right|^2\right)\right]. \tag{3.11}$$

If $r \ge 1$, $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $pr, qr \ge 2$, then also

$$\omega^{2r}(C^*Af(A)B) \le \frac{1}{2}B^{2r}(f,\gamma;A)\left(\left\|\frac{1}{p}\left||A|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left||A^*|^{1-\alpha}C\right|^{2qr}\right\|+\omega^r\left(\left||A^*|^{1-\alpha}C\right|^2\left||A|^{\alpha}B\right|^2\right)\right). \tag{3.12}$$

Proof. If we take the power $r>0$ in (2.7) written for $y=x$ then we get, by Young and McCarthy inequalities that

$$\begin{aligned}
|\langle C^*Af(A)Bx,x\rangle|^{2r} &\le B^{2r}(f,\gamma;A)\left\langle\left||A|^{\alpha}B\right|^2x,x\right\rangle^r\left\langle\left||A^*|^{1-\alpha}C\right|^2x,x\right\rangle^r\\
&\le B^{2r}(f,\gamma;A)\left[\frac{1}{p}\left\langle\left||A|^{\alpha}B\right|^2x,x\right\rangle^{rp}+\frac{1}{q}\left\langle\left||A^*|^{1-\alpha}C\right|^2x,x\right\rangle^{rq}\right]\\
&\le B^{2r}(f,\gamma;A)\left[\frac{1}{p}\left\langle\left||A|^{\alpha}B\right|^{2rp}x,x\right\rangle+\frac{1}{q}\left\langle\left||A^*|^{1-\alpha}C\right|^{2rq}x,x\right\rangle\right]\\
&= B^{2r}(f,\gamma;A)\left[\left\langle\frac{1}{p}\left||A|^{\alpha}B\right|^{2rp}+\frac{1}{q}\left||A^*|^{1-\alpha}C\right|^{2rq}x,x\right\rangle\right]
\end{aligned}$$

for $x \in H$ with $\|x\|=1$.
By taking the supremum over $\|x\|=1$, then we get that

$$\begin{aligned}
\omega^{2r}(C^*Af(A)B) &= \sup_{\|x\|=1}|\langle C^*Af(A)Bx,x\rangle|^{2r}\\
&\le B^{2r}(f,\gamma;A)\sup_{\|x\|=1}\left[\left\langle\left(\frac{1}{p}\left||A|^{\alpha}B\right|^{2rp}+\frac{1}{q}\left||A^*|^{1-\alpha}C\right|^{2rq}\right)x,x\right\rangle\right]\\
&= B^{2r}(f,\gamma;A)\left\|\frac{1}{p}\left||A|^{\alpha}B\right|^{2rp}+\frac{1}{q}\left||A^*|^{1-\alpha}C\right|^{2rq}\right\|,
\end{aligned}$$

which proves (3.10).
If we take the power $r \ge 1$ in (3.6) and by using the convexity of the power function, we get

$$\begin{aligned}
|\langle C^*Af(A)Bx,x\rangle|^{2r} &= B^{2r}(f,\gamma;A)\times\left[\frac{\left\|\left||A|^{\alpha}B\right|^2x\right\|\left\|\left||A^*|^{1-\alpha}C\right|^2x\right\|+\left|\left\langle\left||A^*|^{1-\alpha}C\right|^2\left||A|^{\alpha}B\right|^2x,x\right\rangle\right|}{2}\right]^r\\
&\le B^{2r}(f,\gamma;A)\times\frac{\left\|\left||A|^{\alpha}B\right|^2x\right\|^r\left\|\left||A^*|^{1-\alpha}C\right|^2x\right\|^r+\left|\left\langle\left||A^*|^{1-\alpha}C\right|^2\left||A|^{\alpha}B\right|^2x,x\right\rangle\right|^r}{2}
\end{aligned} \tag{3.13}$$

for $x \in H$ with $\|x\|=1$.
By taking the supremum over $\|x\|=1$, then we get that

$$\begin{aligned}
\omega^{2r}(C^*Af(A)B) &\le B^{2r}(f,\gamma;A)\times\frac{\left\|\left||A|^{\alpha}B\right|^2\right\|^r\left\|\left||A^*|^{1-\alpha}C\right|^2\right\|^r+\omega^r\left(\left||A^*|^{1-\alpha}C\right|^2\left||A|^{\alpha}B\right|^2\right)}{2}\\
&= B^{2r}(f,\gamma;A)\times\frac{\left\||A|^{\alpha}B\right\|^{2r}\left\||A^*|^{1-\alpha}C\right\|^{2r}+\omega^r\left(\left||A^*|^{1-\alpha}C\right|^2\left||A|^{\alpha}B\right|^2\right)}{2},
\end{aligned}$$

which proves (3.11).
Also, observe that

$$\begin{aligned}\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|^{r}\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\|^{r} &\le \frac{1}{p}\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|^{pr}+\frac{1}{q}\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\|^{qr}\\ &= \frac{1}{p}\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|^{2\frac{pr}{2}}+\frac{1}{q}\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\|^{2\frac{qr}{2}}\\ &= \frac{1}{p}\left\langle\left|\left|A\right|^{\alpha}B\right|^{4}x,x\right\rangle^{\frac{pr}{2}}+\frac{1}{q}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{4}x,x\right\rangle^{\frac{qr}{2}}\\ &\le \frac{1}{p}\left\langle\left|\left|A\right|^{\alpha}B\right|^{2pr}x,x\right\rangle+\frac{1}{q}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}x,x\right\rangle\\ &= \left\langle\left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}\right)x,x\right\rangle,\end{aligned}$$

for $x\in H$ with $\|x\|=1$. Then

$$\begin{aligned}&\frac{\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|^{r}\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\|^{r}+\left|\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle\right|^{r}}{2} \le \frac{1}{2}\left[\left\langle\left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}\right)x,x\right\rangle\right.\\ &\qquad\left.+\left|\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle\right|^{r}\right]\end{aligned}$$

and by (3.13)

$$|\langle C^{*}Af(A)Bx,x\rangle|^{2r}\le\frac{1}{2}B^{2r}(f,\gamma;A)\left[\left\langle\left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}\right)x,x\right\rangle+\left|\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle\right|^{r}\right]$$

for $x\in H$ with $\|x\|=1$.
By taking the supremum over $\|x\|=1$, we derive (3.12). □

Remark 3.4. *If we take $r=1$ and $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ in (3.10), then we obtain*

$$\omega^{2}(C^{*}Af(A)B)\le B^{2}(f,\gamma;A)\left\|\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2p}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2q}\right\|,\tag{3.14}$$

which for $p=q=2$ gives

$$\omega^{2}(C^{*}Af(A)B)\le\frac{1}{2}B^{2}(f,\gamma;A)\left\|\left|\left|A\right|^{\alpha}B\right|^{4}+\left|\left|A^{*}\right|^{1-\alpha}C\right|^{4}\right\|.\tag{3.15}$$

If we take $r=1$ and $p=q=2$ in (3.12), then we get

$$\omega^{2}(C^{*}Af(A)B)\le\frac{1}{2}B^{2}(f,\gamma;A)\left(\frac{1}{2}\left\|\left|\left|A\right|^{\alpha}B\right|^{4}+\left|\left|A^{*}\right|^{1-\alpha}C\right|^{4}\right\|+\omega\left(\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}\right)\right).\tag{3.16}$$

If we take $r=2$ and $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ in (3.12), then we get

$$\omega^{4}(C^{*}Af(A)B)\le\frac{1}{2}B^{4}(f,\gamma;A)\left(\left\|\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{4p}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{4q}\right\|+\omega^{2}\left(\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}\right)\right).\tag{3.17}$$

We also have:

Theorem 3.5. *With the assumptions of Theorem 3.1, we have for $r\ge1$, $\lambda\in[0,1]$ that*

$$\omega^{2}(C^{*}Af(A)B)\le B^{2}(f,\gamma;A)\left\|(1-\lambda)\left|\left|A\right|^{\alpha}B\right|^{2r}+\lambda\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2r}\right\|^{1/r}\times\left\|\left|A\right|^{\alpha}B\right\|^{2\lambda}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|^{2(1-\lambda)}\tag{3.18}$$

for all $\alpha\in[0,1]$.
Also, we have

$$\omega^{2}(C^{*}Af(A)B)\le B^{2}(f,\gamma;A)\left\|(1-\lambda)\left|\left|A\right|^{\alpha}B\right|^{2r}+\lambda\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2r}\right\|^{1/r}\times\left\|\lambda\left|\left|A\right|^{\alpha}B\right|^{2r}+(1-\lambda)\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2r}\right\|^{1/r}\tag{3.19}$$

for all $\alpha\in[0,1]$ and $r\ge1$.

Proof. From the first part of (3.6) we have

$$\begin{aligned}|\langle C^{*}Af(A)Bx,x\rangle|^{2} &\leq B^{2}(f,\gamma;A)\left\langle \left||A|^{\alpha}B\right|^{2}x,x\right\rangle \left\langle x,\left||A^{*}|^{1-\alpha}C\right|^{2}x\right\rangle \\ &= B^{2}(f,\gamma;A)\left\langle \left||A|^{\alpha}B\right|^{2}x,x\right\rangle^{1-\lambda}\left\langle x,\left||A^{*}|^{1-\alpha}C\right|^{2}x\right\rangle^{\lambda}\times\left\langle \left||A|^{\alpha}B\right|^{2}x,x\right\rangle^{\lambda}\left\langle x,\left||A^{*}|^{1-\alpha}C\right|^{2}x\right\rangle^{1-\lambda} \\ &\leq B^{2}(f,\gamma;A)\left[(1-\lambda)\left\langle \left||A|^{\alpha}B\right|^{2}x,x\right\rangle+\lambda\left\langle x,\left||A^{*}|^{1-\alpha}C\right|^{2}x\right\rangle\right]\times\left\langle \left||A|^{\alpha}B\right|^{2}x,x\right\rangle^{\lambda}\left\langle x,\left||A^{*}|^{1-\alpha}C\right|^{2}x\right\rangle^{1-\lambda}\end{aligned}$$

for all $x\in H$, $\|x\|=1$.
If we take the power $r\geq 1$, then we get by the convexity of power r that

$$\begin{aligned}|\langle C^{*}Af(A)Bx,x\rangle|^{2r} &\leq B^{2r}(f,\gamma;A)\left[(1-\lambda)\left\langle \left||A|^{\alpha}B\right|^{2}x,x\right\rangle+\lambda\left\langle x,\left||A^{*}|^{1-\alpha}C\right|^{2}x\right\rangle\right]^{r} \\ &\quad\times\left\langle \left||A|^{\alpha}B\right|^{2}x,x\right\rangle^{r\lambda}\left\langle x,\left||A^{*}|^{1-\alpha}C\right|^{2}x\right\rangle^{r(1-\lambda)} \\ &\leq B^{2r}(f,\gamma;A)\left[(1-\lambda)\left\langle \left||A|^{\alpha}B\right|^{2}x,x\right\rangle^{r}+\lambda\left\langle x,\left||A^{*}|^{1-\alpha}C\right|^{2}x\right\rangle^{r}\right] \\ &\quad\times\left\langle \left||A|^{\alpha}B\right|^{2}x,x\right\rangle^{r\lambda}\left\langle x,\left||A^{*}|^{1-\alpha}C\right|^{2}x\right\rangle^{r(1-\lambda)}\end{aligned}\tag{3.20}$$

for all $x\in H$, $\|x\|=1$.
If we use McCarthy inequality for power $r\geq 1$, then we get

$$\begin{aligned}(1-\lambda)\left\langle \left||A|^{\alpha}B\right|^{2}x,x\right\rangle^{r}+\lambda\left\langle x,\left||A^{*}|^{1-\alpha}C\right|^{2}x\right\rangle^{r} &\leq (1-\lambda)\left\langle \left||A|^{\alpha}B\right|^{2r}x,x\right\rangle+\lambda\left\langle x,\left||A^{*}|^{1-\alpha}C\right|^{2r}x\right\rangle \\ &= \left\langle\left[(1-\lambda)\left||A|^{\alpha}B\right|^{2r}+\lambda\left||A^{*}|^{1-\alpha}C\right|^{2r}\right]x,x\right\rangle\end{aligned}$$

and by (3.20)

$$|\langle C^{*}Af(A)Bx,x\rangle|^{2r}\leq B^{2r}(f,\gamma;A)\left[\left\langle\left[(1-\lambda)\left||A|^{\alpha}B\right|^{2r}+\lambda\left||A^{*}|^{1-\alpha}C\right|^{2r}\right]x,x\right\rangle\right]\times\left\langle \left||A|^{\alpha}B\right|^{2}x,x\right\rangle^{r\lambda}\left\langle x,\left||A^{*}|^{1-\alpha}C\right|^{2}x\right\rangle^{r(1-\lambda)}\tag{3.21}$$

for all $x\in H$, $\|x\|=1$.
If we take the supremum over $\|x\|=1$, then we get

$$\begin{aligned}\omega^{2r}(C^{*}Af(A)B) &= \sup_{\|x\|=1}|\langle C^{*}Af(A)Bx,x\rangle|^{2r} \\ &\leq B^{2r}(f,\gamma;A)\sup_{\|x\|=1}\left[\left\langle\left[(1-\lambda)\left||A|^{\alpha}B\right|^{2r}+\lambda\left||A^{*}|^{1-\alpha}C\right|^{2r}\right]x,x\right\rangle\right] \\ &\quad\times\sup_{\|x\|=1}\left\langle \left||A|^{\alpha}B\right|^{2}x,x\right\rangle^{r\lambda}\sup_{\|x\|=1}\left\langle x,\left||A^{*}|^{1-\alpha}C\right|^{2}x\right\rangle^{r(1-\lambda)} \\ &= B^{2r}(f,\gamma;A)\left\|(1-\lambda)\left||A|^{\alpha}B\right|^{2r}+\lambda\left||A^{*}|^{1-\alpha}C\right|^{2r}\right\|\times\left\||A|^{\alpha}B\right\|^{2r\lambda}\left\||A^{*}|^{1-\alpha}C\right\|^{2r(1-\lambda)},\end{aligned}$$

which gives (3.18).
We also have

$$|\langle C^{*}Af(A)Bx,x\rangle|^{2r}\leq B^{2r}(f,\gamma;A)\left[\left\langle\left[(1-\lambda)\left||A|^{\alpha}B\right|^{2r}+\lambda\left||A^{*}|^{1-\alpha}C\right|^{2r}\right]x,x\right\rangle\right]\times\left[\left\langle\left[\lambda\left||A|^{\alpha}B\right|^{2r}+(1-\lambda)\left||A^{*}|^{1-\alpha}C\right|^{2r}\right]x,x\right\rangle\right]$$

for all $x\in H$, $\|x\|=1$, which proves (3.19). □

Remark 3.6. *If we take $r=1$ in Theorem 3.5, then we get*

$$\omega^{2}(C^{*}Af(A)B)\leq B^{2}(f,\gamma;A)\left\|(1-\lambda)\left||A|^{\alpha}B\right|^{2}+\lambda\left||A^{*}|^{1-\alpha}C\right|^{2}\right\|\times\left\||A|^{\alpha}B\right\|^{2\lambda}\left\||A^{*}|^{1-\alpha}C\right\|^{2(1-\lambda)}\tag{3.22}$$

and

$$\omega^{2}(C^{*}Af(A)B)\leq B^{2}(f,\gamma;A)\left\|(1-\lambda)\left||A|^{\alpha}B\right|^{2}+\lambda\left||A^{*}|^{1-\alpha}C\right|^{2}\right\|\times\left\|\lambda\left||A|^{\alpha}B\right|^{2}+(1-\lambda)\left||A^{*}|^{1-\alpha}C\right|^{2}\right\|\tag{3.23}$$

for all $\alpha,\lambda\in[0,1]$.
If we take $\lambda=1/2$ in (3.22), then we obtain

$$\omega^{2}(C^{*}Af(A)B)\leq\frac{1}{2}B^{2}(f,\gamma;A)\left\|\left||A|^{\alpha}B\right|^{2}+\left||A^{*}|^{1-\alpha}C\right|^{2}\right\|\left\||A|^{\alpha}B\right\|\left\||A^{*}|^{1-\alpha}C\right\|.\tag{3.24}$$

If we take $r=2$ in Theorem 3.5, then we get

$$\omega^2(C^*Af(A)B) \le B^2(f,\gamma;A)\left\|(1-\lambda)\left||A|^\alpha B\right|^4+\lambda\left||A^*|^{1-\alpha}C\right|^4\right\|^{1/2}\times\left\||A|^\alpha B\right\|^{2\lambda}\left\||A^*|^{1-\alpha}C\right\|^{2(1-\lambda)} \tag{3.25}$$

and

$$\omega^2(C^*Af(A)B) \le B^2(f,\gamma;A)\left\|(1-\lambda)\left||A|^\alpha B\right|^4+\lambda\left||A^*|^{1-\alpha}C\right|^4\right\|^{1/2}\times\left\|\lambda\left||A|^\alpha B\right|^4+(1-\lambda)\left||A^*|^{1-\alpha}C\right|^4\right\|^{1/2} \tag{3.26}$$

for all $\alpha,\lambda\in[0,1]$.
If we take $\lambda=1/2$ in (3.25), then we obtain

$$\omega^2(C^*Af(A)B) \le \frac{\sqrt{2}}{2}B^2(f,\gamma;A)\left\|\left||A|^\alpha B\right|^4+\left||A^*|^{1-\alpha}C\right|^4\right\|^{1/2}\left\||A|^\alpha B\right\|\left\||A^*|^{1-\alpha}C\right\|. \tag{3.27}$$

4. Inequalities for Trace of Operators

We have the following result for trace of operators:

Theorem 4.1. *Let $r\ge 1/2$, $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $pr,qr\ge 1$. Let $f:G\subset\mathbb{C}\to\mathbb{C}$ be an analytic function on the domain G and $A\in\mathscr{B}(H)$ with $Sp(A)\subset G$ and γ a closed rectifiable path in G and such that $Sp(A)\subset ins(\gamma)$. If $B,C\in\mathscr{B}(H)$ with $|A|^\alpha B\in\mathscr{B}_{2pr}(H)$ and $|A^*|^{1-\alpha}C\in\mathscr{B}_{2qr}(H)$ for $\alpha\in[0,1]$, then $C^*Af(A)B\in\mathscr{B}_{2r}(H)$ and*

$$\|C^*Af(A)B\|_{2r}\le B(f,\gamma;A)\left\||A|^\alpha B\right\|_{2pr}\left\||A^*|^{1-\alpha}C\right\|_{2qr}. \tag{4.1}$$

In particular,

$$\|C^*Af(A)B\|_{2r}\le B(f,\gamma;A)\left\||A|^{1/2}B\right\|_{2pr}\left\||A^*|^{1/2}C\right\|_{2qr} \tag{4.2}$$

for $|A|^{1/2}B\in\mathscr{B}_{2pr}(H)$ and $|A^*|^{1/2}C\in\mathscr{B}_{2qr}(H)$.

Proof. If we take in (2.7) the power $r>0$ and $x=e_i$, $y=f_i$ where $\mathscr{E}=\{e_i\}_{i\in I}$ and $\mathscr{F}=\{f_i\}_{i\in I}$ are orthonormal basis and sum, then we get

$$\sum_{i\in I}|\langle C^*Af(A)Be_i,f_i\rangle|^{2r}\le B^{2r}(f,\gamma;A)\sum_{i\in I}\left\langle\left||A|^\alpha B\right|^2e_i,e_i\right\rangle^r\left\langle\left||A^*|^{1-\alpha}C\right|^2f_i,f_i\right\rangle^r. \tag{4.3}$$

If we use the Hölder's inequality for $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we get

$$\sum_{i\in I}\left\langle\left||A|^\alpha B\right|^2e_i,e_i\right\rangle^r\left\langle\left||A^*|^{1-\alpha}C\right|^2f_i,f_i\right\rangle^r\le\left(\sum_{i\in I}\left\langle\left||A|^\alpha B\right|^2e_i,e_i\right\rangle^{pr}\right)^{1/p}\left(\sum_{i\in I}\left\langle\left||A^*|^{1-\alpha}C\right|^2f_i,f_i\right\rangle^{qr}\right)^{1/q}. \tag{4.4}$$

By the McCarthy inequality for $pr,qr\ge 1$, we have

$$\sum_{i\in I}\left\langle\left||A|^\alpha B\right|^2e_i,e_i\right\rangle^{pr}\le\sum_{i\in I}\left\langle\left||A|^\alpha B\right|^{2pr}e_i,e_i\right\rangle$$

and

$$\sum_{i\in I}\left\langle\left||A^*|^{1-\alpha}C\right|^2f_i,f_i\right\rangle^{qr}\le\sum_{i\in I}\left\langle\left||A^*|^{1-\alpha}C\right|^{2qr}f_i,f_i\right\rangle,$$

therefore

$$\begin{aligned}\left(\sum_{i\in I}\left\langle\left||A|^\alpha B\right|^2e_i,e_i\right\rangle^{pr}\right)^{1/p}\left(\sum_{i\in I}\left\langle\left||A^*|^{1-\alpha}C\right|^2f_i,f_i\right\rangle^{qr}\right)^{1/q}&\le\left(\sum_{i\in I}\left\langle\left||A|^\alpha B\right|^{2pr}e_i,e_i\right\rangle\right)^{1/p}\left(\sum_{i\in I}\left\langle\left||A^*|^{1-\alpha}C\right|^2f_i,f_i\right\rangle^{qr}\right)^{1/q}\\&=\left(\left\||A|^\alpha B\right\|_{2pr}^{2pr}\right)^{1/p}\left(\left\||A^*|^{1-\alpha}C\right\|_{2qr}^{2qr}\right)^{1/q}=\left\||A|^\alpha B\right\|_{2pr}^{2r}\left\||A^*|^{1-\alpha}C\right\|_{2qr}^{2r}.\end{aligned}$$

By (4.3) and (4.4) we derive

$$\sum_{i\in I}|\langle C^*Af(A)Be_i,f_i\rangle|^{2r}\le B^{2r}(f,\gamma;A)\left\||A|^\alpha B\right\|_{2pr}^{2r}\left\||A^*|^{1-\alpha}C\right\|_{2qr}^{2r}. \tag{4.5}$$

Now, if we take the supremum over $\mathscr{E}$ and $\mathscr{F}$ in (4.5), then by (1.23) we get

$$\|C^*Af(A)B\|_{2r}^{2r}\le B^{2r}(f,\gamma;A)\left\||A|^\alpha B\right\|_{2pr}^{2r}\left\||A^*|^{1-\alpha}C\right\|_{2qr}^{2r}$$

and the inequality (4.1) is obtained. □

Remark 4.2. *If we take $r = 1/2$ and $p = q = 2$, then by (4.1) we get*

$$\|C^*Af(A)B\|_1 \le B(f,\gamma;A)\,\||A|^\alpha B\|_2 \left\||A^*|^{1-\alpha}C\right\|_2 \tag{4.6}$$

provided that $|A|^\alpha B \in \mathscr{B}_2(H)$ and $|A^*|^{1-\alpha}C \in \mathscr{B}_2(H)$ for $\alpha \in [0,1]$.
Also, if $r = 1$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by (4.1) we get

$$\|C^*Af(A)B\|_2 \le B(f,\gamma;A)\,\||A|^\alpha B\|_{2p} \left\||A^*|^{1-\alpha}C\right\|_{2q} \tag{4.7}$$

provided that $|A|^\alpha B \in \mathscr{B}_{2p}(H)$ and $|A^*|^{1-\alpha}C \in \mathscr{B}_{2q}(H)$ for $\alpha \in [0,1]$.

We also have:

Theorem 4.3. *Let $r \ge 1/2$, $p, q \ge 1$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Let $f : G \subset \mathbb{C} \to \mathbb{C}$ be an analytic function on the domain G and $A \in \mathscr{B}(H)$ with $Sp(A) \subset G$ and γ a closed rectifiable path in G and such that $Sp(A) \subset ins(\gamma)$. If $|A|^\alpha B \in \mathscr{B}_{2p}(H)$ and $|A^*|^{1-\alpha}C \in \mathscr{B}_{2q}(H)$ for $\alpha \in [0,1]$, then $C^*Af(A)B \in \mathscr{B}_{2r}(H)$ and*

$$\|C^*Af(A)B\|_{2r} \le B(f,\gamma;A)\,\||A|^\alpha B\|_{2p} \left\||A^*|^{1-\alpha}C\right\|_{2q}. \tag{4.8}$$

In particular,

$$\|C^*Af(A)B\|_{2r} \le B(f,\gamma;A)\left\||A|^{1/2}B\right\|_{2p} \left\||A^*|^{1/2}C\right\|_{2q} \tag{4.9}$$

for $|A|^{1/2}B \in \mathscr{B}_{2p}(H)$ and $|A^*|^{1/2}C \in \mathscr{B}_{2q}(H)$.

Proof. Assume that $\mathscr{E} = \{e_i\}_{i\in I}$ and $\mathscr{F} = \{f_i\}_{i\in I}$ are orthonormal basis in H. Observe that we have $\frac{1}{\frac{p}{r}} + \frac{1}{\frac{q}{r}} = 1$ and by Hölder's inequality for $\frac{p}{r}$ and $\frac{q}{r}$ we have

$$\begin{aligned}\sum_{i\in I}\left\langle \left||A|^\alpha B\right|^2 e_i, e_i\right\rangle^r \left\langle \left||A^*|^{1-\alpha}C\right|^2 f_i, f_i\right\rangle^r &= \sum_{i\in I}\left[\left\langle \left||A|^\alpha B\right|^2 e_i, e_i\right\rangle^p\right]^{\frac{r}{p}} \left[\left\langle \left||A^*|^{1-\alpha}C\right|^2 f_i, f_i\right\rangle^q\right]^{\frac{r}{q}} \\ &\le \left(\sum_{i\in I}\left\langle \left||A|^\alpha B\right|^2 e_i, e_i\right\rangle^p\right)^{r/p} \left(\sum_{i\in I}\left\langle \left||A^*|^{1-\alpha}C\right|^2 f_i, f_i\right\rangle^q\right)^{r/q}.\end{aligned} \tag{4.10}$$

By McCarthy inequality for $p, q > 1$ we get

$$\sum_{i\in I}\left\langle \left||A|^\alpha B\right|^2 e_i, e_i\right\rangle^p \le \sum_{i\in I}\left\langle \left||A|^\alpha B\right|^{2p} e_i, e_i\right\rangle$$

and

$$\sum_{i\in I}\left\langle \left||A^*|^{1-\alpha}C\right|^2 f_i, f_i\right\rangle^q \le \sum_{i\in I}\left\langle \left||A^*|^{1-\alpha}C\right|^{2q} f_i, f_i\right\rangle$$

and by (4.10)

$$\begin{aligned}\sum_{i\in I}\left\langle \left||A|^\alpha B\right|^2 e_i, e_i\right\rangle^r \left\langle \left||A^*|^{1-\alpha}C\right|^2 f_i, f_i\right\rangle^r &\le \left(\sum_{i\in I}\left\langle \left||A|^\alpha B\right|^{2p} e_i, e_i\right\rangle\right)^{r/p} \left(\sum_{i\in I}\left\langle \left||A^*|^{1-\alpha}C\right|^{2q} f_i, f_i\right\rangle\right)^{r/q} \\ &= \||A|^\alpha B\|_{2p}^{2r} \left\||A^*|^{1-\alpha}C\right\|_{2q}^{2r}.\end{aligned} \tag{4.11}$$

By (4.3) and (4.11) we get

$$\sum_{i\in I}|\langle C^*Af(A)Be_i, f_i\rangle|^{2r} \le B^{2r}(f,\gamma;A)\,\||A|^\alpha B\|_{2p}^{2r} \left\||A^*|^{1-\alpha}C\right\|_{2q}^{2r}. \tag{4.12}$$

Now, if we take the supremum over $\mathscr{E}$ and $\mathscr{F}$ in (4.12) we get

$$\|C^*Af(A)B\|_{2r}^{2r} \le B^{2r}(f,\gamma;A)\,\||A|^\alpha B\|_{2p}^{2r} \left\||A^*|^{1-\alpha}C\right\|_{2q}^{2r}$$

and the inequality (4.8) is thus proved. □

Remark 4.4. *If we take $p=q=2r=s\geq 1$, then by (4.8) we get*

$$\|C^{*}Af(A)B\|_{s}\leq B(f,\gamma;A)\left\||A|^{\alpha}B\right\|_{2s}\left\||A^{*}|^{1-\alpha}C\right\|_{2s} \tag{4.13}$$

provided that $|A|^{\alpha}B\in\mathscr{B}_{2s}(H)$ and $|A^{*}|^{1-\alpha}C\in\mathscr{B}_{2s}(H)$ for $\alpha\in[0,1]$.
For $\alpha=1/2$ we have

$$\|C^{*}Af(A)B\|_{s}\leq B(f,\gamma;A)\left\||A|^{1/2}B\right\|_{2s}\left\||A^{*}|^{1/2}C\right\|_{2s} \tag{4.14}$$

provided that $|A|^{1/2}B\in\mathscr{B}_{2s}(H)$ and $|A^{*}|^{1/2}C\in\mathscr{B}_{2s}(H)$.
If $r=2$ and $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$, then

$$\|C^{*}Af(A)B\|_{4}\leq B(f,\gamma;A)\left\||A|^{\alpha}B\right\|_{2p}\left\||A^{*}|^{1-\alpha}C\right\|_{2q} \tag{4.15}$$

provided that $|A|^{\alpha}B\in\mathscr{B}_{2p}(H)$ and $|A^{*}|^{1-\alpha}C\in\mathscr{B}_{2q}(H)$ for $\alpha\in[0,1]$.
In particular,

$$\|C^{*}Af(A)B\|_{4}\leq B(f,\gamma;A)\left\||A|^{1/2}B\right\|_{2p}\left\||A^{*}|^{1/2}C\right\|_{2q} \tag{4.16}$$

for $|A|^{1/2}B\in\mathscr{B}_{2p}(H)$ and $|A^{*}|^{1/2}C\in\mathscr{B}_{2q}(H)$.

Theorem 4.5. *Let $f:G\subset\mathbb{C}\to\mathbb{C}$ be an analytic function on the domain G and $A\in\mathscr{B}(H)$ with $Sp(A)\subset G$ and γ a closed rectifiable path in G and such that $Sp(A)\subset ins(\gamma)$.*
If $r\geq 1/2$, $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, $pr,qr\geq 1$ and $\left||A|^{\alpha}B\right|^{2pr},\left||A^{*}|^{1-\alpha}C\right|^{2qr}\in\mathscr{B}_{1}(H)$, then $C^{*}Af(A)B\in\mathscr{B}_{2r}(H)$ and

$$\omega_{2r}^{2r}(C^{*}Af(A)B)\leq B^{2r}(f,\gamma;A)\,tr\left(\frac{1}{p}\left||A|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left||A^{*}|^{1-\alpha}C\right|^{2qr}\right). \tag{4.17}$$

If $r\geq 1$ and $|A|^{\alpha}B,|A^{*}|^{1-\alpha}C\in\mathscr{B}_{4r}(H)$, then $C^{*}Af(A)B\in\mathscr{B}_{2r}(H)$ and

$$\begin{aligned}\omega_{2r}^{2r}(C^{*}Af(A)B)&\leq\frac{1}{2}B^{2r}(f,\gamma;A)\left(\||A|^{\alpha}B\|_{4r}^{2r}\left\||A^{*}|^{1-\alpha}C\right\|_{4r}^{2r}+\omega_{r}^{r}\left(\left||A^{*}|^{1-\alpha}C\right|^{2}\left||A|^{\alpha}B\right|^{2}\right)\right)\\&\leq\frac{1}{2}B^{2r}(f,\gamma;A)\left(\||A|^{\alpha}B\|_{4r}^{2r}\left\||A^{*}|^{1-\alpha}C\right\|_{4r}^{2r}+\left\|\left||A^{*}|^{1-\alpha}C\right|^{2}\left||A|^{\alpha}B\right|^{2}\right\|_{r}^{r}\right).\end{aligned} \tag{4.18}$$

If $r\geq 1$, $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, $pr,qr\geq 2$ and $|A|^{\alpha}B$, $|A^{*}|^{1-\alpha}C\in\mathscr{B}_{1}(H)$ then $C^{*}Af(A)B\in\mathscr{B}_{2r}(H)$ and

$$\begin{aligned}\omega_{2r}^{2r}(C^{*}Af(A)B)&\leq\frac{1}{2}B^{2r}(f,\gamma;A)\left[tr\left(\frac{1}{p}\left||A|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left||A^{*}|^{1-\alpha}C\right|^{2qr}\right)+\omega_{r}^{r}\left(\left||A^{*}|^{1-\alpha}C\right|^{2}\left||A|^{\alpha}B\right|^{2}\right)\right]\\&\leq\frac{1}{2}B^{2r}(f,\gamma;A)\left[tr\left(\frac{1}{p}\left||A|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left||A^{*}|^{1-\alpha}C\right|^{2qr}\right)+\left\|\left||A^{*}|^{1-\alpha}C\right|^{2}\left||A|^{\alpha}B\right|^{2}\right\|_{r}^{r}\right].\end{aligned} \tag{4.19}$$

Proof. From (2.7) for $y=x$ we have that

$$|\langle C^{*}Af(A)Bx,x\rangle|^{2}\leq B^{2}(f,\gamma;A)\left\langle\left||A|^{\alpha}B\right|^{2}x,x\right\rangle\left\langle\left||A^{*}|^{1-\alpha}C\right|^{2}x,x\right\rangle \tag{4.20}$$

for $x\in H$ with $\|x\|=1$.
If we take the power $r>0$, we get, by Young and McCarthy inequalities, that

$$\begin{aligned}|\langle C^{*}Af(A)Bx,x\rangle|^{2r}&\leq B^{2r}(f,\gamma;A)\left\langle\left||A|^{\alpha}B\right|^{2}x,x\right\rangle^{r}\left\langle\left||A^{*}|^{1-\alpha}C\right|^{2}x,x\right\rangle^{r}\\&\leq B^{2r}(f,\gamma;A)\left[\frac{1}{p}\left\langle\left||A|^{\alpha}B\right|^{2}x,x\right\rangle^{pr}+\frac{1}{q}\left\langle\left||A^{*}|^{1-\alpha}C\right|^{2}x,x\right\rangle^{qr}\right]\\&\leq B^{2r}(f,\gamma;A)\left[\frac{1}{p}\left\langle\left||A|^{\alpha}B\right|^{2pr}x,x\right\rangle+\frac{1}{q}\left\langle\left||A^{*}|^{1-\alpha}C\right|^{2qr}x,x\right\rangle\right]\\&=B^{2r}(f,\gamma;A)\left\langle\left(\frac{1}{p}\left||A|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left||A^{*}|^{1-\alpha}C\right|^{2qr}\right)x,x\right\rangle\end{aligned}$$

for $x\in H$ with $\|x\|=1$.
If $\mathscr{E}=\{e_i\}_{i\in I}$ is an orthonormal basis, then by taking $x=e_i$ and summing over $i\in I$ we get

$$\begin{aligned}\|C^{*}Af(A)B\|_{\mathscr{E},2r}^{2r}&=\sum_{i\in I}|\langle C^{*}Af(A)Be_i,e_i\rangle|^{2r}\\&\leq B^{2r}(f,\gamma;A)\sum_{i\in I}\left\langle\left(\frac{1}{p}\left||A|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left||A^{*}|^{1-\alpha}C\right|^{2qr}\right)e_i,e_i\right\rangle\\&=B^{2r}(f,\gamma;A)\,\mathrm{tr}\left(\frac{1}{p}\left||A|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left||A^{*}|^{1-\alpha}C\right|^{2qr}\right),\end{aligned}$$

which, by taking the supremum over $\mathscr{E}$, proves (4.17).
By Buzano's inequality we have

$$\left\langle \left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle \left\langle x,\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\rangle \leq \frac{1}{2}\left[\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\|+\left|\left\langle \left|\left|A\right|^{\alpha}B\right|^{2}x,\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\rangle\right|\right]$$
$$=\frac{1}{2}\left[\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\|+\left|\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle\right|\right]$$

for $x\in H$ with $\|x\|=1$.
If we take the power $r\geq 1$ and use the convexity of power function, then we get

$$\left\langle \left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle^{r} \left\langle x,\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\rangle^{r} \leq \left[\frac{\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\|+\left|\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle\right|}{2}\right]^{r}$$
$$\leq\frac{\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|^{r}\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\|^{r}+\left|\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle\right|^{r}}{2}$$
$$=\frac{\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|^{2\frac{r}{2}}\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\|^{2\frac{r}{2}}+\left|\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle\right|^{r}}{2}$$
$$=\frac{\left\langle \left|\left|A\right|^{\alpha}B\right|^{4}x,x\right\rangle^{\frac{r}{2}}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{4}x,x\right\rangle^{\frac{r}{2}}+\left|\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle\right|^{r}}{2}$$

for $x\in H$ with $\|x\|=1$.
Therefore

$$\begin{aligned}\|C^{*}Af(A)B\|_{\mathscr{E},2r}^{2r} &=\sum_{i\in I}\left|\langle C^{*}Af(A)Be_{i},e_{i}\rangle\right|^{2r} \\ &\leq B^{2r}(f,\gamma;A)\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{2}e_{i},e_{i}\right\rangle^{r}\left\langle e_{i},\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}e_{i}\right\rangle^{r} \\ &\leq\frac{1}{2}B^{2r}(f,\gamma;A)\left[\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{4}e_{i},e_{i}\right\rangle^{\frac{r}{2}}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{4}e_{i},e_{i}\right\rangle^{\frac{r}{2}}+\sum_{i\in I}\left|\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}e_{i},e_{i}\right\rangle\right|^{r}\right].\end{aligned}\tag{4.21}$$

Using Cauchy-Schwarz inequality we have

$$\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{4}e_{i},e_{i}\right\rangle^{\frac{r}{2}}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{4}e_{i},e_{i}\right\rangle^{\frac{r}{2}}\leq\left(\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{4}e_{i},e_{i}\right\rangle^{r}\right)^{1/2}\left(\sum_{i\in I}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{4}e_{i},e_{i}\right\rangle^{r}\right)^{1/2}$$
$$\leq\left(\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{4r}e_{i},e_{i}\right\rangle\right)^{1/2}\left(\sum_{i\in I}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{4r}e_{i},e_{i}\right\rangle\right)^{1/2}$$
$$=\left\|\left|A\right|^{\alpha}B\right\|_{4r}^{2r}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{4r}^{2r},$$

where for the last inequality we used McCarthy's result for $r\geq 1$.
By taking the supremum over $\mathscr{E}$, we get the desired result (4.18).
Further, if we use Young's inequality for $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$,

$$ab\leq\frac{1}{p}a^{p}+\frac{1}{q}b^{q},\ a,b\geq 0,$$

then we get

$$\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|^{r}\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\|^{r}\leq\frac{1}{p}\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|^{pr}+\frac{1}{q}\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\|^{qr}$$
$$=\frac{1}{p}\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|^{2\frac{pr}{2}}+\frac{1}{q}\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\|^{2\frac{qr}{2}}$$
$$=\frac{1}{p}\left\langle \left|\left|A\right|^{\alpha}B\right|^{4}x,x\right\rangle^{\frac{pr}{2}}+\frac{1}{q}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{4}x,x\right\rangle^{\frac{qr}{2}}$$
$$\leq\frac{1}{p}\left\langle \left|\left|A\right|^{\alpha}B\right|^{2pr}x,x\right\rangle+\frac{1}{q}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}x,x\right\rangle$$
$$=\left\langle\left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}\right)x,x\right\rangle$$

for $x\in H$ with $\|x\|=1$.
Therefore

$$\begin{aligned}\|C^{*}Af(A)B\|_{\mathscr{E},2r}^{2r}&=\sum_{i\in I}|\langle C^{*}Af(A)Be_i,e_i\rangle|^{2r}\\&\le B^{2r}(f,\gamma;A)\sum_{i\in I}\left\langle \left||A|^{\alpha}B\right|^{2}e_i,e_i\right\rangle^{r}\left\langle e_i,\left||A^{*}|^{1-\alpha}C\right|^{2}e_i\right\rangle^{r}\\&\le\frac{1}{2}B^{2r}(f,\gamma;A)\left[\sum_{i\in I}\left\langle\left(\frac{1}{p}\left||A|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left||A^{*}|^{1-\alpha}C\right|^{2qr}\right)e_i,e_i\right\rangle+\sum_{i\in I}\left|\left\langle\left||A^{*}|^{1-\alpha}C\right|^{2}\left||A|^{\alpha}B\right|^{2}e_i,e_i\right\rangle\right|^{r}\right]\\&=\frac{1}{2}B^{2r}(f,\gamma;A)\left[\operatorname{tr}\left(\frac{1}{p}\left||A|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left||A^{*}|^{1-\alpha}C\right|^{2qr}\right)+\left\|\left||A^{*}|^{1-\alpha}C\right|^{2}\left||A|^{\alpha}B\right|^{2}\right\|_{\mathscr{E},r}^{r}\right],\end{aligned}$$

which proves, by taking the supremum over $\mathscr{E}$, the desired inequality (4.19). □

Remark 4.6. *Let $\alpha\in[0,1]$. If $r=1/2$, $p,q=2$ and $\left||A|^{\alpha}B\right|^{2}$, $\left||A^{*}|^{1-\alpha}C\right|^{2}\in\mathscr{B}_1(H)$, then $C^{*}Af(A)B\in\mathscr{B}_1(H)$ and by (4.17) we get*

$$\omega_1(C^{*}Af(A)B)\le\frac{1}{2}B(f,\gamma;A)\,tr\left(\left||A|^{\alpha}B\right|^{2}+\left||A^{*}|^{1-\alpha}C\right|^{2}\right).\tag{4.22}$$

If $r=1$ and $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then by (4.17) we obtain

$$\omega_2^{2}(C^{*}Af(A)B)\le B^{2}(f,\gamma;A)\,tr\left(\frac{1}{p}\left||A|^{\alpha}B\right|^{2p}+\frac{1}{q}\left||A^{*}|^{1-\alpha}C\right|^{2q}\right),\tag{4.23}$$

provided that $\left||A|^{\alpha}B\right|^{2p}$, $\left||A^{*}|^{1-\alpha}C\right|^{2q}\in\mathscr{B}_1(H)$.
If we take $r=1$ in (4.18), then we get

$$\begin{aligned}\omega_2^{2}(C^{*}Af(A)B)&\le\frac{1}{2}B^{2}(f,\gamma;A)\left(\||A|^{\alpha}B\|_4^{2}\left\||A^{*}|^{1-\alpha}C\right\|_4^{2}+\omega_1\left(\left||A^{*}|^{1-\alpha}C\right|^{2}\left||A|^{\alpha}B\right|^{2}\right)\right)\\&\le\frac{1}{2}B^{2}(f,\gamma;A)\left(\||A|^{\alpha}B\|_4^{2}\left\||A^{*}|^{1-\alpha}C\right\|_4^{2}+\left\|\left||A^{*}|^{1-\alpha}C\right|^{2}\left||A|^{\alpha}B\right|^{2}\right\|_1\right),\end{aligned}\tag{4.24}$$

provided that $|A|^{\alpha}B$, $|A^{*}|^{1-\alpha}C\in\mathscr{B}_4(H)$.
If $r=1$ and $p=q=2$ in (4.19), then we get for $\left||A|^{\alpha}B\right|^{2p}$, $\left||A^{*}|^{1-\alpha}C\right|^{2q}\in\mathscr{B}_1(H)$ that

$$\begin{aligned}\omega_2^{2}(C^{*}Af(A)B)&\le\frac{1}{4}B^{2}(f,\gamma;A)\left[tr\left(\left||A|^{\alpha}B\right|^{2p}+\left||A^{*}|^{1-\alpha}C\right|^{2q}\right)+\frac{1}{2}B^{2}(f,\gamma;A)\,\omega_1\left(\left||A^{*}|^{1-\alpha}C\right|^{2}\left||A|^{\alpha}B\right|^{2}\right)\right.\\&\le\frac{1}{4}B^{2}(f,\gamma;A)\,tr\left(\left||A|^{\alpha}B\right|^{2p}+\left||A^{*}|^{1-\alpha}C\right|^{2q}\right)+\frac{1}{2}B^{2}(f,\gamma;A)\left\|\left||A^{*}|^{1-\alpha}C\right|^{2}\left||A|^{\alpha}B\right|^{2}\right\|_1.\end{aligned}\tag{4.25}$$

We also have:

Theorem 4.7. *With the assumptions of Theorem 4.5, we have for $r\ge1$, $\lambda\in[0,1]$ that*

$$\omega_{2r}^{2r}(C^{*}Af(A)B)\le B^{2r}(f,\gamma;A)\left\|(1-\lambda)\left||A|^{\alpha}B\right|^{2r}+\lambda\left||A^{*}|^{1-\alpha}C\right|^{2r}\right\|\times\||A|^{\alpha}B\|_{2r}^{2r\lambda}\left\||A^{*}|^{1-\alpha}C\right\|_{2r}^{2r(1-\lambda)},\tag{4.26}$$

provided that $|A|^{\alpha}B$, $|A^{*}|^{1-\alpha}C\in\mathscr{B}_{2r}(H)$.
In particular,

$$\omega_{2r}^{2r}(C^{*}Af(A)B)\le\frac{1}{2}B^{2r}(f,\gamma;A)\left\|\left||A|^{\alpha}B\right|^{2r}+\left||A^{*}|^{1-\alpha}C\right|^{2r}\right\|\times\||A|^{\alpha}B\|_{2r}^{r}\left\||A^{*}|^{1-\alpha}C\right\|_{2r}^{r}.\tag{4.27}$$

Proof. If $\mathscr{E}=\{e_i\}_{i\in I}$ is an orthonormal basis, then by taking $x=e_i$ in (3.21) and summing over $i\in I$ we get

$$\begin{aligned}\sum_{i\in I}|\langle C^{*}Af(A)Be_i,e_i\rangle|^{2r}&\le B^{2r}(f,\gamma;A)\sum_{i\in I}\left[\left\langle\left[(1-\lambda)\left||A|^{\alpha}B\right|^{2r}+\lambda\left||A^{*}|^{1-\alpha}C\right|^{2r}\right]e_i,e_i\right\rangle\right]\\&\quad\times\left\langle\left||A|^{\alpha}B\right|^{2}e_i,e_i\right\rangle^{r\lambda}\left\langle\left||A^{*}|^{1-\alpha}C\right|^{2}e_i,e_i\right\rangle^{r(1-\lambda)}\\&\le B^{2r}(f,\gamma;A)\left\|(1-\lambda)\left||A|^{\alpha}B\right|^{2r}+\lambda\left||A^{*}|^{1-\alpha}C\right|^{2r}\right\|\times\sum_{i\in I}\left\langle\left||A|^{\alpha}B\right|^{2}e_i,e_i\right\rangle^{r\lambda}\left\langle\left||A^{*}|^{1-\alpha}C\right|^{2}e_i,e_i\right\rangle^{r(1-\lambda)}.\end{aligned}\tag{4.28}$$

If we use Hölder's inequality for $p=\frac{1}{\lambda}$, $q=\frac{1}{1-\lambda}$, then we have

$$\begin{aligned}\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{2}e_{i},e_{i}\right\rangle^{r\lambda}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}e_{i},e_{i}\right\rangle^{r(1-\lambda)} &\leq \left(\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{2}e_{i},e_{i}\right\rangle^{r}\right)^{\lambda}\left(\sum_{i\in I}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}e_{i},e_{i}\right\rangle^{r}\right)^{1-\lambda}\\ &\leq \left(\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{2r}e_{i},e_{i}\right\rangle\right)^{\lambda}\left(\sum_{i\in I}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2r}e_{i},e_{i}\right\rangle\right)^{1-\lambda}\\ &=\left\|\left|A\right|^{\alpha}B\right\|_{2r}^{2r\lambda}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2r}^{2r(1-\lambda)}.\end{aligned}$$

By taking the supremum over $\mathscr{E}$, we get the desired result (4.26). □

Remark 4.8. *If we take $r=1$ in Theorem 4.7, then we get for $\alpha\in[0,1]$ that*

$$\omega_{2}^{2}\left(C^{*}Af\left(A\right)B\right)\leq B^{2}\left(f,\gamma;A\right)\left\|\left(1-\lambda\right)\left|\left|A\right|^{\alpha}B\right|^{2}+\lambda\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\right\|\times\left\|\left|A\right|^{\alpha}B\right\|_{2}^{2\lambda}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2}^{2(1-\lambda)},\tag{4.29}$$

provided that $\left|A\right|^{\alpha}B$, $\left|A^{*}\right|^{1-\alpha}C\in\mathscr{B}_{2}\left(H\right)$.
In particular,

$$\omega_{2}^{2}\left(C^{*}Af\left(A\right)B\right)\leq\frac{1}{2}B^{2}\left(f,\gamma;A\right)\left\|\left|\left|A\right|^{\alpha}B\right|^{2}+\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\right\|\times\left\|\left|A\right|^{\alpha}B\right\|_{2}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2}.\tag{4.30}$$

5. Some Examples

Consider the exponential function $f(A)=\exp A$, $A\in\mathscr{B}(H)$. Assume that $A\in\mathscr{B}(H)$ and $\|A\|<R$ for some $R>0$. Observe that for $t\in[0,1]$,

$$\left|\exp\left(Re^{2\pi it}\right)\right|=\left|\exp\left[R\left(\cos\left(2\pi t\right)+i\sin\left(2\pi t\right)\right)\right]\right|=\exp\left[R\cos\left(2\pi t\right)\right]$$

and then by (2.20) we get for B, $C\in\mathscr{B}(H)$ that

$$\left|\left\langle C^{*}A\exp\left(A\right)Bx,y\right\rangle\right|\leq\frac{R}{R-\|A\|}\int_{0}^{1}\exp\left[R\cos\left(2\pi t\right)\right]dt\times\left\langle \left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle^{1/2}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}y,y\right\rangle^{1/2}\tag{5.1}$$

for x, $y\in H$
The *modified Bessel function of the first kind* $I_{\nu}(z)$ for real number ν can be defined by the power series as [1, p. 376]

$$I_{\nu}(z)=\left(\frac{1}{2}z\right)^{\nu}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{4}z^{2}\right)^{k}}{k!\Gamma\left(\nu+k+1\right)},$$

where Γ is the *gamma function*. For $\nu=0$ we have $I_{0}(z)$ given by

$$I_{0}(z)=\sum_{k=0}^{\infty}\frac{\left(\frac{1}{4}z^{2}\right)^{k}}{\left(k!\right)^{2}}.$$

An integral formula for real number ν is

$$I_{\nu}(z)=\frac{1}{\pi}\int_{0}^{\pi}e^{z\cos\theta}\cos\left(\nu\theta\right)d\theta-\frac{\sin\left(\nu\pi\right)}{\pi}\int_{0}^{\infty}e^{-z\cosh t-\nu t}dt,$$

which simplifies for ν an integer n to

$$I_{n}(z)=\frac{1}{\pi}\int_{0}^{\pi}e^{z\cos\theta}\cos\left(n\theta\right)d\theta.$$

For $n=0$ we have

$$I_{0}(z)=\frac{1}{\pi}\int_{0}^{\pi}e^{z\cos\theta}d\theta.$$

If we change the variable $\theta=2\pi t$, then $dt=\frac{1}{2\pi}d\theta$ and

$$\begin{aligned}\int_{0}^{1}\exp\left[R\cos\left(2\pi t\right)\right]dt&=\frac{1}{2\pi}\int_{0}^{2\pi}\exp\left[R\cos\theta\right]d\theta\\&=\frac{1}{2}\left(\frac{1}{\pi}\int_{0}^{\pi}\exp\left[R\cos\theta\right]d\theta+\frac{1}{\pi}\int_{\pi}^{2\pi}\exp\left[R\cos\theta\right]d\theta\right)\\&=\frac{1}{2}\left(I_{0}(R)+I_{0}(-R)\right)=I_{0}(R).\end{aligned}$$

From (5.1) we then get

$$|\langle C^{*}A\exp(A)Bx,y\rangle| \leq \frac{RI_{0}(R)}{R-\|A\|}\times\left\langle\left||A|^{\alpha}B\right|^{2}x,x\right\rangle^{1/2}\left\langle\left||A^{*}|^{1-\alpha}C\right|^{2}y,y\right\rangle^{1/2} \tag{5.2}$$

for $\alpha\in[0,1]$, $x, y\in H$, $A, B, C\in\mathscr{B}(H)$ with $\|A\|<R$.
By taking $B=C=I$ in (5.2) we get for $\|A\|<R$ that

$$|\langle A\exp(A)x,y\rangle| \leq \frac{RI_{0}(R)}{R-\|A\|}\left\langle|A|^{2\alpha}x,x\right\rangle^{1/2}\left\langle|A^{*}|^{2(1-\alpha)}y,y\right\rangle^{1/2} \tag{5.3}$$

for $x, y\in H$. In particular,

$$|\langle A\exp(A)x,y\rangle| \leq \frac{RI_{0}(R)}{R-\|A\|}\langle|A|x,x\rangle^{1/2}\langle|A^{*}|y,y\rangle^{1/2} \tag{5.4}$$

for $x, y\in H$.
If A is invertible and take $C=I$, $B=A^{-1}$ in (5.2), then we get for $\|A\|<R$ that

$$|\langle\exp(A)x,y\rangle| \leq \frac{RI_{0}(R)}{R-\|A\|}\left\langle|A|^{-2(1-\alpha)}x,x\right\rangle^{1/2}\left\langle|A^{*}|^{2(1-\alpha)}y,y\right\rangle^{1/2} \tag{5.5}$$

for $x, y\in H$. In particular,

$$|\langle\exp(A)x,y\rangle| \leq \frac{RI_{0}(R)}{R-\|A\|}\left\langle|A|^{-1}x,x\right\rangle^{1/2}\langle|A^{*}|y,y\rangle^{1/2} \tag{5.6}$$

for $x, y\in H$.
If $0<A$ with $\|A\|<R$ and we take $B=A^{-\beta}$, $C=A^{-1+\beta}$, $\alpha, \beta\in[0,1]$, then by (5.2) we derive

$$|\langle\exp(A)x,y\rangle| \leq \frac{RI_{0}(R)}{R-\|A\|}\left\langle A^{2(\alpha-\beta)}x,x\right\rangle^{1/2}\left\langle A^{2(\beta-\alpha)}y,y\right\rangle^{1/2} \tag{5.7}$$

for $x, y\in H$.
By Theorem 3.1 we get the norm inequality

$$\|C^{*}A\exp(A)B\| \leq \frac{RI_{0}(R)}{R-\|A\|}\left\||A|^{\alpha}B\right\|\left\||A^{*}|^{1-\alpha}C\right\|. \tag{5.8}$$

We also have the numerical radius inequalities

$$\omega(C^{*}A\exp(A)B) \leq \frac{1}{2}\frac{RI_{0}(R)}{R-\|A\|}\left\|\left||A|^{\alpha}B\right|^{2}+\left||A^{*}|^{1-\alpha}C\right|^{2}\right\| \tag{5.9}$$

and

$$\omega^{2}(C^{*}A\exp(A)B) \leq \frac{1}{2}\left(\frac{RI_{0}(R)}{R-\|A\|}\right)^{2}\times\left[\left\||A|^{\alpha}B\right\|^{2}\left\||A^{*}|^{1-\alpha}C\right\|^{2}+\omega\left(\left||A^{*}|^{1-\alpha}C\right|^{2}\left||A|^{\alpha}B\right|^{2}\right)\right]. \tag{5.10}$$

Let $r\geq 1/2$, $p, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $pr, qr\geq 1$. If $B, C\in\mathscr{B}(H)$ with $|A|^{\alpha}B\in\mathscr{B}_{2pr}(H)$ and $|A^{*}|^{1-\alpha}C\in\mathscr{B}_{2qr}(H)$ for $\alpha\in[0,1]$, then $C^{*}A\exp(A)B\in\mathscr{B}_{2r}(H)$ and by (4.1)

$$\|C^{*}A\exp(A)B\|_{2r} \leq \frac{RI_{0}(R)}{R-\|A\|}\left\||A|^{\alpha}B\right\|_{2pr}\left\||A^{*}|^{1-\alpha}C\right\|_{2qr}. \tag{5.11}$$

If $r\geq 1/2$, $p, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, $pr, qr\geq 1$ and $\left||A|^{\alpha}B\right|^{2pr}, \left||A^{*}|^{1-\alpha}C\right|^{2qr}\in\mathscr{B}_{1}(H)$, then $C^{*}A\exp(A)B\in\mathscr{B}_{2r}(H)$ and by (4.17)

$$\begin{aligned}&\omega_{2r}^{2r}(C^{*}A\exp(A)B) \\ &\leq\left(\frac{RI_{0}(R)}{R-\|A\|}\right)^{2r}\operatorname{tr}\left(\frac{1}{p}\left||A|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left||A^{*}|^{1-\alpha}C\right|^{2qr}\right).\end{aligned} \tag{5.12}$$

By using the power series

$$f(z):=\ln(1-z)^{-1}=\sum_{n=1}^{\infty}\frac{1}{n}z^{n}$$

that is convergent on open disk $D(0,1)$, we can define for all elements A in $\mathscr{B}(H)$ with $\|A\|<1$,

$$\ln(I-A)^{-1}:=\sum_{n=1}^{\infty}\frac{1}{n}A^{n}.$$

We observe that for $|z|<1$

$$\left|\ln(1-z)^{-1}\right|\leq\sum_{n=1}^{\infty}\frac{1}{n}|z|^{n}=\ln(1-|z|)^{-1}.$$

Now if we assume that $A, B, C \in \mathscr{B}(H)$ and $\|A\| < R < 1$, then by (2.22) we get

$$\left|\left\langle C^{*}A\ln(I-A)^{-1}Bx,y\right\rangle\right| \leq \frac{R\ln(1-R)^{-1}}{R-\|A\|} \times \left\langle \left||A|^{\alpha}B\right|^{2}x,x\right\rangle^{1/2}\left\langle \left||A^{*}|^{1-\alpha}C\right|^{2}y,y\right\rangle^{1/2} \quad (5.13)$$

for $\alpha \in [0,1]$, $x, y \in H$.
By taking $B = C = I$ in (5.13) we get for $\|A\| < R < 1$ that

$$\left|\left\langle A\ln(I-A)^{-1}x,y\right\rangle\right| \leq \frac{R\ln(1-R)^{-1}}{R-\|A\|}\left\langle |A|^{2\alpha}x,x\right\rangle^{1/2}\left\langle |A^{*}|^{2(1-\alpha)}y,y\right\rangle^{1/2} \quad (5.14)$$

for $x, y \in H$. In particular,

$$\left|\left\langle A\ln(I-A)^{-1}x,y\right\rangle\right| \leq \frac{R\ln(1-R)^{-1}}{R-\|A\|}\langle |A|x,x\rangle^{1/2}\langle |A^{*}|y,y\rangle^{1/2} \quad (5.15)$$

for $x, y \in H$.
If A is invertible and take $C = I$, $B = A^{-1}$ in (5.13), then we get for $\|A\| < R < 1$ that

$$\left|\left\langle \ln(I-A)^{-1}x,y\right\rangle\right| \leq \frac{R\ln(1-R)^{-1}}{R-\|A\|}\left\langle |A|^{-2(1-\alpha)}x,x\right\rangle^{1/2}\left\langle |A^{*}|^{2(1-\alpha)}y,y\right\rangle^{1/2} \quad (5.16)$$

for $x, y \in H$. In particular,

$$\left|\left\langle \ln(I-A)^{-1}x,y\right\rangle\right| \leq \frac{R\ln(1-R)^{-1}}{R-\|A\|}\left\langle |A|^{-1}x,x\right\rangle^{1/2}\langle |A^{*}|y,y\rangle^{1/2}. \quad (5.17)$$

If $0 < A$ with $\|A\| < R < 1$ and we take $B = A^{-\beta}$, $C = A^{-1+\beta}$, $\beta \in [0,1]$ in (5.13), then we derive

$$\left|\left\langle \ln(I-A)^{-1}x,y\right\rangle\right| \leq \frac{R\ln(1-R)^{-1}}{R-\|A\|}\left\langle A^{2(\alpha-\beta)}x,x\right\rangle^{1/2}\left\langle A^{2(\beta-\alpha)}y,y\right\rangle^{1/2} \quad (5.18)$$

for $\alpha \in [0,1]$, $x, y \in H$.
One can state some norm, numerical radius and p-Schatten norm inequalities for $A\ln(I-A)^{-1}$, however the details are omitted.

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