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PARAMETRIC GENERALIZATION OF THE MODIFIED BERNSTEIN-KANTOROVICH OPERATORS

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Abstract. In the current article, a parametrization of the modified Bernstein-Kantorovich operators is studied. Then the Korovkin theorem, approximation properties and central moments of these operators are investigated. The rate of approximation of the operators is obtained by the help of modulus of continuity, functions from Lipschitz class and Peetre-$\mathcal{K}$ functional. Finally, some numerical examples are illustrated to show the effectiveness of the newly defined operators.

1. Introduction

Approximation theory has an important place in studies in the field of mathematics. Let f be a continuous function on the interval $[a,b]$ and then for every $\varepsilon > 0$, there is a polynomial p that satisfies the $\|f(x) - p(x)\| < \varepsilon$ condition. This theorem was given by Weierstrass [19] in 1885. In 1912, Bernstein [3] proved the approximation theorem defined by Weierstrass on the closed interval $[0,1]$. A generalization of Bernstein operators was made by Chen et al. [7] in 2017. Fuat Usta [18] defined modified Bernstein operators in 2020 as

$$B_\eta^*(g;x) = \frac{1}{\eta}\sum_{\zeta=0}^{\eta}\binom{\eta}{\zeta}(\zeta-\eta x)^2 x^{\zeta-1}(1-x)^{\eta-\zeta-1} g\left(\frac{\zeta}{\eta}\right).$$

By definition of the operator $B_\eta^*(g;x)$, he obtained the following equalities

$$B_\eta^*(1;x) = 1,$$

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$$\begin{aligned} B_\eta^*(t;x) &= \frac{\eta-2}{\eta}x+\frac{1}{\eta}, \\ B_\eta^*(t^2;x) &= \frac{(\eta^2-7\eta+6)}{\eta^2}x^2+\frac{5\eta-6}{\eta^2}x+\frac{1}{\eta^2}. \end{aligned}$$

Certain examples of articles on parametric generalizations of operators can be found in [2], [4], [5], [6], [8], [7], [9], [10], [12], [13], [14], [16], [17], [20] and [21].

The θ parameterization of modified Bernstein operators were defined for every $g \in C[0,1]$ by Sofyalıoğlu et al. [15] as

$$B_{\eta,\theta}^*(g;x)=\sum_{\zeta=0}^{\eta}\rho_{\eta,\zeta}^{(\theta)}(x)g\left(\frac{\zeta}{\eta}\right), \tag{1}$$

where $\eta \geq 1$, $0 \leq \theta \leq 1$, $x \in (0,1)$ and

$$\begin{aligned} \rho_{1,0}^{(\theta)}(x) &= x,\ \rho_{1,1}^{(\theta)}(x)=1-x, \\ \rho_{\eta,\zeta}^{(\theta)}(x) &= \left\{\frac{1}{\eta-1}\begin{pmatrix}\eta-2\\ \zeta\end{pmatrix}(\zeta-(\eta-1)x)^2(1-\theta)x\right. \\ &\quad +\frac{1}{\eta-1}\begin{pmatrix}\eta-2\\ \zeta-2\end{pmatrix}(\zeta-1-(\eta-1)x)^2(1-\theta)(1-x) \\ &\quad \left.+\frac{1}{\eta}\begin{pmatrix}\eta\\ \zeta\end{pmatrix}(\zeta-\eta x)^2\theta x(1-x)\right\}x^{\zeta-2}(1-x)^{\eta-\zeta-2}, \quad \eta \geq 2 \end{aligned}$$

with binomial coefficients

$$\begin{pmatrix}\eta\\ \zeta\end{pmatrix}=\begin{cases}\frac{\eta!}{(\eta-\zeta)!\zeta!} & \text{if } 0 \leq \zeta \leq \eta \\ 0 & \text{otherwise}\end{cases}.$$

In this paper, we give the Kantorovich type of parametric generalizations of the modified Bernstein operators created by Sofyalıoğlu et al. [15]. Later, we study approximation properties of the operators. Then we give central moments and rate of convergence.

Now, we define the parametric generalization of the modified Bernstein-Kantorovich operators

$$K_{\eta,\theta}^*(g;x)=\sum_{\zeta=0}^{\eta}\rho_{\eta,\zeta}^{(\theta)}(x)\int_{\frac{\zeta}{\eta}}^{\frac{\zeta+1}{\eta}}g(t)dt, \tag{2}$$

where $\eta \geq 1$, $0 \leq \theta \leq 1$, $x \in (0,1)$ and

$$\begin{aligned} \rho_{1,0}^{(\theta)}(x) &= x,\ \rho_{1,1}^{(\theta)}(x)=1-x, \\ \rho_{\eta,\zeta}^{(\theta)}(x) &= \left\{\frac{\eta}{\eta-1}\begin{pmatrix}\eta-2\\ \zeta\end{pmatrix}(\zeta-(\eta-1)x)^2(1-\theta)x\right. \\ &\quad +\frac{\eta}{\eta-1}\begin{pmatrix}\eta-2\\ \zeta-2\end{pmatrix}(\zeta-1-(\eta-1)x)^2(1-\theta)(1-x) \end{aligned}$$

$$+\left(\begin{array}{c}\eta\\ \zeta\end{array}\right)(\zeta-\eta x)^2\theta x(1-x)\Big\} x^{\zeta-2}(1-x)^{\eta-\zeta-2},\quad \eta\geq 2$$

with binomial coefficients

$$\left(\begin{array}{c}\eta\\ \zeta\end{array}\right)=\left\{\begin{array}{cc}\frac{\eta!}{(\eta-\zeta)!\zeta!} & \text{if } 0\leq\zeta\leq\eta\\ 0 & \text{otherwise}\end{array}\right. .$$

Choosing $\theta=1$, it is seen that the operators $B^*_{\eta,\theta}(g;x)$ turn into $B^*_{\eta}(g;x)$ given by Usta [18].
The following equalities are going to use in the proof of the next theorem

$$\left(\begin{array}{c}\eta-2\\ \zeta\end{array}\right)=\left(1-\frac{\zeta}{\eta-1}\right)\left(\begin{array}{c}\eta-1\\ \zeta\end{array}\right), \tag{3}$$

$$\left(\begin{array}{c}\eta-2\\ \zeta-1\end{array}\right)=\frac{\zeta}{\eta-1}\left(\begin{array}{c}\eta-1\\ \zeta\end{array}\right). \tag{4}$$

Theorem 1. *The parametric generalization of the modified Bernstein-Kantorovich operators can be expressed as*

$$\begin{aligned}K^*_{\eta,\theta}(g;x) &= (1-\theta)\sum_{\zeta=0}^{\eta-1}\frac{\eta}{\eta-1}\left[\left(1-\frac{\zeta}{\eta-1}\right)\int_{\frac{\zeta}{\eta}}^{\frac{\zeta+1}{\eta}}g(t)dt+\frac{\zeta}{\eta-1}\int_{\frac{\zeta+1}{\eta}}^{\frac{\zeta+2}{\eta}}g(t)dt\right]\\ &\quad\times\left(\begin{array}{c}\eta-1\\ \zeta\end{array}\right)(\zeta-(\eta-1)x)^2x^{\zeta-1}(1-x)^{\eta-\zeta-2}\\ &\quad+\theta\sum_{\zeta=0}^{\eta}\left(\begin{array}{c}\eta\\ \zeta\end{array}\right)(\zeta-\eta x)^2x^{\zeta-1}(1-x)^{\eta-\zeta-1}\int_{\frac{\zeta}{\eta}}^{\frac{\zeta+1}{\eta}}g(t)dt.\end{aligned}$$

Proof. We rewrite the Eqn. (2) in more explicit form as

$$\begin{aligned}K^*_{\eta,\theta}(g;x) &= (1-\theta)\left[\sum_{\zeta=0}^{\eta}\frac{\eta}{\eta-1}\left(\begin{array}{c}\eta-2\\ \zeta\end{array}\right)(\zeta-(\eta-1)x)^2x^{\zeta-1}(1-x)^{\eta-\zeta-2}\right.\\ &\quad\times\int_{\frac{\zeta}{\eta}}^{\frac{\zeta+1}{\eta}}g(t)dt\\ &\quad+\sum_{\zeta=0}^{\eta}\frac{\eta}{\eta-1}\left(\begin{array}{c}\eta-2\\ \zeta-2\end{array}\right)(\zeta-1-(\eta-1)x)^2x^{\zeta-2}(1-x)^{\eta-\zeta-1}\\ &\quad\left.\times\int_{\frac{\zeta}{\eta}}^{\frac{\zeta+1}{\eta}}g(t)dt\right]\\ &\quad+\theta\sum_{\zeta=0}^{\eta}\left(\begin{array}{c}\eta\\ \zeta\end{array}\right)(\zeta-\eta x)^2x^{\zeta-1}(1-x)^{\eta-\zeta-1}\int_{\frac{\zeta}{\eta}}^{\frac{\zeta+1}{\eta}}g(t)dt.\end{aligned}$$

In other words,

$$K^*_{\eta,\theta}(g;x) = (1-\theta)(\mu_1+\mu_2) + \theta \sum_{\zeta=0}^{\eta} \binom{\eta}{\zeta} (\zeta-\eta x)^2 x^{\zeta-1}(1-x)^{\eta-\zeta-1} \times \int_{\frac{\zeta}{\eta}}^{\frac{\zeta+1}{\eta}} g(t)dt, \tag{5}$$

where μ_1 and μ_2 are

$$\mu_1 = \sum_{\zeta=0}^{\eta-2} \frac{\eta}{\eta-1} \binom{\eta-2}{\zeta} (\zeta-(\eta-1)x)^2 x^{\zeta-1}(1-x)^{\eta-\zeta-2} \int_{\frac{\zeta}{\eta}}^{\frac{\zeta+1}{\eta}} g(t)dt,$$

$$\mu_2 = \sum_{\zeta=1}^{\eta} \frac{\eta}{\eta-1} \binom{\eta-2}{\zeta-2} (\zeta-1-(\eta-1)x)^2 x^{\zeta-2}(1-x)^{\eta-\zeta-1} \times \int_{\frac{\zeta}{\eta}}^{\frac{\zeta+1}{\eta}} g(t)dt.$$

When we choose the term $\zeta = \eta$ and $\zeta = \eta - 1$ respectively, we get $\mu_1 = 0$. Similarly, replacing $\zeta = 0$ gives $\mu_2 = 0$.
Therefore, we obtain

$$\mu_2 = \sum_{\zeta=0}^{\eta-2} \frac{\eta}{\eta-1} \binom{\eta-2}{\zeta-1} (\zeta-(\eta-1)x)^2 x^{\zeta-1}(1-x)^{\eta-\zeta-2} \int_{\frac{\zeta+1}{\eta}}^{\frac{\zeta+2}{\eta}} g(t)dt.$$

By using Eqn. (3) and Eqn. (4), we have

$$\mu_1+\mu_2 = \sum_{\zeta=0}^{\eta-2} \frac{\eta}{\eta-1} \left[\binom{\eta-2}{\zeta} \int_{\frac{\zeta}{\eta}}^{\frac{\zeta+1}{\eta}} g(t)dt + \binom{\eta-2}{\zeta-1} \int_{\frac{\zeta+1}{\eta}}^{\frac{\zeta+2}{\eta}} g(t)dt \right] \times (\zeta-(\eta-1)x)^2 x^{\zeta-1}(1-x)^{\eta-\zeta-2}.$$

If we rewrite the above equation in (5), we achieve the desired result. □

2. Auxiliary Results

Lemma 1. *For every $x \in (0,1)$, the operator $K^*_{\eta,\theta}(e_m;x)$ has the following identities:*

$$K^*_{\eta,\theta}(e_0;x) = 1,$$

$$K^*_{\eta,\theta}(e_1;x) = \frac{\eta-2}{\eta}x + \frac{3}{2\eta},$$

$$K^*_{\eta,\theta}(e_2;x) = \frac{(3\eta^3-18\eta^2-3\eta+18)-\theta(6\eta^2-42\eta+36)}{3\eta^2(\eta-1)}x^2$$

$$+\frac{(18\eta^2-6\eta-24)-\theta(36\eta-48)}{3\eta^2(\eta-1)}x$$
$$+\frac{\left(\eta^2+5\eta+6\right)-12\theta}{3\eta^2(\eta-1)},$$

where $e_m = t^m$ *for* $m = 0, 1, 2$.

Proof. We briefly mention the results of $K^*_{\eta,\theta}(e_m;x)$, where $e_m = t^m$, $m = 0, 1, 2$. For $e_0 = 1$, we write

$$\begin{aligned} K^*_{\eta,\theta}(1;x) &= (1-\theta)\sum_{\zeta=0}^{\eta-1}\frac{1}{\eta-1}\left[\left(1-\frac{\zeta}{\eta-1}+\frac{\zeta}{\eta-1}\right)\right]\binom{\eta-1}{\zeta} \\ &\quad\times(\zeta-(\eta-1)x)^2x^{\zeta-1}(1-x)^{\eta-\zeta-2} \\ &\quad+\theta\sum_{\zeta=0}^{\eta}\frac{1}{\eta}\binom{\eta}{\zeta}(\zeta-\eta x)^2x^{\zeta-1}(1-x)^{\eta-\zeta-1} \\ &= (1-\theta)B^*_\eta(1;x)+\theta B^*_\eta(1;x) \\ &= 1. \end{aligned}$$

For $e_1 = t$, we have

$$\begin{aligned} K^*_{\eta,\theta}(t;x) &= (1-\theta)\sum_{\zeta=0}^{\eta-1}\frac{\eta}{\eta-1}\left[\left(1-\frac{\zeta}{\eta-1}\right)\int_{\frac{\zeta}{\eta}}^{\frac{\zeta+1}{\eta}}tdt+\frac{\zeta}{\eta-1}\int_{\frac{\zeta+1}{\eta}}^{\frac{\zeta+2}{\eta}}tdt\right] \\ &\quad\times\binom{\eta-1}{\zeta}(\zeta-(\eta-1)x)^2x^{\zeta-1}(1-x)^{\eta-\zeta-2} \\ &\quad+\theta\sum_{\zeta=0}^{\eta}\binom{\eta}{\zeta}(\zeta-\eta x)^2x^{\zeta-1}(1-x)^{\eta-\zeta-1}\int_{\frac{\zeta}{\eta}}^{\frac{\zeta+1}{\eta}}tdt. \end{aligned}$$

Since $\int_{\frac{\zeta}{\eta}}^{\frac{\zeta+1}{\eta}}tdt=\frac{2\zeta+1}{2\eta^2}$ and $\int_{\frac{\zeta+1}{\eta}}^{\frac{\zeta+2}{\eta}}tdt=\frac{2\zeta+3}{2\eta^2}$,

$$\begin{aligned} K^*_{\eta,\theta}(t;x) &= (1-\theta)\sum_{\zeta=0}^{\eta-1}\frac{\eta}{\eta-1}\left[\left(1-\frac{\zeta}{\eta-1}\right)\left(\frac{2\zeta+1}{2\eta^2}\right)+\frac{\zeta}{\eta-1}\left(\frac{2\zeta+3}{2\eta^2}\right)\right] \\ &\quad\times\binom{\eta-1}{\zeta}(\zeta-(\eta-1)x)^2x^{\zeta-1}(1-x)^{\eta-\zeta-2} \\ &\quad+\theta\sum_{\zeta=0}^{\eta}\binom{\eta}{\zeta}(\zeta-\eta x)^2x^{\zeta-1}(1-x)^{\eta-\zeta-1}\left(\frac{2\zeta+1}{2\eta^2}\right) \\ &= (1-\theta)B^*_\eta(t;x)+\frac{1-\theta}{2\eta}+\theta B^*_\eta(t;x)+\frac{\theta}{2\eta} \\ &= \frac{\eta-2}{\eta}x+\frac{3}{2\eta}. \end{aligned}$$

For $e_2 = t^2$, we have

$$\begin{aligned} K^*_{\eta,\theta}(t^2;x) &= (1-\theta)\sum_{\zeta=0}^{\eta-1}\frac{\eta}{\eta-1}\left[\left(1-\frac{\zeta}{\eta-1}\right)\int_{\frac{\zeta}{\eta}}^{\frac{\zeta+1}{\eta}} t^2dt + \frac{\zeta}{\eta-1}\int_{\frac{\zeta+1}{\eta}}^{\frac{\zeta+2}{\eta}} t^2dt\right] \\ &\quad \times \binom{\eta-1}{\zeta}(\zeta-(\eta-1)x)^2 x^{\zeta-1}(1-x)^{\eta-\zeta-2} \\ &\quad +\theta\sum_{\zeta=0}^{\eta}\binom{\eta}{\zeta}(\zeta-\eta x)^2 x^{\zeta-1}(1-x)^{\eta-\zeta-1}\int_{\frac{\zeta}{\eta}}^{\frac{\zeta+1}{\eta}} t^2dt. \end{aligned}$$

Since $\int_{\frac{\zeta}{\eta}}^{\frac{\zeta+1}{\eta}} t^2dt = \frac{3\zeta^2+3\zeta+1}{3\eta^3}$ and $\int_{\frac{\zeta+1}{\eta}}^{\frac{\zeta+2}{\eta}} t^2dt = \frac{3\zeta^2+9\zeta+7}{3\eta^3}$,

$$\begin{aligned} K^*_{\eta,\theta}(t^2;x) &= (1-\theta)\sum_{\zeta=0}^{\eta-1}\frac{\eta}{\eta-1}\left[\left(1-\frac{\zeta}{\eta-1}\right)\left(\frac{3\zeta^2+3\zeta+1}{3\eta^3}\right)\right. \\ &\quad \left.+\frac{\zeta}{\eta-1}\left(\frac{3\zeta^2+9\zeta+7}{3\eta^3}\right)\right] \\ &\quad \times \binom{\eta-1}{\zeta}(\zeta-(\eta-1)x)^2 x^{\zeta-1}(1-x)^{\eta-\zeta-2} \\ &\quad +\theta\sum_{\zeta=0}^{\eta}\binom{\eta}{\zeta}(\zeta-\eta x)^2 x^{\zeta-1}(1-x)^{\eta-\zeta-1}\left(\frac{3\zeta^2+3\zeta+1}{3\eta^3}\right) \\ &= \left(\frac{1-\theta}{\eta-1}+\frac{(1-\theta)\,\eta}{\eta-1}+\theta\right)B^*_\eta(t^2;x) \\ &\quad +\left(\frac{1-\theta}{\eta-1}+\frac{1-\theta}{\eta\,(\eta-1)}+\frac{\theta}{\eta}\right)B^*_\eta(t;x)+\frac{1-\theta}{3\eta}+\frac{\theta}{3\eta} \\ &= \frac{(3\eta^3-18\eta^2-3\eta+18)-\theta(6\eta^2-42\eta+36)}{3\eta^2(\eta-1)}x^2 \\ &\quad +\frac{(18\eta^2-6\eta-24)-\theta(36\eta-48)}{3\eta^2(\eta-1)}x \\ &\quad +\frac{\left(\eta^2+5\eta+6\right)-12\theta}{3\eta^2(\eta-1)}. \end{aligned}$$

□

Lemma 2. *For every $x \in (0,1)$, we have the central moments as*

$$\begin{aligned} K^*_{\eta,\theta}(t-x;x) &= \frac{-4x+3}{2\eta}, \\ K^*_{\eta,\theta}((t-x)^2;x) &= \frac{1}{3\eta^2(\eta-1)}\left\{18x^2-24x+6+\eta(5+3x-15x^2)\right. \end{aligned}$$

$$+\eta^2(1+9x-3x^2)$$
$$-\theta[12-48x+36x^2+6\eta^2x^2+\eta(36x-42x^2)]\big\}.$$

Proof. For the sake of brevity, central moments can be expressed as

$$\begin{aligned} K^*_{\eta,\theta}(t-x;x) &= K^*_{\eta,\theta}(e_1;x) - xK^*_{\eta,\theta}(e_0;x), \\ K^*_{\eta,\theta}((t-x)^2;x) &= K^*_{\eta,\theta}(e_2;x) - 2xK^*_{\eta,\theta}(e_1;x) + x^2K^*_{\eta,\theta}(e_0;x). \end{aligned}$$

The proof is completed by using these equalities. □

Let $C[0,1]$ be the Banach space of all continuous functions g on $[0,1]$ with the norm

$$\|g\| = \max_{x\in(0,1)} |g(x)|.$$

Theorem 2. *For every $x \in (0,1)$ and $g \in C[0,1]$*

$$\left\|K^*_{\eta,\theta}(g;x) - g(x)\right\| \to 0, \tag{6}$$

uniformly as $\eta \to \infty$.

Proof. In the light of Lemma 1, we have

$$\lim_{\eta\to\infty} K^*_{\eta,\theta}(e_i;x) = t^i, \qquad i = 0,1,2.$$

By Korovkin theorem [11] the proof is completed. □

3. Rate of convergence

The modulus of continuity is given by

$$\omega(g,\delta) := \sup_{|t-x|\le\delta} \sup_{x\in(0,1)} |g(t)-g(x)|, \quad \delta > 0,$$

where $g \in C[0,1]$. Following feature of the modulus of continuity [1]

$$|g(t)-g(x)| \le \left(1 + \frac{|t-x|}{\delta}\right)\omega(g,\delta)$$

will be used in the proof of the next theorem.

Theorem 3. *For every $x \in (0,1)$ and $g \in C[0,1]$,*

$$|K^*_{\eta,\theta}(g;x) - g(x)| \le 2\omega(g;\delta_\eta). \tag{7}$$

Here,

$$\begin{aligned} \delta_\eta(x) &= \left[K^*_{\eta,\theta}((t-x)^2;x)\right]^{\frac{1}{2}} \\ &= \left\{\frac{1}{3\eta^2(\eta-1)}\left\{-9\eta^4x + 9\eta^3x + 18x^2 - 24x + 6 + \eta^2(1+18x-3x^2)\right.\right. \\ &\quad \left.\left. +\eta(5-6x-15x^2) - \theta[12-48x+36x^2+6\eta^2x^2+\eta(36x-42x^2)]\right\}\right\}^{1/2}. \end{aligned}$$

Proof. For $K^*_{\eta,\theta}$, we write

$$\begin{aligned}\left|K^*_{\eta,\theta}(g;x)-g(x)\right| &= \left|K^*_{\eta,\theta}(g(t)-g(x);x)\right| \\ &\leq K^*_{\eta,\theta}(\left|g(t)-g(x)\right|;x) \\ &\leq \omega\left(g;\delta\right)\left\{K^*_{\eta,\theta}(1;x)+\frac{1}{\delta}K^*_{\eta,\theta}(\left|t-x\right|;x)\right\} \\ &\leq \omega\left(g;\delta\right)\left\{1+\frac{1}{\delta}\left[K^*_{\eta,\theta}((t-x)^2;x)\right]^{\frac{1}{2}}\right\}.\end{aligned}$$

If we select

$$\delta=\delta_\eta=\left[K^*_{\eta,\theta}((t-x)^2;x)\right]^{\frac{1}{2}},$$

then we get

$$\left|K^*_{\eta,\theta}(g;x)-g(x)\right|\leq 2\omega\left(g;\left[K^*_{\eta,\theta}((t-x)^2;x)\right]^{\frac{1}{2}}\right),$$

which is the desired result. □

Here, we investigate the rate of convergence of $K^*_{\eta,\theta}(g;x)$ by using functions of Lipschitz class. Let's recall that a function $g\in Lip_M\left(\varsigma\right)$ on $(0,1)$ if the inequality

$$\left|g\left(t\right)-g\left(x\right)\right|\leq M\left|t-x\right|^{\varsigma}\ ;\ \forall t,x\in(0,1) \tag{8}$$

holds.

Theorem 4. *Let* $x\in(0,1)$, $g\in Lip_M\left(\varsigma\right)$, $0<\varsigma\leq 1$, *then we get*

$$\left|K^*_{\eta,\theta}(g;x)-g(x)\right|\leq M\delta^{\varsigma}_{\eta}\left(x\right),$$

where

$$\begin{aligned}\delta_\eta(x) &= \left[K^*_{\eta,\theta}((t-x)^2;x)\right]^{\frac{1}{2}} \\ &= \left\{\frac{1}{3\eta^2(\eta-1)}\left\{-9\eta^4x+9\eta^3x+18x^2-24x+6+\eta^2(1+18x-3x^2)\right.\right. \\ &\quad \left.\left.+\eta(5-6x-15x^2)-\theta[12-48x+36x^2+6\eta^2x^2+\eta(36x-42x^2)]\right\}\right\}^{1/2}.\end{aligned}$$

Proof. Let $x\in(0,1)$, $g\in Lip_M\left(\varsigma\right)$ and $0<\varsigma\leq 1$. From the linearity and monotonicity of the operators $K^*_{\eta,\theta}$, we have

$$\begin{aligned}\left|K^*_{\eta,\theta}(g;x)-g\left(x\right)\right| &\leq K^*_{\eta,\theta}(\left|g\left(t\right)-g\left(x\right)\right|;x) \\ &\leq MK^*_{\eta,\theta}(\left|t-x\right|^{\varsigma};x).\end{aligned}$$

By putting $p=\frac{2}{\varsigma}$, $q=\frac{2}{2-\varsigma}$ in the Hölder inequality, we obtain

$$\begin{aligned}\left|K^*_{\eta,\theta}\left(g;x\right)-g\left(x\right)\right| &\leq M\left[K^*_{\eta,\theta}(\left(t-x\right)^2;x)\right]^{\frac{\varsigma}{2}} \\ &\leq M\delta^{\varsigma}_{\eta}\left(x\right).\end{aligned}$$

By choosing

$$\delta_\eta(x)=\left[K^*_{\eta,\theta}((t-x)^2;x)\right]^{\frac{1}{2}}$$

the proof is completed. □

Lastly, we will give the rate of convergence of our operator $K^*_{\eta,\theta}(g;x)$ by means of Peetre-$\mathcal{K}$ functionals. First of all, we give the following lemma.

Lemma 3. *For $x \in (0,1)$ and $g \in C[0,1]$, we get*

$$|K^*_{\eta,\theta}(g;x)| \leq ||g||. \tag{9}$$

Proof. For $K^*_{\eta,\theta}$,

$$\begin{aligned} |K^*_{\eta,\theta}(g;x)| &= \left| \sum_{\zeta=0}^{\eta} \rho^{(\theta)}_{\eta,\zeta}(x) \int_{\frac{\zeta}{\eta}}^{\frac{\zeta+1}{\eta}} g(t)dt \right| \\ &\leq \sum_{\zeta=0}^{\eta} \rho^{(\theta)}_{\eta,\zeta}(x) \left| \int_{\frac{\zeta}{\eta}}^{\frac{\zeta+1}{\eta}} g(t)dt \right| \\ &\leq \sum_{\zeta=0}^{\eta} \rho^{(\theta)}_{\eta,\zeta}(x) \int_{\frac{\zeta}{\eta}}^{\frac{\zeta+1}{\eta}} |g(t)|\, dt \\ &\leq ||g|| K^*_{\eta,\theta}(1;x) \\ &= ||g||. \end{aligned}$$

□

$C^2[0,1]$ is the space of the functions g, for which g, g' and g'' are continuous on $[0,1]$. The norm on the space $C^2[0,1]$ is given by

$$\|h\|_{C^2[0,1]} := \|h\|_{C[0,1]} + \|h'\|_{C[0,1]} + \|h''\|_{C[0,1]}.$$

Now, we define classical Peetre-$\mathcal{K}$ functional as follows:

$$\mathcal{K}(g,\lambda) := \inf_{h \in C^2[0,1]} \{\|g-h\| + \lambda \|h''\|\}$$

where $\lambda > 0$.

Theorem 5. *Let $x \in (0,1)$ and $g \in C[0,1]$. Then we have for all $\eta \in \mathbb{N}$,*

$$|K^*_{\eta,\theta}(g;x) - g(x)| \leq 2\mathcal{K}(g;\lambda_\eta(x)),$$

where

$$\begin{aligned} \lambda_\eta(x) &= \frac{1}{6\eta^2(\eta-1)} \left|10\eta^2 - 4\eta + 6 - 12\theta + \left(-24 + 15\eta - 3\eta^2 - 6\theta(6\eta - 8)\right)x \right. \\ &\quad \left. + \left(18 - 15\eta - 3\eta^2 - 6\theta\left(6 - 7\eta + \eta^2\right)\right)x^2\right|. \end{aligned}$$

Proof. For a given function $h \in C^2[0,1]$, we have the following Taylor expansion

$$h(t) = h(x) + (t-x)h'(x) + \int_x^t (t-s)h''(s)ds, \ \ t \in (0,1). \tag{10}$$

Applying $K^*_{\eta,\theta}$ operator to the Eqn. (10), we get

$$\begin{aligned}\left|K^*_{\eta,\theta}(h;x)-h(x)\right| &= \left|K^*_{\eta,\theta}\left((t-x)h'(x);x\right)\right| + \left|K^*_{\eta,\theta}\left(\int_x^t (t-s)h''(s)ds;x\right)\right| \\ &\leq ||h'||\left|K^*_{\eta,\theta}\left(t-x;x\right)\right| + ||h''||\left|K^*_{\eta,\theta}\left(\int_x^t (t-s)ds;x\right)\right| \\ &\leq ||h'||\left|K^*_{\eta,\theta}\left(t-x;x\right)\right| + ||h''||\frac{1}{2}K^*_{\eta,\theta}\left((t-x)^2;x\right).\end{aligned}$$

So,

$$\left|K^*_{\eta,\theta}(h;x)-h(x)\right| \leq \lambda||h||.$$

Using the above inequality, we get

$$\begin{aligned}\left|K^*_{\eta,\theta}(g;x)-g(x)\right| &= \left|K^*_{\eta,\theta}(g;x)-g(x)+K^*_{\eta,\theta}(h;x)-K^*_{\eta,\theta}(h;x)+h(x)-h(x)\right| \\ &\leq ||g-h||\left|K^*_{\eta,\theta}(1;x)\right| + ||g-h|| + \left|K^*_{\eta,\theta}(h;x)-h(x)\right| \\ &\leq 2\left(||g-h||+\lambda||h||\right) \\ &= 2\mathcal{K}\left(g;\lambda\right).\end{aligned}$$

As a result, by choosing

$$\begin{aligned}\lambda=\lambda_\eta\left(x\right) &= \frac{1}{6\eta^2\left(\eta-1\right)}\left|10\eta^2-4\eta+6-12\theta\right. \\ &\quad +\left(-24+15\eta-3\eta^2-6\theta\left(6\eta-8\right)\right)x \\ &\quad \left.+\left(18-15\eta-3\eta^2-6\theta\left(6-7\eta+\eta^2\right)\right)x^2\right|,\end{aligned}$$

we obtain

$$|K^*_{\eta,\theta}(g;x)-g(x)| \leq 2\mathcal{K}\left(g;\lambda_\eta\right). \tag{11}$$

Thus, the proof is completed. □

4. Graphical Analysis

In this part, we present some graphics to show the convergence of the operators $K^*_{\eta,\theta}$ to the function g. It is already known that, the operators $K^*_{\eta,\theta}(g;x)$ have been defined for $x\in(0,1)$. For this reason, the closed interval is given by $[0+\epsilon,1-\epsilon]$, where $\epsilon=0.0001$.

Example 1. *Let*

$$g(x)=x(x-1)\left(x-\frac{1}{12}\right).$$

Then for $\theta=0.25$, $\theta=0.5$ and $\theta=0.9$, we have plotted the convergence of the new constructed $K^*_{\eta,\theta}$ parametric Bernstein-Kantorovich operators and B^*_η modified Bernstein operators [18] to the function g in Fig. 1 for $\eta=125$.

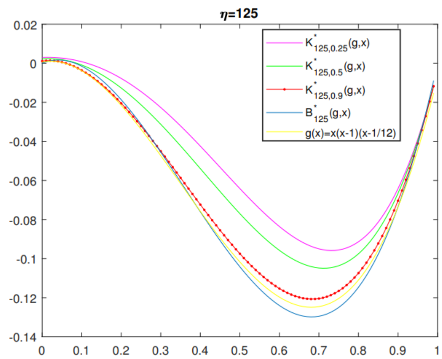

FIGURE 1. Convergence of $K^*_{\eta,\theta}(g;x)$ for different values of θ with fixed $\eta = 125$.

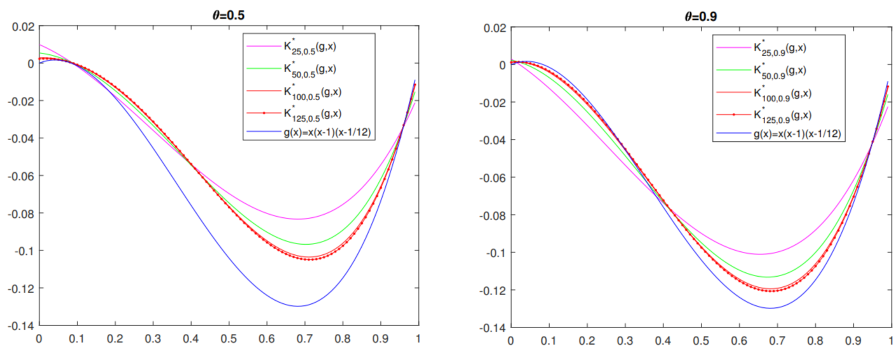

FIGURE 2. Convergence of $K^*_{\eta,\theta}(g;x)$ for different values of η with fixed θ.

In Fig. 2, we have illustrated the convergence of the $K^*_{\eta,\theta}$ operators to the target function $g(x) = x(x-1)\left(x - \frac{1}{12}\right)$ for fixed $\theta = 0.5$ and $\theta = 0.9$, where $\eta \in \{25, 50, 100, 125\}$. The maximum errors for the operators $K^*_{\eta,\theta}$ and B^*_η to the function $g(x) = x(x-1)\left(x - \frac{1}{12}\right)$ are presented in Table 1 for different values of θ and η.

It is obvious from the Table 1 that the best error in the approximation of g by $K^*_{\eta,\theta}$ is achieved when $\theta = 0.999$. Moreover, we note that the error in the approximation of $K^*_{\eta,0.99}(g)$ and $K^*_{\eta,0.999}(g)$ is much smaller than the errors in the approximation $B^*_\eta(g)$, where $\eta \in \{25, 50, 100, 125\}$.

Example 2. *As a second example, we choose*

$$g(x) = xe^{-3x}$$

TABLE 1. Error for approximation of the parametric Bernstein-Kantorovich operators $K^*_{\eta,\theta}$ and modified Bernstein operators B^*_η.

θ	η	$\|\|B^*_\eta(g) - g\|\|$	$\|\|K^*_{\eta,\theta}(g) - g\|\|$
0.99	25	0.0296	0.0262
0.99	50	0.0155	0.0134
0.99	100	0.0079	0.0069
0.99	125	0.0063	0.0056
0.999	25	0.0296	0.0258
0.999	50	0.0155	0.0131
0.999	100	0.0079	0.0066
0.999	125	0.0063	0.0053

and $x \in (0,1)$. *Then for* $\theta = 0.79$, $\theta = 0.89$ *and* $\theta = 0.99$, *we have plotted the convergence of the* $K^*_{\eta,\theta}$ *Bernstein-Kantorovich operators to the function* g *in Fig. 3 for* $\eta = 170$.

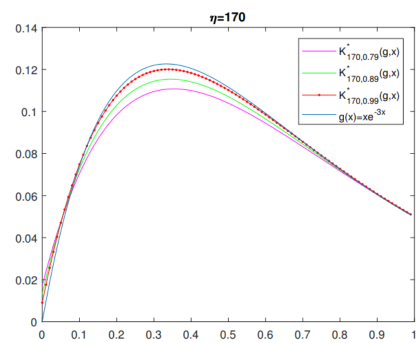

FIGURE 3. Convergence of $K^*_{\eta,\theta}(g;x)$ for $\theta = 0.79$, $\theta = 0.89$ and $\theta = 0.99$.

In Fig. 4, we have presented $K^*_{\eta,\theta}(g;x)$ for fixed $\theta = 0.9$ and $\theta = 0.99$, where $\eta \in \{25, 100, 125, 170\}$.

The error estimation for newly constructed operators $K^*_{\eta,\theta}$ to the function $g(x) = xe^{-3x}$ is presented in Table 2 for different values of θ and η.

It is evident from the Table 2 that the best error in the approximation of g by $K^*_{\eta,\theta}$ is achieved when $\theta = 0.99$ and $\eta = 170$.

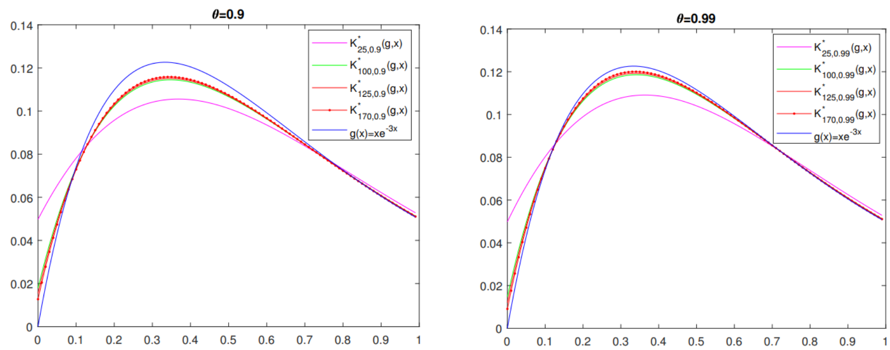

FIGURE 4. Convergence of $K^*_{\eta,\theta}(g;x)$ for $\theta = 0.9$ and $\theta = 0.99$.

TABLE 2. Error for approximation of the $K^*_{\eta,\theta}$ for $\theta = 0.79, 0.89, 0.99$.

θ	η	$\|\|K^*_{\eta,\theta}(g) - g\|\|$
0.79	25	0.0496
0.79	125	0.0194
0.79	170	0.0171
0.89	25	0.0194
0.89	125	0.0157
0.89	170	0.0130
0.99	25	0.0171
0.99	125	0.0119
0.99	170	0.0090

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