Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 53 (5) (2024), 1305 – 1311
DOI : 10.15672/hujms.1339373

Research Article

# On a sampling problem for a Bargmann-Fock space

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**Abstract**

The purpose of the present article is to provide geometric sufficient conditions for discrete points to be a sampling sequence for a generalized Hilbert Bargmann-Fock space in several complex variables.

**Mathematics Subject Classification (2020).** 30H05, 30H20, 31C10, 32A15, 94A20

**Keywords.** Beurling density, generalized Bargmann-Fock spaces, plurisubharmonic functions, relatively separated sequence, sampling sequences

## 1. Introduction

Sampling properties in Bergman and Fock type spaces have been studied in the 90s by Seip and co-authors [12, 14–18]. Generalization of these results to general Fock spaces in one complex variable were provided by Berndtsson and Ortega-Cerdà [1]. Later, Lindhom furnishes necessary conditions for sampling a sequence by a function in a weighted Bargmann-Fock spaces in several complex variables, the weight being given by $\exp(-\varphi)$ where $\varphi$ is a suitable plurisubharmonic function [11]. Recently, there are quite important recent results by Gröchenig *et al* [4, 6] improving Lindholm's results.

The aim of the present article is to provide sufficient conditions for sampling a sequence by a holomorphic function and square-integrable with respect to the suitable measure $\exp(-\varphi(z))dm(z)$ such that $dm(z)$ are the Lebesgue complex measure and a $C^2$-plurisubharmonic function in $\mathbb{C}^n$, respectively, i.e., the associated Levi-form is positive semi-definite.

Let us recall some classical definitions and known results on density conditions for sampling sequences.

**Definition 1.1.** The generalized Bargmann-Fock space in $\mathbb{C}^n$ is defined as

$$F^2_\varphi(\mathbb{C}^n) := \left\{ f \in \mathcal{H}(\mathbb{C}^n) : ||f||^2_{F^2_\varphi(\mathbb{C}^n)} = \int_{\mathbb{C}^n} |f(z)|^2 \exp\left(-\varphi(z)\right) dm(z) < \infty \right\},$$

such that $dm(z)$ represents the Lebesgue measure on $\mathbb{C}^n$, $\mathcal{H}(\mathbb{C}^n)$ stands for the set of holomorphic functions on $\mathbb{C}^n$, $\varphi$ is a real-valued $C^2$-plurisubharmonic function on $\mathbb{C}^n$.

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Received: 08.08.2023; Accepted: 13.10.2023

**Remark 1.2.** We recall that if $\varphi$ is a $C^2$-plurisubharmonic function, then $i\partial\bar{\partial}\varphi(z)$ is a closed positive current of bidegree (1,1), e.g., [10, Proposition 3.3.5]. Concerning properties of plurisubharmonic functions and positive currents in several complex variables, we can have a look to the non-exhaustive surveys [9, 10].

By adopting the following notations that $A \lesssim B$ means that $A$ is less, up to a multiplicative constant, to $B$, and $A \approx B$ when $A \lesssim B$ and $A \gtrsim B$, we have the following definition on the sampling sequence.

**Definition 1.3.** We say that a $\mathbb{C}^n$-valued sequence $\mathcal{A} = (a_j)_{j\in\mathbb{N}}$ is a $F^2_\varphi(\mathbb{C}^n)$-sampling if for any $f \in F^2_\varphi(\mathbb{C}^n)$, we have

$$||f||^2_{F^2_\varphi(\mathbb{C}^n)} \lesssim ||f(a)||^2_{l^2_{\varphi,\mathcal{A}}} \lesssim ||f||^2_{F^2_\varphi(\mathbb{C}^n)},$$

such that $f(a) = (f(a_k))_{k\in\mathbb{N}}$. and $||f(a)||^2_{l^2_{\varphi,\mathcal{A}}} = \sum_{a_k\in\mathcal{A}} |f(a_k)|^2 \exp(-\varphi(a_k))$.

$\lesssim$ means less up to a multiplicative constant.

Berndtsson and Ortega-Cerdà state that a sequence $\mathcal{A}$ is $F^2_\psi(\mathbb{C})$-sampling whenever $\mathcal{A}$ is uniformly separated sequence and dense with respect to $\psi$ (a subharmonic function), i.e., $\dfrac{\#(\mathbb{D}(z,r)\cap\mathcal{A})}{r^2} > \Delta\psi(z) + \delta$ for some some $r > 0$ and $\delta > 0$, where $\Delta$ is the Laplacian operator and $\mathbb{D}(z,r)$ represents the complex disk of center $z \in \mathbb{C}$ with radius $r$, and $\#(\mathbb{D}(z,r)\cap\mathcal{A})$ is the counting function [1, Theorem 1, part (b)].

Then, Ortega-Cerdà and Seip [13, Theorem 1] state that a sequence $\mathcal{A}$ is $F^p_\psi(\mathbb{C})$-sampling for $p \in [1,\infty)$ if and only if

$$\liminf_{r\to\infty}\left(\inf_{z\in\mathbb{C}}\left(\frac{\#(\mathcal{A}\cap\mathbb{D}(z,r))}{\int_{\mathbb{D}(z,r)}\Delta\psi(\omega)}\right)\right) > \frac{2}{\pi}.$$

Next, Lindholm in [11, Theorem 1] considered $\varphi$ a two-homogeneous plurisubharmonic function on $\mathbb{C}^n$ and $C^2$ outside the origin and states that if a sequence $\Gamma$ is a sampling sequence for $F^p_\varphi(\mathbb{C}^n)$ with $p \in [0,\infty]$, then it contains a uniformly separated sampling subset $\Gamma'$ satisfying

$$D^-_\varphi(\Gamma') := \liminf_{r\to\infty}\left(\inf_{z\in\mathbb{C}^n}\left(\frac{\#(\Gamma'\cap\mathbb{B}(z,r))}{\int_{\mathbb{B}(z,r)}(i\partial\bar{\partial}\varphi(\omega))^n}\right)\right) \geq \frac{1}{\pi^n n!}, \tag{1.1}$$

Lindholm pretends that inequality (1.1) should be strict. We recall that $D^-_\varphi(\Gamma')$ is called the lower density associated to the sequence $\Gamma'$ with respect to the $C^2$-plurisubharmonic function $\varphi$ on $\mathbb{C}^n$.

Recently, Gröchenig, Haimi, Ortega Cerdá and Romero show that inequality is (1.1) strict. Precisely, they consider the following type lower weighted Beurling density of $\mathcal{A}$.

$$\mathscr{D}^-_\varphi(\mathcal{A}) = \liminf_{r\to\infty}\left(\inf_{z\in\mathbb{C}^n}\left(\frac{\#\left(\mathcal{A}\cap\mathbb{B}(z,r)\right)}{\int_{\mathbb{B}(z,r)}K_\varphi(\omega,\omega)\exp(-2\varphi(\omega))dm(\omega)}\right)\right), \tag{1.2}$$

such that $K_\varphi(\cdot,\cdot)$ stands for reproducing kernel of $F^2_\varphi(\mathbb{C}^n)$ [6, Theorem 1.1]. Next, they state that if $\varphi$ is a two-homogeneous plurisubharmonic function [4] and $i\partial\bar{\partial}\varphi$ is equivalent to $i\partial\bar{\partial}|z|^2$, e.g., $\varphi(z) = \sum_{l=1}^{n} \lambda_l |z_l|^2$ such that $\lambda_{k_1} \neq \lambda_{k_2}$ and $k_1 \neq k_2 \in \{1, \ldots, n\}$, then they observe that it is possible to compare $D^-_\varphi(\mathcal{A})$ with $\mathscr{D}^-_\varphi(\mathcal{A})$. Precisely, they state

$$D^-_\varphi(\mathcal{A}) = \frac{1}{\pi^n n!} \mathscr{D}^-_\varphi(\mathcal{A}).$$

Then, they state that if $\mathcal{A}$ is a sampling set for $F^2_\varphi(\mathbb{C}^n)$, then $\mathscr{D}^-_\varphi(\mathcal{A}) > 1$ [6, Theorem 1.2 (a)].

The aim of the present article is to provide sufficient density conditions for having a $F^2_\varphi(\mathbb{C}^n)$-sampling sequence such that $\varphi$ is a $C^2$-plurisubharmonic function on $\mathbb{C}^n$ when $\mathcal{A}$ is relatively separated with respect to the ball of center $z \in \mathbb{C}^n$ and radius one, i.e., the number

$$rel(\mathcal{A}) := \sup \{\# (\mathcal{A} \cap \mathbb{B}(z, 1)),\ z \in \mathbb{C}^n\}$$

is finite. Furthermore, we suppose the following kind of density condition

$$\nu(z) * \mathcal{X}_r(z) \geq \Delta\varphi(z) + \eta \text{ for } z \in \mathbb{C}^n, \tag{1.3}$$

for some positive real number $r$ and $\eta > 0$ such that

$$\nu(z) = \sum_{a_j \in \mathcal{A}} \frac{1}{\exp(\varepsilon^{2n-2})} \mathcal{X}_{\mathbb{B}(0,\varepsilon)}(z - a_j),$$

where $\mathcal{A} = (a_k)_{k\in\mathbb{N}}$ is a sequence in $\mathbb{C}^n$, $\varepsilon$ is a positive number, and $\mathcal{X}_r(z) = \frac{1}{r^{2n}} \mathcal{X}_{\mathbb{B}(0,r)}(z)$, and $\mathcal{X}_{\mathbb{B}(0,r)}(\cdot)$ represents the indicator function on $\mathbb{B}(0, r)$, the complex open ball of radius $r$ and of center zero.

Therefore, we show our following sampling theorem providing sufficient conditions for sampling a sequence by a $C^2$-plurisubharmonic function $\varphi(z)$.

**Theorem 1.4.** *Let $\varphi$ be a real-valued $C^2$-plurisubharmonic function on $\mathbb{C}^n$ and satisfy both (1.3) and $i\partial\bar{\partial}\varphi(z)$ be equivalent to $i\partial\bar{\partial}|z|^2$. Then $\mathcal{A} = (a_k)_{k\in\mathbb{N}}$ a relatively separated sequence is a $F^2_\varphi(\mathbb{C}^n)$-sampling.*

**The structure of the article**

The second section focuses on a meaningful lemma on a local holomorphic function with optimal assessment in $\mathbb{C}^n$. The third section is devoted to the proof of Theorem 1.4.

## 2. On a meaningful lemma

Berndtsson and Ortega-Cerdá show a result on a local holomorphic function with good estimates on $\mathbb{D}(a, \rho)$, the disk of center $a \in \mathbb{C}$ and of radius $\rho$. To be precise, they consider $\psi$, a subharmonic function in $\mathbb{D}(a, \rho)$ such that its Laplacian is bounded, then they state that there is $\mathfrak{C}$, a positive constant and $f$, a holomorphic function on $\mathbb{D}(a, \rho)$ such that $f(a) = 0$ and

$$|\psi(z) - \psi(a) - \Re f(z)| \leq \mathfrak{C} \text{ for all } z \in \mathbb{D}(a, \rho). \tag{2.1}$$

Concerning the proof of (2.1), they employ the classical Riesz Decomposition Theorem (RDT) in one-dimensional complex coordinate space, e.g., see [7, Theorem 3.9 p.104] (or [5, Chap.I, p.47], [8, Theorem 3.5.11]), which states that a subharmonic function is the sum of the Newtonian potential (for a Borel measure) plus a harmonic function $u$ and used the fact that $u$ is the real part of a holomorphic function. Thus to prove a version of a local holomorphic function in several complex variables with optimal assessments on $\mathbb{B}(a_k, \delta)$, we cannot use the RDT in complex $n$-space with $n > 1$ due to the fact that in general a harmonic function $u$ is not a pluriharmonic function, so there is no reason for

$u$ to be equal at the real part of some holomorphic function. Therefore, to dodge this impasse, we use the following lemma.

**Lemma 2.1.** *[6, Lemma 2.4] Let* $\theta = \sum_{1\le j,k\le n} \theta_{jk} dz_j \wedge d\bar{z}_k$ *be a positive, d-closed* $(1,1)$*-current satisfying* $\theta \le M i\partial\bar{\partial}|z|^2$*. Then there exists* $u : \mathbb{C}^n \to \mathbb{C}$ *solving the equation* $i\partial\bar{\partial}u = \theta$*, and such that*

$$|u(z)| \le CM(1+|z|)^2 \log(1+|z|), \tag{2.2}$$

*where the constant* $C$ *depends only on the dimension* $n$*.*

Where $M$ is a positive constant and the proof is based on using Poincaré's lemma and on [2, Theorem 9]. Now, let us state our local holomorphic optimal assessment in $\mathbb{C}^n$.

**Lemma 2.2.** *Let* $\mathcal{A} = (a_k)_{k\in\mathbb{N}}$ *be a sequence in* $\mathbb{C}^n$*,* $\varphi$ *be a real* $C^2$*-plurisubharmonic function on* $\mathbb{C}^n$*, and* $i\partial\bar{\partial}\varphi(z) \approx \partial\bar{\partial}|z|^2$*. Then there is a holomorphic function* $G_k$ *on* $\mathbb{B}(a_k,\rho)$ *for* $\rho > 0$ *such that* $G_k(a_k) = 0$ *and a positive constant* $C_1$ *such that:*

$$\sup_{z\in\mathbb{B}(a_k,\rho)} |\varphi(z) - \varphi(a_k) - 2\Re G_k(z)| \le C_1. \tag{2.3}$$

***Proof.*** The fact that $\varphi$ is a $C^2$-plurisubharmonic function on $\mathbb{C}^n$ thus by Remark 1.2, we have that $i\partial\bar{\partial}\varphi(z)$ is a closed positive current of bidegree $(1,1)$ and by assumption $i\partial\bar{\partial}\varphi$ is equivalent to $i\partial\bar{\partial}|z|^2$. Whence, by applying Lemma 2.1 there is a function $\varphi_1$ on $\mathbb{C}^n$ satisfying both $i\partial\bar{\partial}\varphi_1(z) = i\partial\bar{\partial}\varphi(z)$ and the extra size assumption inequality (2.2). Therefore, the function $u = \varphi - \varphi_1$ is pluriharmonic and it is the real part of a holomorphic function $\mathcal{H}$, i.e., $\varphi - \Re\mathcal{H} = \varphi_1$. Let us choose the holomorphic function $2G_k(z) := \mathcal{H}(z) - \mathcal{H}(a_k)$ and by using the fact that the function $(1+|z|^2)^2\log(1+|z|)$ is a bounded continuous function for $z \in \mathbb{B}(a_k,\rho)$, we have the existence of a positive constant $C_1$ such that:

$$|\varphi(z) - \varphi(a_k) - 2\Re G_k(z)| \;=\; |\varphi_1(z) - \varphi_1(a_k)| \le C_1.$$

□

## 3. The proof of Theorem 1.4

*Proof of Theorem 1.4.* Our approach is based on the techniques used for proving [1, Theorem 1, part (b)]. Therefore, let us consider $\mathfrak{g}(z) = (\nu(z) - \nu(z) * \mathcal{X}_r(z)) * E(z)$ such that $E(z) \approx |z|^{2-2n}$ is the fundamental solution of the Laplacian operator on $\mathbb{C}^n$ for $n \ge 2$, thus we have

$$\Delta\mathfrak{g}(z) = \nu(z) - \nu(z) * \mathcal{X}_r(z). \tag{3.1}$$

Let us consider $\psi(z) = \mathfrak{g}(z) + \varphi(z)$, then by employing the fundamental solution of the Laplacian operator in $\mathbb{C}^n$, the expression of $\mathfrak{g}$, and the fact that $\mathcal{A}$ is relatively separated, there is a positive constant $C_\varepsilon$ relying on $\varepsilon$ such that

$$|\psi(z) - \varepsilon^{2-2n} - \varphi(z)| \le C_\varepsilon, \text{ for } z \in \mathbb{B}(a_j,\varepsilon) \text{ and } a_j \in \mathcal{A}. \tag{3.2}$$

Let $h \in F^2_\varphi(\mathbb{C}^n)$, and $U(z) = |h(z)|^2\exp(-\psi(z))$, since that $\log(|h(z)|^2)$ is subharmonic, i.e., its Laplacian is positive, then in one side we have

$$\Delta\log(U(z)) = \Delta(\log(|h(z)|^2)) - \Delta\psi(z) \ge -\Delta\psi(z).$$

Then, by using a direct calculus, we have

$$\begin{aligned} -\Delta\psi(z) \;&\le\; \Delta\log(U(z)) = \frac{\Delta U(z)}{U(z)} - \frac{1}{U^2(z)}\left|\frac{\partial U(z)}{\partial z}\right|^2 \\ &\le\; \frac{\Delta U(z)}{U(z)}. \end{aligned}$$

Whence $\Delta U(z) \geq -U(z)\Delta\psi(z)$, thus

$$\int_{\mathbb{C}^n} U(z)\Delta\psi(z)dm(z) \geq -\int_{\mathbb{C}^n} \Delta U(z)dm(z). \tag{3.3}$$

Form the expression of $U$, we have that $U$ is integrable on $\mathbb{C}^n$, thus by employing a smooth function with compact support in $\mathbb{C}^n$ and the dominated convergence theorem, the right-hand side of (3.3) is positive, thus we have $\int_{\mathbb{C}^n} U(z)\Delta\psi(z)dm(z) \geq 0$. Now, by employing (3.1) the fact that $\psi(z) = \mathfrak{g}(z) + \varphi(z)$, and inequality (1.3), we have $\Delta\psi(z) \leq \nu(z) - \eta$.

Whence, from (3.3), we have

$$0 \leq \int_{\mathbb{C}^n} U(z)\Delta\psi(z)dm(z) \leq \int_{\mathbb{C}^n} |h(z)|^2 \exp(-\psi(z))(\nu(z) - \eta)dm(z). \tag{3.4}$$

Or

$$\eta \int_{\mathbb{C}^n} |h(z)|^2 \exp(-\psi(z))dm(z) \leq \int_{\mathbb{C}^n} |h(z)|^2 \exp(-\psi(z))\nu(z)dm(z). \tag{3.5}$$

Then, by using the fact that $\nu(z) = \sum_{a_j \in \mathcal{A}} \frac{1}{\exp(\varepsilon^{2n-2})} \mathcal{X}_{\mathbb{B}(0,\varepsilon)}(z - a_j)$ and (3.2), inequality (3.5) becomes

$$\eta \int_{\mathbb{C}^n} |h(z)|^2 \exp(-\psi(z))dm(z) \lesssim \sum_{a_j \in \mathcal{A}} \int_{|z-a_j|<\varepsilon} |h(z)|^2 \exp(-\varphi(z))dm(z). \tag{3.6}$$

Whence, by using inequality (2.3) of Lemma 2.2, we have

$$\begin{aligned} &\int_{|z-a_j|<\varepsilon} |h(z)|^2 \exp(-\varphi(z))dm(z) \\ = &\int_{|z-a_j|<\varepsilon} |h(z) \exp(-G_j(z))|^2 \exp(-\varphi(z) + 2\Re G_j(z))dm(z) \\ \lesssim &\int_{|z-a_j|<\varepsilon} |g_j(z)|^2 \exp(-\varphi(a_j))dm(z), \end{aligned} \tag{3.7}$$

where $g_j(z) = h(z) \exp(-G_j(z))$ is a holomorphic function, then it is complex differentiable. Consequently, we have

$$\begin{aligned} \int_{|z-a_j|<\varepsilon} |g_j(z)|^2 \exp(-\varphi(a_j))dm(z) \quad \leq \quad & 2\varepsilon^2 |h(a_j)|^2 \exp(-\varphi(a_j)) \\ & +2\varepsilon^4 \exp(-\varphi(a_j) \sup_{|z-a_j|<\varepsilon} |Dg_j(z)|^2, \end{aligned} \tag{3.8}$$

such that $Dg_j(z) = \frac{\partial g_j(z)}{\partial z_1^{\alpha_1} \partial z_2^{\alpha_2} \ldots \partial z_n^{\alpha_n}}$ where $\sum_{l=1}^{n} \alpha_l = 1$ and $(\alpha_l)_{1 \leq l \leq n} \in \mathbb{N}^n$.
Below, we apply Cauchy integral formula, e.g., [3, Chapter I, 4.1 Theorem], for showing that $\exp(-\varphi(a_j) \sup_{|z-a_j|<\varepsilon} |Dg_j(z)|^2$ is less, up to a multiplicative constant, to $\int_{|z-a_j|<\varepsilon} |g_j(\tau)|^2 \exp(-\varphi(a_j))d\tau$. Thus, let $\mathbb{P}(a_j, \boldsymbol{\varepsilon})$ be the polydisc of polyradius $\boldsymbol{\varepsilon} = (\overbrace{\varepsilon, \ldots, \varepsilon}^{n-times}) \in (0, \infty)^n$ and of center $a_j = (a_j^{(1)}, a_j^{(2)}, \ldots, a_j^{(n)}) \in \mathbb{C}^n$. Precisely, $\mathbb{P}(a_j, \boldsymbol{\varepsilon}) = \{z = (z_k)_{1 \leq k \leq n} \in \mathbb{C}^n : |z_k - a_j^{(k)}| < \varepsilon\}$ such that $\overline{\mathbb{P}}(a_j, \boldsymbol{\varepsilon})$ and $T_{a_j,\boldsymbol{\varepsilon}}$ are the closure and the boundary of $\mathbb{P}(a_j, \boldsymbol{\varepsilon})$, respectively.

The fact that $g_j$ is a holomorphic function then there is $\xi \in \overline{\mathbb{P}}(a_j, \boldsymbol{\varepsilon})$ such that $\sup_{z\in\overline{\mathbb{P}}(a_j,\boldsymbol{\varepsilon})} |Dg_j(z)|^2 = |Dg_j(\xi)|^2$, and we have

$$\begin{aligned}\exp(-\varphi(a_j)) \sup_{|z-a_j|<\varepsilon} |Dg_j(z)|^2 &\lesssim \exp(-\varphi(a_j))\,|Dg_j(\xi)|^2 \\ &\lesssim \frac{1}{(2\pi)^n}\int_{T_{a_j,\boldsymbol{\varepsilon}}} |g_j(\tau)|^2 \exp(-\varphi(a_j))d\tau \\ &\lesssim \int_{|z-a_j|<\varepsilon} |g_j(\tau)|^2 \exp(-\varphi(a_j))d\tau. \qquad (3.9)\end{aligned}$$

Now, by utilizing (3.7)-(3.9) and summing up over all pair disjoint balls $(\mathbb{B}(a_j,\varepsilon))_{j\geq 1}$, inequality (3.6) becomes

$$\begin{aligned}\eta\int_{\mathbb{C}^n} |h(z)|^2 \exp(-\varphi(z))dm(z) \lesssim \exp(\varepsilon^2) \sum_{a_j\in\mathcal{A}} |h(a_j)|^2 \exp(-\varphi(a_j)) \\ + \exp(\varepsilon^4)\int_{\mathbb{C}^n} |h(z)|^2 \exp(-\varphi(z))dm(z).\end{aligned} \qquad (3.10)$$

Whence by taking close to zero and $\eta > 2$, we have

$$\int_{\mathbb{C}^n} |h(z)|^2 \exp(-\varphi(z))dm(z) \lesssim \sum_{a_j\in\mathcal{A}} |h(a_j)|^2 \exp(-\varphi(a_j)) = ||h(a)||^2_{l^2_{\varphi,\mathcal{A}}}. \qquad (3.11)$$

Now, by employing the assumption that $i\partial\bar{\partial}\varphi(z) \approx i\partial\bar{\partial}(|z|^2)$, we apply [11, Lemma 7] (with $p = 2$) that for each $a_j$, we have

$$|h(a_j)|^2 \exp(-\varphi(a_j)) \lesssim \int_{\mathbb{B}(a_j,1)} |h(z)|^2 \exp(-\varphi(z))dm(z). \qquad (3.12)$$

Then, the fact that $\mathcal{A}$ is relatively separated, and thanks to (3.12), we have

$$\begin{aligned}||h(a)||^2_{l^2_{\varphi,\mathcal{A}}} &\lesssim rel(\mathcal{A})\int_{\mathcal{A}+\mathbb{B}(0,1)} |h(z)|^2 \exp(-\varphi(z))dm(z) \\ &\lesssim \int_{\mathbb{C}^n} |h(z)|^2 \exp(-\varphi(z))dm(z). \qquad (3.13)\end{aligned}$$

Whence, by combining Inequalities (3.11) and (3.13), we have

$$\int_{\mathbb{C}^n} |h(z)|^2 \exp(-\varphi(z))dm(z) \lesssim ||h(a)||^2_{l^2_{\varphi,\mathcal{A}}} \lesssim \int_{\mathbb{C}^n} |h(z)|^2 \exp(-\varphi(z))dm(z).$$

The proof of Theorem 1.4 is complete.

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