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Ambarzumyan Theorem for Conformable Type Sturm-Liouville Problem on Time Scales

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Abstract: In this study, we give an Ambarzumyan type theorem for a Sturm-Liouville dynamic equation which includes conformable type derivative on time scales with conformable Robin boundary conditions. Under certain conditions, we prove that potential function can be determined by using only first eigenvalue.

Key words: Ambarzumyan Theorem, Conformable Derivative, Sturm-Liouville Equation, Time Scale.

Zaman Skalasında Uyumlu Tip Sturm-Liouville Problemi için Ambarzumyan Teoremi

Öz: Bu çalışmada, uygun Robin sınır koşullarına sahip zaman skalasında uyumlu türev içeren bir Sturm-Liouville dinamik denklemi için Ambarzumyan tipi bir teorem veriyoruz. Belirli koşullar altında potansiyel fonksiyonun yalnızca birinci özdeğer kullanılarak belirlenebileceğini kanıtlıyoruz.

Anahtar kelimeler: Ambarzumyan Teoremi, Uyumlu Türev, Sturm-Liouville Denklemi, Zaman Skalası.

1. Introduction

A time scale $\mathbb{T}$ is a non-empty, arbitrary, closed subset of $\mathbb{R}$. Time scale theory was introduced by Hilger to combine continuous and discrete analysis [1]. This theory allows generalization of existing concepts and better interpretation of physical phenomena. Especially after 2000s, this way of thinking has received a lot of attention and has applied quickly to numerous areas in mathematics. Similarly, the reflections of this idea in spectral theory began to be seen in a short time. Sturm-Liouville theory on $\mathbb{T}$ was firstly studied by Erbe and Hilger in 1993 [2]. In this context, the properties of the eigenvalues and eigenfunctions of the Sturm-Liouville problem were discussed in many studies on $\mathbb{T}$ for different type derivatives (see [3]-[12]). As in classical spectral theory, it is very valuable to obtain results about inverse problems in this theory. Therefore, it is aimed to provide an important literature contribution to the studies on inverse problems for Conformable Sturm-Liouville equation on $\mathbb{T}$ in our study.

The inverse spectral problem is the problem of obtaining the coefficient functions of an operator using various data. One of these data is the set of eigenvalues obtained under the given conditions of the problem. Although there are many studies on inverse problems for different operators in classical case, there isn't much study related to inverse spectral problems on $\mathbb{T}$ with conformable derivative. Before expressing our results for inverse problems involving conformable derivatives on $\mathbb{T}$, it would be useful to briefly mention the first study done on this subject for classical case.

Ambarzumyan's theorem is the first known work in the literature for the steady state Sturm-Liouville problem by Ambarzumyan, who has made very important studies in mathematical physics [13]. He considered following problem of critical importance in mathematical physics and proved that if q is continuous on $(0,1)$ and eigenvalues of the problem,

$$\begin{cases} -y''(t) + q(t)y(t) = \lambda y(t), t \in (0,1) \\ \qquad\qquad y'(0) = y'(1) = 0, \end{cases}$$

are given as $\lambda_n = n^2\pi^2, n \geq 0$, then $q \equiv 0$. In fact, the result obtained here is an exceptional case [13]. In general, a single spectrum is not sufficient to obtain the potential function. Notwithstanding, this study opens an important path for mathematicians studying on spectral theory.

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After this study, Ambarzumian type theorems have been generalized in various ways for different equations and problems. Although this theorem was an exceptional case, it gave an idea to the mathematicians working on this subject at that time that the operator could be determined using spectral data. Based on this idea, two spectra, a spectrum, a normative constant set, and finally a nodal point set were used. Freiling and Yurko have recently interpreted Ambarzumyan's theorem in a different way [14]. They proved that it is sufficient to determine q by only the first eigenvalue instead of whole spectrum and formulated Ambarzumyan's theorem as below:

- $q \equiv \lambda_0$ provided that $\lambda_0 = \int_0^1 q(t)dt$.

Later, Yurko generalized this different approach to a large class of self-adjoint differential operators with arbitrary self-adjoint boundary conditions [15]. This special type of Ambarzumyan theorem was proved by Özkan in 2018 for the Sturm-Liouville dynamical equation on $\mathbb{T}$ (see [16]).

In this study, we discussed and proved the theorem discussed by Özkan from a different aspect using conformable derivative on $\mathbb{T}$. Thus, we have determined how the conformable derivative, which is one of the most useful and functional versions of the fractional derivative, works in the Ambarzumyan theorem.

2. Preliminaries

Before expressing the main results, it needs to be reminded some basic notions on $\mathbb{T}$ [17-19]. Let $a = \inf \mathbb{T}$ and $b = \sup \mathbb{T}$. Forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined by

$$\sigma(t) = inf\{s \in \mathbb{T} : s > t\}, \rho(t) = sup\{s \in \mathbb{T} : s < t\}$$

respectively, for $t \in \mathbb{T}$ where $a < t < b, t < \sup \mathbb{T},\ \inf \emptyset = \sup \mathbb{T},\ \sup \emptyset = \inf \mathbb{T}$ and $\emptyset$ indicates empty set. If $\mathbb{T}$ is bounded, one can write $\sigma(b) = b$ and $\rho(a) = a$. The related forward-step function is defined by [18,19]

$$\mu: \mathbb{T}^\kappa \to \mathbb{R}^+, \mu(t) = \sigma(t) - t.$$

Here, $\mathbb{T}^\kappa = \mathbb{T} \setminus \{b\}$ is bounded above and b is left-scattered; otherwise $\mathbb{T}^\kappa = \mathbb{T}$. This is often used when trading with a first-order delta derivative. Similarly, the n-dimensional version of this set can be defined. $f: \mathbb{T} \to \mathbb{R}$ is right side continuous at $t \in \mathbb{T}$ if there is some $\delta > 0$ such that $|f(t) - f(s)| < \varepsilon$ for all $s \in [t, t + \delta)$ and $\varepsilon > 0$. The set of all these functions on $\mathbb{T}$ is denoted by $C_{rd}(\mathbb{T})$. One can define $f^\Delta(t)$ to be the value for $t \in \mathbb{T}^\kappa$, if one exists, there is a neighborhood U of t such that for all $s \in U$ and $\varepsilon > 0$

$$|[f^\sigma(t) - f(s)] - f^\Delta(t)(\sigma(t) - s)| < \varepsilon|\sigma(t) - s|.$$

f is Δ-differentiable on $\mathbb{T}^\kappa$ if $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. Let $f \in C_{rd}(\mathbb{T})$. Then, there exists a function F such that $F^\Delta(t) = f(t)$, and Δ integral is constructed by

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

Now, let's examine the problem on which we have built this study and expressed by Özkan on $\mathbb{T}$ [16].

Let $\mathbb{T}$ be bounded, $a = \inf \mathbb{T}$ and $b = \sup \mathbb{T}$. Ozkan considered $\ell = \ell(q, h_a, h_b)$ generated by below Sturm–Liouville dynamic equation

$$\ell y = -y^{\Delta\Delta}(t) + q(t)y^\sigma(t) = \lambda y^\sigma(t), t \in \mathbb{T}^{\kappa^2}, \tag{2.1}$$

subject to boundary conditions

$$y^\Delta(a) - h_a y(a) = 0, \tag{2.2}$$
$$y^\Delta(\rho(a)) - h_b y(\rho(a)) = 0, \tag{2.3}$$

where $q(t)$ is real-valued, continuous on $\mathbb{T}, h_a, h_b \in \mathbb{T}, a \neq \rho(b), 1 + h_a\mu(a) \neq 0, 1 + h_b\mu(\rho(b)) \neq 0$ and λ is a spectral parameter [16]. The following theorem is the Freiling-Yurko version of the Ambarzumyan theorem on the time scale.

Theorem 2.1. [16] Let λ_1 be the first eigenvalue of $(2.1)-(2.3)$. If

$$\lambda_1 = \frac{1}{\rho(b)-a}\left\{h_a - h_b + \int_a^{\rho(b)} q(t)\Delta t\right\},$$

then $q(t) \equiv \lambda_1$ on $\mathbb{T}^{\kappa^2}$ and $h_a = h_b = 0$.

The proof of this theorem was made by Özkan using the basic properties of the time scale and the dynamic equation structure. In the next section, properties of conformable derivative and integral will be expressed and Theorem 2.1 will be generalized to conformable derivative on $\mathbb{T}$.

3. Main Results

Different versions of the fractional derivative have been described and generalized over the years. The most common fractional derivatives are Riemann-Liouville, Caputo, Grünwald-Letnikov, Wely, Riesz. For more information about the characteristics of these fractional concepts, we refer to [20].

For these fractional derivative types, different ideas have been put forward over time due to the difficulties in the applications and the difficulties experienced. Recently, Khalil et al. give a new fractional derivative "conformable" [21]. Unlike other types, this new version satisfies properties "derivative of product and quotient of two functions". Apart from this, the chain rule, which has an important place in applications, has taken a simpler form in this derivative.

Definition 3.1. [21,22] Conformable fractional derivative of $f: [0,\infty) \to \mathbb{R}$ with order α, is defined by

$$(T_\alpha f)(t) = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \tag{3.1}$$

for all $t > 0$ and $0 < \alpha \leq 1$. If f is α-differentiable on some $(0,\alpha)$, then $\lim_{t\to 0^+}(T_\alpha f)(t)$ exists and

$$(T_\alpha f)(0) = \lim_{t\to 0^+}(T_\alpha f)(t). \tag{3.2}$$

In the next theorem, we will recall the necessary and important properties of conformable derivative.

Theorem 3.2. [21] Let f, g be α-conformable differentiable at $t > 0$ and $0 < \alpha \leq 1$. Then

i. $T_\alpha(af + bg) = a(T_\alpha f) + b(T_\alpha g), \forall a, b \in \mathbb{R}$.
ii. $T_\alpha(t^p) = pt^{p-\alpha}, \forall p \in \mathbb{R}$.
iii. $T_\alpha(\lambda) = 0$, for $f(t) = \lambda, \lambda \in \mathbb{R}$.
iv. $T_\alpha(fg) = f(T_\alpha g) + g(T_\alpha f)$.
v. $T_\alpha\left(\frac{f}{g}\right) = \frac{g(T_\alpha f) - f(T_\alpha g)}{g^2}$.
vi. $(T_\alpha f)(t) = t^{1-\alpha}\frac{df}{dt}(t)$ when f is α- conformable differentiable.

Definition 3.3. α - conformable integral of regulated function $h: \mathbb{T} \to \mathbb{R}$ is defined by [23]

$$\int h(t)\Delta^\alpha t = \int h(t)t^{\alpha-1}\Delta t. \tag{3.3}$$

α-Conformable fractional integral of h reduces to classical Conformable fractional integral for $\mathbb{T} = \mathbb{R}$ and $\alpha = 1$ [21]. And, it reduces to indefinite integral on $\mathbb{T}$ for $\alpha = 1$ [21]. If indefinite α-Conformable fractional integral of h order α is

$$H_\alpha(t) = \int h(t)\Delta^\alpha t,$$

then, Cauchy α-Conformable fractional integral of h is [24]

$\int_a^b h(t)\Delta^\alpha t = H_\alpha(b) - H_\alpha(a),$

for all $a, b \in \mathbb{T}$.
Now it is time to define our conformable problem and prove Ambarzumyan theorem on $\mathbb{T}$. Consider conformable Sturm-Liouville boundary value problem by

$$Ly = -T_\alpha\big(T_\alpha y(t)\big) + q(t)y^\sigma(t) = \lambda y^\sigma(t),\ t \in [a, \rho(b)] \tag{3.4}$$
$$T_\alpha\big(y(a)\big) - h_a y(a) = 0, \tag{3.5}$$
$$T_\alpha\Big(y\big(\rho(b)\big)\Big) - h_b y\big(\rho(b)\big) = 0, \tag{3.6}$$

where $a \neq \rho(b)$, $1 + h_a\mu(a)a^{\alpha-1} \neq 0$, $1 + h_b\mu\big(\rho(b)\big)(\rho(b))^{\alpha-1} \neq 0$, λ is spectral parameter and $L = L(q, h_a, h_b)$.

Definition 3.4. [16] The values of λ when equation (3.4) has non-zero solutions satisfying (3.5) and (3.6) are eigenvalues and corresponding non-trivial solutions are eigenfunctions. Additionally, the set of all eigenvalues for $(3.4) - (3.6)$ is real and below bounded [18].

Definition 3.5. [16] A solution y of (3.4) has a zero at $t \in \mathbb{T}$ if $y(t) = 0$, and it has a node between t and $\sigma(t)$ if $y(t)y\big(\sigma(t)\big) < 0$.

Lemma 3.6. [3] Eigenvalues of $(3.4) - (3.6)$ can be arranged as $-\infty < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$, and an eigenfunction corresponding to λ_{k+1} has exactly k generalized zeros on (a, b).

Proof. Assume that x and y are solutions to the equations $ly + \lambda y^\sigma = 0$ and $lx + \rho_v x^\sigma = 0$, respectively where $\rho_v \in \mathbb{R}$, $1 \leq v \leq k$, and also $\tilde{x} = x + \widetilde{x_1}$, $S = \frac{\tilde{x}}{y}(xT_\alpha y - yT_\alpha x)$. If $\lambda_{k+1} < \infty$, $y = y(.,\lambda)$, let's assume $x = 0$, $x_v = x_v$, $1 \leq v \leq k$, $k \in \mathbb{N}_0$, $p_v = \lambda_v$, $\lambda = \lambda_{k+1}$. In this situation, $S_\alpha(a) = S_\alpha\big(\rho(b)\big) = 0$, and the rest of the proof is the similar in [3].

Lemma 3.7. $y^\sigma(a) \neq 0$ and $y^\sigma(\rho(b)) \neq 0$ provided that $y(t)$ is eigenfunction of $(3.4) - (3.6)$.

Proof. We get

$$y^\sigma(a) = y(a) + \mu(a)a^{1-\alpha}T_\alpha\big(y(a)\big) = y(a)[1 + h_a\mu(a)a^{\alpha-1}],$$

and

$$y^\sigma(\rho(b)) = y\big(\rho(b)\big)[1 + h_b\mu\big(\rho(b)\big)\big(\rho(b)\big)]^{\alpha-1}.$$

Here by the properties of time scale calculus, we allege that $y^\sigma(a) \neq 0$ and $y^\sigma(\rho(b)) \neq 0$. Otherwise, either $T_\alpha\big(y(a)\big) = 0$ or $T_\alpha(\rho(b)) = 0$ holds by (3.4) ve (3.5). It implies that $y(t) = 0$ which contradicts to be an eigenfunction. Hence, the assumptions $1 + h_a\mu(a)a^{\alpha-1} \neq 0$, $1 + h_b\mu\big(\rho(b)\big)\big(\rho(b)\big)^{\alpha-1} \neq 0$ completes the proof. Now, together with the following theorem, the Freiling-Yurko version of the Ambarzumyan theorem will be expressed in a time scale including conformable derivative.

Theorem 3.8. Let λ_1 be the first eigenvalue of the problem $(3.4) - (3.6)$.

$$\lambda_1 = \frac{\alpha}{\big(\rho(b)\big)^\alpha - a^\alpha}\Big\{h_a - h_b + \int_a^{\rho(b)} q(t)\Delta^\alpha t\Big\},$$

İmplies that $q(t) = \lambda_1$.

Proof. By (3.4), we obtain

$\frac{T_\alpha(T_\alpha y_1(t))}{y_1^\sigma(t)} = q(t) - \lambda_1,$
$T_\alpha\left(\frac{T_\alpha(y_1)}{y_1}\right) = \frac{T_\alpha(T_\alpha y_1)y_1 - T_\alpha y_1 T_\alpha y_1}{y_1^\sigma y_1}$

$$= \frac{T_\alpha(T_\alpha y_1)}{y_1^\sigma} - \frac{(T_\alpha y_1)^2}{y_1^\sigma y_1}.$$

Since $\frac{T_\alpha(T_\alpha y_1)}{y_1^\sigma} = q(t) - \lambda_1$, it yields

$$\frac{(T_\alpha y_1)^2}{y_1^\sigma y_1} = -T_\alpha\left(\frac{T_\alpha y_1}{y_1}\right) + q(t) - \lambda_1.$$

If we apply conformable integration to both sides of above equality from a to $\rho(b)$, we get

$$\int_a^{\rho(b)} \frac{(T_\alpha y_1)^2}{y_1^\sigma y_1} \Delta^\alpha t = \frac{T_\alpha(y_1(a))}{y_1(a)} - \frac{T_\alpha\left(y_1(\rho(b))\right)}{y_1(\rho(b))} + \int_a^{\rho(b)} [q(t) - \lambda_1]\Delta^\alpha t.$$

On the other hand, since $\frac{(T_\alpha y_1)^2}{y_1^\sigma y_1} > 0$ and $\Delta^\alpha t = t^{\alpha-1}\Delta t$, the result of the above expression is obtained as follows

$$\int_a^{\rho(b)} \frac{(T_\alpha y_1)^2}{y_1^\sigma y_1} \Delta^\alpha t = h_a - h_b + \int_a^{\rho(b)} q(t)\Delta^\alpha t - \frac{\lambda_1}{\alpha}\left[\left(\rho(b)\right)^\alpha - a^\alpha\right] = 0.$$

Considering the conditions of the problem, we get

$$T_\alpha y_1(t) \equiv 0 \Longrightarrow y_1(t) = c \Longrightarrow q(t) = \lambda_1.$$

This completes the proof.

Example 3.9.
Let us consider below conformable Sturm-Liouville problem

$$-T_{0.5}\left(T_{0.5}y(t)\right) + q(t)y^\sigma(t) = \lambda y^\sigma(t),\ t \in [1, \rho(3)]$$
$$T_{0.5}(y(1)) - y(1) = 0, (h_a = 1)$$
$$T_{0.5}\left(y\left(\rho(3)\right)\right) - y\left(\rho(3)\right) = 0, (h_b = 1).$$

Using Theorem 3.7 and properties of delta conformable derivative, we get

$$\lambda_1 = \frac{0.5}{\left(\rho(3)\right)^{0.5} - 1^{0.5}}\left\{1 - 1 + \int_1^{\rho(3)} q(t)\Delta^{0.5} t\right\}$$
$$= \frac{1}{2(\sqrt{\rho(3)} - 1)}\int_1^{\rho(3)} q(t)\mathrm{t}^{-0.5}\Delta t$$
$$= \frac{1}{2(\sqrt{\rho(3)} - 1)}\int_1^{\rho(3)} \frac{q(t)}{\sqrt{t}}\Delta t$$

Then, it implies

$$q(t) = \lambda_1.$$

Here, the order of conformable derivative and the time scale studied can be chosen arbitrarily.

4.Conclusions

In this study, a Freiling-Yurko type Amabarzumyan theorem, which has been proved on the time scale before, is considered and proven to include a conformable derivative on the time scale. The results obtained are very important in terms of the application of the fractional derivative to inverse spectral theory on the time scale. The theorem proved is made more concrete with an example for some special cases. This study can be done for other kinds of fractional derivative and inverse problems other than Ambarzumyan theorem can be proved.

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