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CHOLESKY ALGORITHM OF A LUCAS TYPE MATRIX

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Abstract. Many generalizations have been made for Fibonacci and Lucas number sequences and many properties have been found about these sequences. In the article [13], the authors obtained many features of these sequences with the Cholesky decomposition algorithm, using the 2 x 2 matrix belonging to a generalization of the Fibonacci sequence. In this study, it is shown that many different features can be found by using a 2 x 2 matrix belonging to the Lucas number sequence with the same method.

1. Introduction

Most identities for the Fibonacci number sequence F_n and the Lucas number sequence L_n are obtained by changing the recursion relations and/or initial conditions of the sequences and making sequence generalizations ([2]- [5], [9]- [15], [17], [19]- [27]).

The Fibonacci numbers F_n are defined by a quadratic recurrence relation:

$$F_{n+2} = F_{n+1} + F_n, \ n \geq 0 \tag{1}$$

with initial conditions $F_0 = 0$ and $F_1 = 1$, see [15]. Binet formula for the numbers F_n is

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{2}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. From here, it can be noted that and

$$\begin{aligned} \alpha\beta &= -1, \\ \alpha + \beta &= 1, \\ \alpha - \beta &= \sqrt{5}. \end{aligned}$$

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We also recall [15] that

$$F_n = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-j-1}{j},$$

where $\lfloor \cdot \rfloor$ denotes the greatest integer function. Using the Binet formula, we can write the following equation for negative indices:

$$F_{-n} := (-1)^{n+1} F_n.$$

Analogously, the numbers L_n are defined by a quadratic recurrence relation:

$$L_{n+2} = L_{n+1} + L_n, \ n \geq 0$$

with initial conditions $L_0 = 2, L_1 = 1$, see [17]. Binet formula for the numbers L_n is

$$L_n = \alpha^n + \beta^n. \tag{3}$$

Also the F_n and L_n numbers satisfied following identity

$$L_n = F_{n-1} + F_{n+1}. \tag{4}$$

Morever, from above equalities we have that

$$L_{-n} = (-1)^n L_n.$$

In [8] and [28], the *Cholesky decomposition* (*Cholesky factorization*) is defined as: If $A \in \mathbb{R}_n^n$ is symmetric positive definite matrix, then there exists a unique lower triangular matrix $G \in \mathbb{R}_n^n$ with positive diagonal entries such that $A = GG^T$. Here G^T is the transpose matrix of the G. The calculation of G and G^T matrices is called the *Cholesky algorithm*.

Matrix method is also very useful method to obtain the properties of Fibonacci and Lucas sequences, see [6], [13], [16], [18], [22], [24], [26]. In particular, Horadam and Flipponi obtained some new features for Fibonacci and Lucas sequences by using the matrix M_k which is created by the Cholesky matrix decomposition algorithm [13]. While doing this work they used the k-Fibonacci generalized sequence and the M matrix belonging to this sequence.

We observed that the application of the same method for the M matrix constituting the Lucas sequence creates different sequence properties. In this study, the matrix functions of the xM_k^n matrix sequence, which was created by using the $M = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$ matrix that produced the Lucas sequence, were examined and new results were obtained.

2. Main Results

From [16] let's consider the 2×2 symmetric matrix

$$M = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

which has eigenvalues $\alpha+2$ and $\beta+2$. For a positive integer n,

$$M^n = \begin{cases} 5^{\frac{n-1}{2}} \begin{bmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{bmatrix}, & \text{if } n \text{ is odd,} \\ 5^{\frac{n}{2}} \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}, & \text{if } n \text{ is even,} \end{cases} \tag{5}$$

see [16]. Now let us define the matrix sequence M_k in the following steps.

Let $M_1 := M$, therefore

$$M_1 = M = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

and Cholesky decomposition of M_1 is obtained as

$$M_1 = T_1 T_1^T = \begin{bmatrix} a_1 & 0 \\ c_1 & b_1 \end{bmatrix} \begin{bmatrix} a_1 & c_1 \\ 0 & b_1 \end{bmatrix},$$

where T_1 is a lower triangular matrix and T_1^T is the transpose matrix of T_1. So T_1^T is an upper triangular matrix. The a_1, b_1 and c_1 components of T_1 easily obtained with the matrix equation above. In fact, the system

$$\begin{aligned} a_1^2 &= 3, \\ a_1 c_1 &= 1, \\ b_1^2 + c_1^2 &= 2 \end{aligned}$$

can be written, whose solution is

$$\begin{aligned} a_1 &= \pm\sqrt{3} \\ c_1 &= \frac{1}{a_1} \\ b_1 &= \pm\sqrt{2 - c_1^2} \end{aligned}$$

Any of the four solutions obtained creates a Cholesky decomposition of the symmetric matrix M_1.

We also know that the product of a lower triangular matrix and an upper triangular matrix is generally not commutative, so it is known that the inverse product $T_1^T T_1$ gives a symmetric matrix M_2 similar to but different from M_1 [7]. If we consider the $b_1 = \sqrt{\frac{5}{3}}$ solution, we get

$$M_2 = \frac{1}{3} \begin{bmatrix} 10 & \sqrt{5} \\ \sqrt{5} & 5 \end{bmatrix},$$

when $b_1 = -\sqrt{\frac{5}{3}}$ the off-diagonal components of M_2 are negative.

In contrast, M_2 can be decomposed similarly so that

$$M_2 = T_2 T_2^T = \begin{bmatrix} a_2 & 0 \\ c_2 & b_2 \end{bmatrix} \begin{bmatrix} a_2 & c_2 \\ 0 & b_2 \end{bmatrix},$$

where

$$\begin{aligned} a_2 &= \pm\sqrt{\frac{10}{3}}, \\ c_2 &= \frac{\sqrt{5}}{a_2} = \pm\frac{\sqrt{6}}{6}, \\ b_2 &= \pm\frac{\sqrt{6}}{2}. \end{aligned}$$

The inverse product $T_2^T T_2$ gave rise to a matrix M_3 with the sign of the off-diagonal entries based on b_2.

If we repeat such a procedure indefinitely, we get the sequence $(M_k)_1^\infty$ of the 2×2 symmetric matrices. Henceforth M_k be called the *k-order Lucas-type Cholesky algorithm matrix*.

Due to the unclear sign of Cholesky decomposition, the above matrix sequence is not the only possible result of applications of the Cholesky algorithm to M. However, other possible outcomes may differ only in the sign of the off-diagonal components of the above matrix sequence, in any term of the sequence except the first term. However, from now on we will only consider the positive definite (M_k) matrix sequence.

Since the matrices M_k are similar, they have the same eigenvalues. M_k tends to a diagonal matrix containing these eigenvalues as k tends to infinity.

The following Lemma can be easily obtained from [15] and [27]

Lemma 1. *Let k be a positive integer, then*

i) If k is odd, then $L_{k-1}L_{k+1} = 5F_k^2 + 1$.

ii) If k is even, then $5F_{k+1} = L_{\frac{k}{2}+1}^2 + L_{\frac{k}{2}}^2$.

iii) If k is even, then $L_k^2 + 1 = F_{k+1}\left(L_{\frac{k}{2}-1}^2 + L_{\frac{k}{2}}^2\right)$.

Theorem 1. *Let k be positive integer, then*

$$M_k = \begin{cases} \frac{1}{F_k}\begin{bmatrix} L_{k+1} & 1 \\ 1 & L_{k-1} \end{bmatrix}, & \textit{if } k \textit{ is odd,} \\ \frac{1}{L_k}\begin{bmatrix} L_{\frac{k}{2}+1}^2 + L_{\frac{k}{2}}^2 & \sqrt{5} \\ \sqrt{5} & L_{\frac{k}{2}}^2 + L_{\frac{k}{2}-1}^2 \end{bmatrix}, & \textit{if } k \textit{ is even.} \end{cases}$$

Proof. From the M_1 and M_2 matrices we found earlier, it can be seen that the equality is achieved in the case of $k = 1$ and $k = 2$.

If k is odd:

$$M_k = \frac{1}{F_k}\begin{bmatrix} L_{k+1} & 1 \\ 1 & L_{k-1} \end{bmatrix} = T_k T_k^T$$

hence, using Lemma 1, we obtain

$$T_k = \begin{bmatrix} \frac{\sqrt{L_{k+1}}}{\sqrt{F_k}} & 0 \\ \frac{1}{\sqrt{F_k L_{k+1}}} & \sqrt{5}\frac{\sqrt{F_k}}{\sqrt{L_{k+1}}} \end{bmatrix}.$$

Therefore

$$\begin{aligned} M_{k+1} &= T_k^T T_k = \begin{bmatrix} \frac{\sqrt{L_{k+1}}}{\sqrt{F_k}} & \frac{1}{\sqrt{F_k L_{k+1}}} \\ 0 & \sqrt{5}\frac{\sqrt{F_k}}{\sqrt{L_{k+1}}} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{L_{k+1}}}{\sqrt{F_k}} & 0 \\ \frac{1}{\sqrt{F_k L_{k+1}}} & \sqrt{5}\frac{\sqrt{F_k}}{\sqrt{L_{k+1}}} \end{bmatrix} \\ &= \frac{1}{L_{k+1}}\begin{bmatrix} \frac{L_{k+1}^2+1}{F_k} & \sqrt{5} \\ \sqrt{5} & 5F_k \end{bmatrix}. \end{aligned}$$

Here, using the Lemma 1

$$M_{k+1} = \frac{1}{L_{k+1}}\begin{bmatrix} L^2_{\frac{k+1}{2}+1} + L^2_{\frac{k+1}{2}} & \sqrt{5} \\ \sqrt{5} & L^2_{\frac{k+1}{2}} + L^2_{\frac{k-1}{2}} \end{bmatrix}$$

is obtained.

If k is even:

$$M_k = \frac{1}{L_k}\begin{bmatrix} L^2_{\frac{k}{2}+1} + L^2_{\frac{k}{2}} & \sqrt{5} \\ \sqrt{5} & L^2_{\frac{k}{2}} + L^2_{\frac{k}{2}-1} \end{bmatrix}$$

hence, using Lemma 1, we obtain

$$T_k = \begin{bmatrix} \sqrt{5}\frac{\sqrt{F_{k+1}}}{\sqrt{L_k}} & 0 \\ \frac{1}{\sqrt{L_k}\sqrt{F_{k+1}}} & \frac{\sqrt{L_k}}{\sqrt{F_{k+1}}} \end{bmatrix}.$$

Therefore

$$\begin{aligned} M_{k+1} &= T_k^T T_k = \begin{bmatrix} \sqrt{5}\frac{\sqrt{F_{k+1}}}{\sqrt{L_k}} & \frac{1}{\sqrt{L_k}\sqrt{F_{k+1}}} \\ 0 & \frac{\sqrt{L_k}}{\sqrt{F_{k+1}}} \end{bmatrix} \begin{bmatrix} \sqrt{5}\frac{\sqrt{F_{k+1}}}{\sqrt{L_k}} & 0 \\ \frac{1}{\sqrt{L_k}\sqrt{F_{k+1}}} & \frac{\sqrt{L_k}}{\sqrt{F_{k+1}}} \end{bmatrix} \\ &= \frac{1}{F_{k+1}}\begin{bmatrix} \frac{5F_{k+1}^2+1}{L_k} & 1 \\ 1 & L_k \end{bmatrix} \\ &= \frac{1}{F_{k+1}}\begin{bmatrix} L_{k+2} & 1 \\ 1 & L_k \end{bmatrix}. \end{aligned}$$

Here, the equation $L_{k+2}\, L_k = 5F_{k+1}^2 + 1$ obtained from $L_{2m}\, L_{2n} = 5F_{m+n}^2 + L_{m-n}^2$ in [15, p.109] is used. □

Theorem 2. *If we apply the Cholesky algorithm to M^n, we obtain the followings:*

$$(M^n)_k = \begin{cases} \frac{5^{\frac{n-1}{2}}}{F_k}\begin{bmatrix} L_{n+k} & L_n \\ L_n & L_{n-k} \end{bmatrix}, & \text{if } k \text{ is odd and } n \text{ is odd,} \\ \frac{5^{\frac{n}{2}}}{F_k}\begin{bmatrix} F_{n+k} & F_n \\ F_n & F_{n-k} \end{bmatrix}, & \text{if } k \text{ is odd and } n \text{ is even,} \\ \frac{5^{\frac{n}{2}}}{L_k}\begin{bmatrix} L_{n+k} & F_n\sqrt{5} \\ F_n\sqrt{5} & L_{n-k} \end{bmatrix}, & \text{if } k \text{ is even and } n \text{ is even,} \\ \frac{5^{\frac{n-1}{2}}}{L_k}\begin{bmatrix} 5F_{n+k} & L_n\sqrt{5} \\ L_n\sqrt{5} & 5F_{n-k} \end{bmatrix}, & \text{if } k \text{ is even and } n \text{ is odd.} \end{cases}$$

We can also see that the equation $(M_k)^n = (M^n)_k$ and for simplicity we will use the notation $M_k^n := (M_k)^n = (M^n)_k$.

Proof. It can be easily seen by induction using Theorem 1 and equation (5) . □

Here, suppose the above power equation is true for some value of n, say N. In this case, $(M_k)^N = (M^N)_k$. From this, it can be easily seen that $(M_k)^{N+1} = M_k(M_k)^N = M_k(M^N)_k = (M^{N+1})_k$ so if the above power equation is true for N, it is also true for $N+1$.

2.1. **Functions of the Matrix xM_k^n.** From the theory of functions of matrices [7], if the function f is a function defined on the spectrum of a 2×2 matrix $A=[a_{ij}]$ with distinct eigenvalues λ_1 and λ_2, then

$$f(A) = X = [x_{ij}] = c_0 I + c_1 A, \tag{6}$$

where I is the 2×2 identity matrix and the coefficients c_0 and c_1 are given by the solution of the system

$$\begin{aligned} c_0 + c_1\lambda_1 &= f(\lambda_1), \\ c_0 + c_1\lambda_2 &= f(\lambda_2). \end{aligned}$$

Therefore

$$\begin{aligned} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} &= c_0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_1\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \\ &= \begin{bmatrix} c_0 + c_1 a_{11} & c_1 a_{12} \\ c_1 a_{21} & c_0 + c_1 a_{22} \end{bmatrix}. \end{aligned}$$

From the last equation, we get

$$\begin{aligned} x_{11} &= c_0 + c_1 a_{11}, \\ x_{12} &= c_1 a_{12}, \end{aligned}$$

$$\begin{aligned} x_{21} &= c_1 a_{21}, \\ x_{22} &= c_0 + c_1 a_{22}. \end{aligned}$$

In equation (6), let us write λ_1 and λ_2 instead of A and find c_0 and c_1 values

$$\begin{aligned} c_0 &= \frac{(\beta+2)^n f(x(\alpha+2)^n) - (\alpha+2)^n f(x(\beta+2)^n)}{(\beta+2)^n - (\alpha+2)^n}, \\ c_1 &= \frac{f(x(\beta+2)^n) - f(x(\alpha+2)^n)}{(\beta+2)^n - (\alpha+2)^n} \end{aligned}$$

and then

$$\begin{aligned} x_{11} &= [(a_{11} - \lambda_1) f(\lambda_2) - (a_{11} - \lambda_2) f(\lambda_1)]/(\lambda_2 - \lambda_1), \\ x_{12} &= a_{12}[f(\lambda_2) - f(\lambda_1)]/(\lambda_2 - \lambda_1), \\ x_{21} &= a_{21}[f(\lambda_2) - f(\lambda_1)]/(\lambda_2 - \lambda_1), \\ x_{22} &= [(a_{22} - \lambda_1) f(\lambda_2) - (a_{22} - \lambda_2) f(\lambda_1)]/(\lambda_2 - \lambda_1). \end{aligned}$$

Lemma 2. *Let k and n be arbitrary positive integers. For x an arbitrary quantity, let us consider the matrix xM_k^n having eigenvalues*

$$\begin{aligned} \lambda_1 &= x(\alpha+2)^n, \\ \lambda_2 &= x(\beta+2)^n. \end{aligned}$$

Proof. It is easily seen by induction. □

To express the y_{ij} components of $Y = [y_{ij}] = f(xM_k^n)$ in separate formulas, we can give the following theorem with

$$\lambda := \frac{(\beta+2)^n f(x(\alpha+2)^n) - (\alpha+2)^n f(x(\beta+2)^n)}{(\beta+2)^n - (\alpha+2)^n}$$

and

$$\phi := \frac{f(x(\beta+2)^n) - f(x(\alpha+2)^n)}{(\beta+2)^n - (\alpha+2)^n}.$$

Theorem 3. *Let k and n be arbitrary positive integers.*

i) If n is even and k is odd, then

$$Y = \frac{5^{\frac{n}{2}}}{F_k} \begin{bmatrix} \lambda F_k + \phi F_{n+k} & \phi F_n \\ \phi F_n & \lambda F_k + \phi F_{n-k} \end{bmatrix}.$$

ii) If n is odd and k is odd, then

$$Y = \frac{5^{\frac{n-1}{2}}}{F_k} \begin{bmatrix} \lambda F_k + \phi L_{n+k} & \phi L_n \\ \phi L_n & \lambda F_k + \phi L_{n-k} \end{bmatrix}.$$

iii) If n is odd and k is even, then

$$Y=\frac{5^{\frac{n-1}{2}}}{L_k}\begin{bmatrix}\lambda L_k+5\phi F_{n+k} & \sqrt{5}\phi L_n\\ \sqrt{5}\phi L_n & \lambda L_k+5\phi F_{n-k}\end{bmatrix}.$$

iv) If n is even and k is even, then

$$Y=\frac{5^{\frac{n}{2}}}{L_k}\begin{bmatrix}\lambda L_k+\phi L_{n+k} & \sqrt{5}\phi F_n\\ \sqrt{5}\phi F_n & \lambda L_k+\phi L_{n-k}\end{bmatrix}.$$

Proof. Taking xM_k^n as matrix A in equation (6) and applying the above steps using Lemma 2 the desired result is obtained. □

Theorem 4. *If f is the matrix inversion function then*

$$(xM_k^n)^{-1}=\begin{cases}\dfrac{5^{\frac{-n-1}{2}}}{xF_k}\begin{bmatrix}L_{n-k} & -L_n\\ -L_n & L_{n+k}\end{bmatrix}, & \text{if } k \text{ is odd and } n \text{ is odd,}\\ \dfrac{5^{\frac{-n}{2}}}{xF_k}\begin{bmatrix}F_{n-k} & -F_n\\ -F_n & F_{n+k}\end{bmatrix}, & \text{if } k \text{ is odd and } n \text{ is even,}\\ \dfrac{5^{\frac{-n}{2}}}{xL_k}\begin{bmatrix}L_{n-k} & -F_n\sqrt{5}\\ -F_n\sqrt{5} & L_{n+k}\end{bmatrix}, & \text{if } k \text{ is even and } n \text{ is even,}\\ \dfrac{5^{\frac{-n-1}{2}}}{xL_k}\begin{bmatrix}5F_{n-k} & -L_n\sqrt{5}\\ -L_n\sqrt{5} & 5F_{n+k}\end{bmatrix}, & \text{if } k \text{ is even and } n \text{ is odd.}\end{cases}$$

Proof. It can be easily seen using the identity $(xM_k^n)^{-1}=\frac{1}{x}M_k^{-n}$, $(x\neq 0)$. □

3. Relations with Some Finite Series

In this section, sums of some finite series containing F_n and L_n are found using some properties of the Lucas-type Cholesky algorithm matrix M_k.

Lemma 3. *If k is a positive integer, then*

$$M_k^2=5M_k-5I, \tag{7}$$

and

$$M_k^{-1}=I-\frac{1}{5}M_k. \tag{8}$$

Proof. Using equation (1), it easily be obtained from equations Theorem 1 and Theorem 2. □

Lemma 4. *If x is an arbitrary quantity with the constraints $x \neq \dfrac{1}{\alpha^n}$ and $x \neq \dfrac{1}{\beta^n}$ then*

$$(xM_k^n - I)^{-1} = \begin{cases} \dfrac{\left(5^{\frac{n+1}{2}} F_n x - 1\right) I - xM_k^n}{5^n x^2 - 5^{\frac{n+1}{2}} F_n x + 1}, & \text{if } n \text{ is odd,} \\ \dfrac{\left(5^{\frac{n}{2}} L_n x - 1\right) I - xM_k^n}{5^n x^2 - 5^{\frac{n}{2}} L_n x + 1}, & \text{if } n \text{ is even.} \end{cases}$$

Proof. It can be easily seen using equations (2), (3), (4) and Lemma 3 and the following equations

$$\begin{aligned} L_{k+n} - 5F_nF_k &= -L_{n-k} \quad && \text{if } k \text{ is odd and } n \text{ is odd [15, p. 111, 83.],} \\ F_{k+n} - F_kL_n &= -F_{n-k} && \text{if } k \text{ is odd and } n \text{ is even [15, p. 118, 58.],} \\ L_{k+n} - L_nL_k &= -L_{n-k} && \text{if } k \text{ is even and } n \text{ is even [15, p. 111, 83.],} \\ F_{k+n} - L_kF_n &= -F_{n-k} && \text{if } k \text{ is even and } n \text{ is odd [15, p. 118, 58.].} \end{aligned}$$

□

Lemma 5. *For positive numbers k and n the following equality holds*

$$M_k^n = \sum_{j=0}^{n} 5^{-j} \binom{n}{j} M_k^{2j}.$$

Proof. From equation (7) we can write $(M_k^2 + 5I)^n = (5M_k)^n$, from which the proof can be obtained by using the binomial expansion. □

Theorem 5. *i) Let n be a nonnegative even integer and k be an arbitrary positive integer. Then we have*

$$F_{n\mp k} = 5^{-\frac{n}{2}} \sum_{j=0}^{n} \binom{n}{j} F_{2j\mp k},$$

$$L_{n\mp k} = 5^{-\frac{n}{2}} \sum_{j=0}^{n} \binom{n}{j} L_{2j\mp k}.$$

ii) Let n be a nonnegative odd integer and k be an arbitrary positive integer. Then we have

$$F_{n\mp k} = 5^{\frac{-n-1}{2}} \sum_{j=0}^{n} \binom{n}{j} F_{2j\mp k},$$

$$L_{n\mp k} = 5^{\frac{-n+1}{2}} \sum_{j=0}^{n} \binom{n}{j} L_{2j\mp k}.$$

Proof. If n is even positive integer and k is odd positive integer, then from Theorem 2 and Lemma 5,

$$M_k^n = \frac{5^{\frac{n}{2}}}{F_k}\begin{bmatrix} F_{n+k} & F_n \\ F_n & F_{n-k} \end{bmatrix} = \sum_{j=0}^{n} 5^{-j}\binom{n}{j}\frac{5^j}{F_k}\begin{bmatrix} F_{2j+k} & F_{2j} \\ F_{2j} & F_{2j-k} \end{bmatrix},$$

hence,

$$5^{\frac{n}{2}}\begin{bmatrix} F_{n+k} & F_n \\ F_n & F_{n-k} \end{bmatrix} = \begin{bmatrix} \sum\limits_{j=0}^{n}\binom{n}{j}F_{2j+k} & \sum\limits_{j=0}^{n}\binom{n}{j}F_{2j} \\ \sum\limits_{j=0}^{n}\binom{n}{j}F_{2j} & \sum\limits_{j=0}^{n}\binom{n}{j}F_{2j-k} \end{bmatrix},$$

therefore,

$$F_{n\mp k} = 5^{-\frac{n}{2}}\sum_{j=0}^{n}\binom{n}{j}F_{2j\mp k}.$$

Other equations are obtained in a similar way. □

Lemma 6. *For positive integers k, n, s the following equality holds*

$$M_k^{2n+s} = 5^n\sum_{j=0}^{n}(-1)^{n+j}\binom{n}{j}M_k^{s+j}.$$

Proof. From equation (7), we can write

$$(5M_k - 5I)^n M_k^s = M_k^{2n+s} \tag{9}$$

from which the proof can be obtained by using the binomial expansion. □

Theorem 6. *For positive integers n and s the following equality holds*

$$L_{2n+s} = \sum_{j=0}^{n}\binom{n}{j}\begin{cases} (-1)^{n+1}5^{\frac{j+1}{2}}F_{s+j}, & \textit{if } j \textit{ is odd,} \\ (-1)^n 5^{\frac{j}{2}}L_{s+j}, & \textit{if } j \textit{ is even,} \end{cases},$$

$$F_{2n+s} = \sum_{j=0}^{n}\binom{n}{j}\begin{cases} (-1)^{n+1}5^{\frac{j-1}{2}}L_{s+j}, & \textit{if } j \textit{ is odd,} \\ (-1)^n 5^{\frac{j}{2}}F_{s+j}, & \textit{if } j \textit{ is even.} \end{cases}.$$

Proof. It can be easily seen with Lemma 6 and Theorem 2. □

Theorem 7. *For positive integers k and n the followings holds*

$$L_{n\pm k}=\begin{cases} \sum_{j=0}^{n}\binom{n}{j}\begin{cases}-5^{\frac{n-j}{2}}L_{j\mp k}, & \text{if } j \text{ is odd,}\\ 5^{\frac{n-j+1}{2}}F_{j\mp k}, & \text{if } j \text{ is even,}\end{cases} & \text{if } k \text{ is odd and } n \text{ is odd,}\\ \sum_{j=0}^{n}\binom{n}{j}\begin{cases}5^{\frac{n-j}{2}}L_{j\mp k}, & \text{if } j \text{ is odd,}\\ -5^{\frac{n-j+1}{2}}F_{j\mp k}, & \text{if } j \text{ is even,}\end{cases} & \text{if } k \text{ is even and } n \text{ is odd,}\\ \sum_{j=0}^{n}\binom{n}{j}\begin{cases}5^{\frac{n-j+1}{2}}F_{j\mp k}, & \text{if } j \text{ is odd,}\\ -5^{\frac{n-j}{2}}L_{j\mp k}, & \text{if } j \text{ is even,}\end{cases} & \text{if } k \text{ is odd and } n \text{ is even,}\\ \sum_{j=0}^{n}\binom{n}{j}\begin{cases}-5^{\frac{n-j+1}{2}}F_{j\mp k}, & \text{if } j \text{ is odd,}\\ 5^{\frac{n-j}{2}}L_{j\mp k}, & \text{if } j \text{ is even,}\end{cases} & \text{if } k \text{ is even and } n \text{ is even,}\end{cases}$$

$$F_{n\pm k}=\begin{cases} \sum_{j=0}^{n}\binom{n}{j}\begin{cases}5^{\frac{n-j}{2}}F_{j\mp k}, & \text{if } j \text{ is odd,}\\ -5^{\frac{n-j-1}{2}}L_{j\mp k}, & \text{if } j \text{ is even,}\end{cases} & \text{if } k \text{ is odd and } n \text{ is odd,}\\ \sum_{j=0}^{n}\binom{n}{j}\begin{cases}-5^{\frac{n-j}{2}}F_{j\mp k}, & \text{if } j \text{ is odd,}\\ 5^{\frac{n-j-1}{2}}L_{j\mp k}, & \text{if } j \text{ is even,}\end{cases} & \text{if } k \text{ is even and } n \text{ is odd,}\\ \sum_{j=0}^{n}\binom{n}{j}\begin{cases}-5^{\frac{n-j-1}{2}}L_{j\mp k}, & \text{if } j \text{ is odd,}\\ 5^{\frac{n-j}{2}}F_{j\mp k}, & \text{if } j \text{ is even,}\end{cases} & \text{if } k \text{ is odd and } n \text{ is even,}\\ \sum_{j=0}^{n}\binom{n}{j}\begin{cases}5^{\frac{n-j-1}{2}}L_{j\mp k}, & \text{if } j \text{ is odd,}\\ -5^{\frac{n-j}{2}}F_{j\mp k}, & \text{if } j \text{ is even,}\end{cases} & \text{if } k \text{ is even and } n \text{ is even.}\end{cases}$$

Proof. Using equation (8) we can write $(I-\frac{1}{5}M_k)^n=(M_k^n)^{-1}$. Here,

$$(I-\frac{1}{5}M_k)^n=\sum_{j=0}^{n}\binom{n}{j}(-1)^j\,\frac{1}{5^j}M_k^j=(M_k^n)^{-1}.$$

Let n,k be odd positive integers.

$$\sum_{\substack{j=0\\ j\text{ odd}}}^{n}\binom{n}{j}(-1)^j\,\frac{1}{5^j}\frac{5^{\frac{j-1}{2}}}{F_k}\begin{bmatrix} L_{j+k} & L_j\\ L_j & L_{j-k}\end{bmatrix}+\sum_{\substack{j=0\\ j\text{ even}}}^{n}\binom{n}{j}(-1)^j\,\frac{1}{5^j}\frac{5^{\frac{j}{2}}}{F_k}\begin{bmatrix} F_{j+k} & F_j\\ F_j & F_{j-k}\end{bmatrix}$$

$$=\frac{5^{\frac{-(n+1)}{2}}}{F_k}\begin{bmatrix} L_{n-k} & -L_n\\ -L_n & L_{n+k}\end{bmatrix},$$

hence,

$$\begin{bmatrix} L_{n-k} & -L_n\\ -L_n & L_{n+k}\end{bmatrix}=\sum_{j=0}^{n}\binom{n}{j}(-1)^j\begin{cases}5^{\frac{n-j}{2}}\begin{bmatrix} L_{j+k} & L_j\\ L_j & L_{j-k}\end{bmatrix}, & \text{if } j \text{ is odd,}\\ 5^{\frac{n-j+1}{2}}\begin{bmatrix} F_{j+k} & F_j\\ F_j & F_{j-k}\end{bmatrix}, & \text{if } j \text{ is even,}\end{cases},$$

from which the following result is obtained

$$\begin{aligned} L_{n-k} &= \sum_{j=0}^{n} \binom{n}{j} \begin{cases} -5^{\frac{n-j}{2}} L_{j+k}, & \textit{if } j \textit{ is odd,} \\ 5^{\frac{n-j+1}{2}} F_{j+k}, & \textit{if } j \textit{ is even,} \end{cases} \\ L_{n+k} &= \sum_{j=0}^{n} \binom{n}{j} \begin{cases} -5^{\frac{n-j}{2}} L_{j-k}, & \textit{if } j \textit{ is odd,} \\ 5^{\frac{n-j+1}{2}} F_{j-k}, & \textit{if } j \textit{ is even.} \end{cases} \end{aligned}$$

Other equations are obtained in a similar way. □

Theorem 8. *Let h, k and n be positive integers and*

$$\theta(n) := 5^{\frac{n+1}{2}} F_n x - 1, \quad \vartheta(n) := 5^{\frac{n}{2}} L_n x - 1.$$

i) If n is odd, then

$$\begin{aligned} \sum_{j=0}^{h} x^j M_k^{nj} &= \frac{\theta(n)I - xM_k^n}{5^n x^2 - \theta(n)} \left(x^{h+1} M_k^{n(h+1)} - I \right) \\ &= -\frac{x^{h+2} M_k^{n(h+2)} - xM_k^n - \theta(n) \left(xM_k^n\right)^{h+1} + \theta(n)I}{5^n x^2 - \theta(n)}. \end{aligned}$$

ii) If n is even, then

$$\begin{aligned} \sum_{j=0}^{h} x^j M_k^{nj} &= \frac{\vartheta(n)I - xM_k^n}{5^n x^2 - \vartheta(n)} \left(x^{h+1} M_k^{n(h+1)} - I \right) \\ &= -\frac{x^{h+2} M_k^{n(h+2)} - xM_k^n - \vartheta(n) \left(xM_k^n\right)^{h+1} + \vartheta(n)I}{5^n x^2 - \vartheta(n)}. \end{aligned}$$

Proof.

$$(xA^n - I) \sum_{j=0}^{h} x^j A^{nj} = x^{h+1} A^{n(h+1)} - I, \tag{10}$$

is valid for every square matrix A. Using equation (10) and Lemma 4, i) and ii) can easily be shown. □

Theorem 9. *Let n and s be arbitrary integers where $x \neq \dfrac{1}{\alpha^n}$ and $x \neq \dfrac{1}{\beta^n}$, the following equations are satisfied:*

i)

$$\sum_{j=0}^{h} x^j F_{nj+s} = \frac{(-1)^{n-1} x^{h+2} F_{nh+s} + x^{h+1} F_{n(h+1)+s} - (-1)^s x F_{n-s} - F_s}{(-1)^{n-1} x^2 + L_n x - 1},$$

ii)

$$\sum_{j=0}^{h} x^j L_{nj+s} = \frac{(-1)^{n-1}x^{h+2}L_{nh+s} + x^{h+1}L_{n(h+1)+s} + (-1)^s x L_{n-s} - L_s}{(-1)^{n-1}x^2 + L_n x - 1}.$$

Proof. The equation i) can be obtained by using the Lemma 4 and Theorem 2. By substitute $s \pm 1$ for s in equation i) we obtained ii). □

4. Relationships with Some Infinite Series

In this section, we consider a method using functions of the matrix xM_k^n to find sums of infinite series containing F_n and L_n. Under certain restrictions, some sum formulas can be computed using the results given in Section 3.

Theorem 10. *If*

$$-\frac{1}{\alpha^n} < x < \frac{1}{\alpha^n}$$

then,

$$\begin{aligned} \sum_{j=0}^{\infty} x^j F_{nj+s} &= \frac{(-1)^{s-1}xF_{n-s} - F_s}{(-1)^{n-1}x^2 + L_n x - 1}, \\ \sum_{j=0}^{\infty} x^j L_{nj+s} &= \frac{(-1)^{s}xL_{n-s} - L_s}{(-1)^{n-1}x^2 + L_n x - 1}. \end{aligned}$$

Proof. If the limits of i) and ii) in Theorem 9 are taken on both sides as h goes to infinity, we get the equations. □

4.1. Calculation of Certain Functions of xM_k^n.

In [7] and [13] we see that the authors obtain some identity with the matrix functions. Similarly, we can examine some series of Fibonacci and Lucas sequences using the xM_k^n matrices.

Theorem 11. *For positive numbers k, n the following equality holds*

$$Y = \exp(xM_k^n) = \sum_{j=0}^{\infty} \frac{x^j M_k^{jn}}{j!}.$$

Proof. If we take $A = xM_k^n$ in the equation given in [7, p. 113] for the exponential function of a matrix A, we get the result. □

Theorem 12. *For positive integers k and n the following identities holds*

$$\sum_{j=0}^{\infty} \frac{x^j L_{jn+k}}{j!} = \alpha^k \exp(x\alpha^n) + \beta^k \exp(x\beta^n),$$

$$\sum_{j=0}^{\infty} \frac{x^j L_{jn}}{j!} = \exp(x\alpha^n) + \exp(x\beta^n),$$

$$\sum_{j=0}^{\infty} \frac{x^j L_{jn-k}}{j!} = (-1)^k[\alpha^k \exp(x\alpha^n) + \beta^k \exp(x\beta^n)],$$

$$\sum_{j=0}^{\infty} \frac{x^j F_{jn+k}}{j!} = \frac{\alpha^k \exp(x\alpha^n) - \beta^k \exp(x\beta^n)}{\alpha - \beta},$$

$$\sum_{j=0}^{\infty} \frac{x^j F_{jn}}{j!} = \frac{\exp(x\alpha^n) - \exp(x\beta^n)}{\alpha - \beta},$$

$$\sum_{j=0}^{\infty} \frac{x^j F_{jn-k}}{j!} = (-1)^{k-1} \left[\frac{\alpha^k \exp(x\beta^n) - \beta^k \exp(x\alpha^n)}{\alpha - \beta}\right].$$

Proof. When f is an exponential function, if we replace Y in Theorem 3 by its equivalent given in Theorem 11, we obtain these identities from the matrix equation. □

The technique presented above allows us to consider a very large number of infinite series involving F_n and L_n by considering power series expansions ([1], [7], [21]) of other functions of the matrix xM_n^k. Finally, let us examine the expansion of $\tan^{-1} y$.

Theorem 13. *Under the constraint*

$$-\frac{1}{\alpha^n} \leq x \leq \frac{1}{\alpha^n}$$

we have

$$\sum_{j=1}^{\infty} \frac{(-1)^{j+1} x^{2j-1} L_{n(2j-1)+s}}{2j-1} = \alpha^s \tan^{-1}(x\alpha^n) + \beta^s \tan^{-1}(x\beta^n).$$

5. Conclusion

In this work, many identities for Fibonacci and Lucas sequences have been obtained. Although some of these are identities that can be obtained more simply in different ways, they are not found in the literature. What we really want to do here is to show how productive the Cholesky decomposition method is.

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