
	<b>SAKARYA ÜNİVERSİTESİ FEN BİLİMLERİ ENSTİTÜSÜ DERGİSİ</b> <i>SAKARYA UNIVERSITY JOURNAL OF SCIENCE</i>		
	e-ISSN: 2147-835X Dergi sayfası: <a href="http://dergipark.gov.tr/saufenbilder">http://dergipark.gov.tr/saufenbilder</a>		
	<u>Geliş/Received</u> 09-03-2017 <u>Kabul/Accepted</u> 01-08-2017	<u>Doi</u> 10.16984/saufenbilder.297047	

## The de Groot Dual of time scales

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### ABSTRACT

In this paper, we investigate the de Groot dual topology of time scales. The de Groot dual topology is related to the concept of potential infinity instead of actual infinity. Whenever the real number line  $\mathbb{R}$  denotes time then its dual space  $\mathbb{R}^*$  is compact and this provides insight that time is unbounded but finite in the sense of compact. On the other hand time scales are arbitrary non-empty closed subsets of  $\mathbb{R}$  (not only the real intervals or discrete sets) and include the real numbers.  $\mathbb{R}^*$  has the usual topology on every bounded time scales but its topological structure differs when time scales are unbounded. Therefore, we state the topological properties of a time scale with respect the de Groot dual topology and determine the connectedness conditions of it. Moreover, we illustrate our results with known examples of discrete and continuous time scales.

**Keywords:** Time-scale, de Groot Dual Topology, Topological Properties

## Zaman skalalarının de Groot Duali

### ÖZ

Bu çalışmada, zaman skalasının de Groot dual topolojisini inceledik. De Groot dual topolojisi fiili sonsuzluk yerine potansiyel sonsuzluk ile ilgilidir.  $\mathbb{R}$  reel sayı doğrusu zamanı göstermek üzere onun de Groot duali olan  $\mathbb{R}^*$  kompakttır ve zamanın sınırsız ancak kompaktlık açısından sonlu olduğu fikrini verir. Diğer taraftan zaman skalaları da sadece reel aralıklar veya ayrık kümeleri değil  $\mathbb{R}$ 'nin tüm kapalı alt kümeleridir ve reel sayıları da içermektedir.  $\mathbb{R}^*$  tüm sınırlı zaman skalaları üzerinde alışılmış topolojiye sahip olur fakat zaman skalaları sınırsız iken topolojik yapısı farklılaşır. Bu nedenle zaman skalasının de Groot dual topolojisine göre topolojik özelliklerini inceledik ve bağlantılılık koşullarını belirledik. Ayrıca sonuçlarımızı bilinen ayrık ve sürekli zaman skalaları ile örnekledik.

**Anahtar Kelimeler:** Zaman Skalası, de Groot Dual Topolojisi, Topolojik Özellikler

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## 1. INTRODUCTION

Let  $(X, \tau)$  be a topological space. The topology generated by the family of all compact saturated sets of  $X$  taken as the closed base is called de Groot dual (or co-compact) topology and denoted by  $\tau^d$ . A set  $A \subset X$  is said to be saturated if it is the intersection of open sets. In  $T_1$ -spaces every set is saturated. Hence the dual operator  $d$  coincides with the well-known compactness operator  $\rho$  of de Groot et al [1] and the dual operator  $d$  is known as de Groot dual. The idea of the dual topology appeared in [2] and this notion is treated systematically by [3-9].

It is known that the de Groot dual topology  $\tau^d$  is weaker than the original topology  $\tau$  and also it is compact, superconnected,  $T_1$  and non-Hausdorff [2]. Inspiring from this point an alternative topology for Minkowski space-time is proposed and it is mentioned that the de Groot dual topology may be regarded as more fundamental for the natural world than the Euclidean topology of Minkowski space-time in [6]

On the other hand, the theory of time scales, which goes back to its founder S. Hilger [10], is an area of mathematics that has recently received a lot of attention. Since then remarkable number of papers appeared in the theory [11, 12]. The time scale and its applications to dynamic equations are introduced by M. Bohner and A. Peterson in [13, 14]. Also, this theory can be found in [15, 16, 17] which summarize the basic notions. In the way of time scales, the results related to the set of real numbers or to the set of integers are revealed. The

Fell topology on the space of time scale for dynamic equations and the convergence of time scales under the Fell topology are studied in [18] and [19], respectively.

[8] has inspired the present study to execute the topological properties of a time scale with respect the de Groot dual

## 2. PRELIMINARIES

For the convenience of the reader, some common definitions and notations of the time scales calculus are given in this section.

**Definition 1.** *A time scale is an arbitrary nonempty closed subset  $\mathbb{T}$  of the real numbers  $\mathbb{R}$ . The set  $\mathbb{T}$  inherits the standard topology of  $\mathbb{R}$  [10].*

Examples of time scales include the real numbers  $\mathbb{R}$ , the natural numbers  $\mathbb{N}$ , the integers  $\mathbb{Z}$ , the Cantor set, and any finite union of closed intervals of  $\mathbb{R}$ .

**Definition 2.** *Let  $\mathbb{T}$  be a time scale. For any  $t \in \mathbb{T}$  the mappings  $\parallel$*

$\sigma, \rho: \mathbb{T} \rightarrow \mathbb{T}$ , such that

$$\sigma(t) = \inf \{s \in \mathbb{T} : s > t\} \text{ and}$$

$$\rho(t) = \sup \{s \in \mathbb{T} : s < t\}$$

*are called jump operators. In the case  $\mathbb{T}$  is bounded above, we denote the definition by  $\sigma(\max \mathbb{T}) := \max \mathbb{T}$  and hence  $\rho(\min \mathbb{T}) := \min \mathbb{T}$  if  $\mathbb{T}$  is bounded below, [15], [17].*

If  $\sigma(t) > t$ , we say that  $t$  is right-scattered, while if  $\rho(t) < t$  we say that  $t$  is left-scattered [15, 17]. Points that are both right-scattered and left-scattered are isolated. Also if  $t < \sup \mathbb{T}$  and  $\sigma(t) = t$ , then  $t$  is called right-dense, if  $t > \inf \mathbb{T}$  and  $\rho(t) = t$  then  $t$  is called left-dense. Points that are both right-dense and left-dense at the same time are called dense [15].

Finally, if  $f : \mathbb{T} \rightarrow \mathbb{R}$  is a function, then we define the function  $f^\sigma : \mathbb{T} \rightarrow \mathbb{R}$

$$f^\sigma(t) = f(\sigma(t))$$

for all  $t \in \mathbb{T}$  [15].

**Definition 3.** The mapping  $\mu : \mathbb{T} \rightarrow [0, \infty)$  such that  $\mu(t) := \sigma(t) - t$  is called graininess function [15].

**Example 1.** Let us consider the two examples  $\mathbb{T} = \mathbb{R}$  and  $\mathbb{T} = \mathbb{Z}$ .

i. If  $\mathbb{T} = \mathbb{R}$ , then we have for any  $t \in \mathbb{R}$

$$\sigma(t) = \inf \{s \in \mathbb{R} : s > t\} = \inf (t, \infty) = t$$

and similarly  $\rho(t) = t$ . Hence every point  $t \in \mathbb{R}$  is dense. The graininess function  $\mu$  is found as  $\mu(t) = 0$  for all  $t \in \mathbb{T}$ .

ii. If  $\mathbb{T} = \mathbb{Z}$ , then we have for any  $t \in \mathbb{Z}$

$$\sigma(t) = \inf \{s \in \mathbb{Z} : s > t\} = \inf \{t+1, t+2, \dots\} = t+1$$

and similarly  $\rho(t) = t-1$ . Hence every point  $t \in \mathbb{Z}$  is isolated. The graininess function  $\mu$  is concluded to be  $\mu(t) = 1$  for all  $t \in \mathbb{T}$  [15].

### 3. THE DE GROOT DUAL OF TIME SCALES

Let  $\mathfrak{F}$  be a closed topology base of the usual topological space of the real numbers  $\mathbb{R}$  and  $\mathfrak{F}^d$  be the collection of all compact sets in  $\mathbb{R}$ . Hence  $\mathfrak{F}^d$  becomes a closed topology base of the de Groot dual topological space on  $\mathbb{R}$  which will be denoted by  $\mathbb{R}^*$ . It is well known that  $\mathbb{R}$  is a non-compact,  $T_2$  - space. However,  $\mathbb{R}^*$  is compact, non-Hausdorff,  $T_1$  - space [2].

Let us denote the hyperspace of closed subsets by  $CL(\mathbb{R}) = \{\mathbb{T} \subset \mathbb{R} \mid \mathbb{T} \neq \emptyset \text{ and } \mathbb{T} \text{ is closed in } \mathbb{R}\}$ .

Any subset  $\mathbb{T} \in CL(\mathbb{R})$  is a time scale (or measure chain) since the concept of time scale is introduced as arbitrary non-empty closed subset of  $\mathbb{R}$  [18, 19]. Then each time scale is compact in  $\mathbb{R}^*$ . A time scale needs not to be always finite in  $\mathbb{R}$ , however it is finite in the sense of compact in  $\mathbb{R}^*$ .  $\mathbb{R}^*$  has the usual topology of  $\mathbb{R}$  on every bounded time scale. On the other hand, topological properties of an unbounded time scale differ with respect to the de Groot dual topology.

**Theorem 1.** Let  $\mathbb{T}$  be a time scale. Then

$$cl_* \mathbb{T} = \begin{cases} \mathbb{T}, & \text{if } \sigma(\max \mathbb{T}) = \max \mathbb{T} \text{ and } \rho(\min \mathbb{T}) = \min \mathbb{T} \text{ in } \mathbb{R} \\ \mathbb{R}^*, & \text{if } \sigma(\max \mathbb{T}) \neq \max \mathbb{T} \text{ or } \rho(\min \mathbb{T}) \neq \min \mathbb{T} \text{ in } \mathbb{R} \end{cases}$$

where  $cl_* \mathbb{T}$  denotes closure of  $\mathbb{T}$  in  $\mathbb{R}^*$  and  $\sigma, \rho$  are jump operators.

*Proof.* If  $\sigma(\max \mathbb{T}) = \max \mathbb{T}$  and  $\rho(\min \mathbb{T}) = \min \mathbb{T}$ , then  $\mathbb{T}$  is bounded i.e., compact in  $\mathbb{R}$ . Therefore,  $\mathbb{T}$  is closed in  $\mathbb{R}^*$ . Hence  $\text{cl}_* \mathbb{T} = \mathbb{T}$ . Suppose that  $\text{cl}_* \mathbb{T} \neq \mathbb{R}^*$  when  $\sigma(\max \mathbb{T}) \neq \max \mathbb{T}$  or  $\rho(\min \mathbb{T}) \neq \min \mathbb{T}$ . Then  $\mathbb{T}$  is not dense in  $\mathbb{R}^*$  and there is a non-empty open set  $U$  in  $\mathbb{R}^*$  such that  $\mathbb{T} \cap U = \emptyset$ . Since  $\mathbb{R} \setminus U$  is a closed proper subset of  $\mathbb{R}^*$ , it is compact in  $\mathbb{R}$ . This means that  $\mathbb{R} \setminus U$  is bounded in  $\mathbb{R}^*$ . Thereby this contradicts with  $\mathbb{T}$  is unbounded and  $\mathbb{T} \subset \mathbb{R} \setminus U$ .

**Corollary 1.** *If a time scale unbounded in  $\mathbb{R}$ , then it is dense with respect to the de Groot dual topology.*

**Example 2.**  $\mathbb{T} = \mathbb{N}$  is nowhere dense in  $\mathbb{R}$  but it is dense in  $\mathbb{R}^*$  since it is unbounded. Also this implies that  $\mathbb{R}^*$  is separable.

Cantor set is an another example of time scale which is nowhere dense in  $\mathbb{R}$  and  $\mathbb{R}^*$ , since it is bounded in  $\mathbb{R}$ .

**Theorem 2.** *Let  $\mathbb{T}$  be a time scale. Then*

$$\text{int}_* \mathbb{T} = \begin{cases} \text{int } \mathbb{T}, & \text{if } \sigma(\max(\mathbb{S})) = \max(\mathbb{S}) \text{ and } \rho(\min(\mathbb{S})) = \min(\mathbb{S}) \\ \emptyset, & \text{if } \sigma(\max(\mathbb{R} \setminus \mathbb{T})) \neq \max(\mathbb{R} \setminus \mathbb{T}) \text{ or } \rho(\min(\mathbb{R} \setminus \mathbb{T})) \neq \min(\mathbb{R} \setminus \mathbb{T}) \end{cases}$$

Here,  $\text{int}_* \mathbb{T}$  denotes the interior of  $\mathbb{T}$  in  $\mathbb{R}^*$ ,

$\mathbb{S} = \text{cl}(\mathbb{R} \setminus \mathbb{T})$  and  $\sigma, \rho$  are jump operators.

*Proof.*

Assume that  $\sigma(\max(\text{cl}(\mathbb{R} \setminus \mathbb{T}))) = \max(\text{cl}(\mathbb{R} \setminus \mathbb{T}))$  and  $\rho(\min(\text{cl}(\mathbb{R} \setminus \mathbb{T}))) = \min(\text{cl}(\mathbb{R} \setminus \mathbb{T}))$ . Then

$\text{cl}(\mathbb{R} \setminus \mathbb{T})$  is bounded and it is compact in  $\mathbb{R}$  since it is also closed in  $\mathbb{R}$ . Thus it is closed in  $\mathbb{R}^*$ . Therefore  $\mathbb{R} \setminus \text{cl}(\mathbb{R} \setminus \mathbb{T}) = \text{int } \mathbb{T}$  is an open set in  $\mathbb{R}^*$  and  $\text{int}_* \mathbb{T} = \text{int } \mathbb{T}$ . On the other hand, suppose that  $\text{int}_* \mathbb{T}$  is non-empty subset of  $\mathbb{R}^*$  when  $\sigma(\max(\mathbb{R} \setminus \mathbb{T})) \neq \max(\mathbb{R} \setminus \mathbb{T})$  or  $\rho(\min(\mathbb{R} \setminus \mathbb{T})) \neq \min(\mathbb{R} \setminus \mathbb{T})$  which means that  $\mathbb{R} \setminus \mathbb{T}$  is unbounded. Then  $\mathbb{R} \setminus \text{int}_* \mathbb{T} \neq \mathbb{R}$  shows that  $\mathbb{R} \setminus \text{int}_* \mathbb{T}$  is a proper closed subset of  $\mathbb{R}^*$ . Hence  $\mathbb{R} \setminus \text{int}_* \mathbb{T}$  is compact in  $\mathbb{R}$  i.e., it is bounded in  $\mathbb{R}$ . Besides it is known  $\mathbb{R} \setminus \mathbb{T} \subset \mathbb{R} \setminus \text{int}_* \mathbb{T}$ .  $\mathbb{R} \setminus \text{int}_* \mathbb{T}$  is bounded however  $\mathbb{R} \setminus \mathbb{T}$  is unbounded. This is a contradiction.

**Corollary 2.** *If the closure of complement of a time scale is unbounded in  $\mathbb{R}$ , it has empty interior with respect to the de Groot dual topology.*

**Example 3.**  $\mathbb{T} = [0, \infty)$  is a time scale. Since  $(-\infty, 0]$  is unbounded in  $\mathbb{R}$ ,  $\mathbb{T}$  has no interior in  $\mathbb{R}^*$ .

**Theorem 3.**

$$\partial_* \mathbb{T} = \begin{cases} \mathbb{T}, & \text{if } \mathbb{T} \text{ is bounded in } \mathbb{R}, \\ \text{cl}(\mathbb{R} \setminus \mathbb{T}), & \text{if } \text{cl}(\mathbb{R} \setminus \mathbb{T}) \text{ is bounded in } \mathbb{R}, \\ \mathbb{R}^*, & \text{if } \mathbb{T} \text{ and } \mathbb{R} \setminus \mathbb{T} \text{ are unbounded in } \mathbb{R}. \end{cases}$$

Here,  $\partial_* \mathbb{T}$  denotes the boundary of  $\mathbb{T}$  in  $\mathbb{R}^*$ .

*Proof.* Let time scale  $\mathbb{T}$  be bounded in  $\mathbb{R}$ . According to Theorem 1,  $\text{cl}_* \mathbb{T} = \mathbb{T}$ . Also,  $\text{cl}_*(\mathbb{R} \setminus \mathbb{T}) = \mathbb{R}^*$  since  $\mathbb{R} \setminus \mathbb{T}$  is unbounded. Therefore  $\partial_* \mathbb{T} = \text{cl}_* \mathbb{T} \cap \text{cl}_*(\mathbb{R} \setminus \mathbb{T}) = \mathbb{T}$ . Similarly, if we call  $\text{cl}(\mathbb{R} \setminus \mathbb{T}) = \mathbb{S}$ , then  $\mathbb{S}$  is a time scale

since it is closed in  $\mathbb{R}$ . Again, by virtue of Theorem 3.1 when  $\mathbb{S}$  is bounded in  $\mathbb{R}$ ,  $cl_*(\mathbb{S}) = \mathbb{S}$  and  $cl_*(\mathbb{R} \setminus \mathbb{S}) = \mathbb{R}^*$ . It is easily seen that  $\partial_*\mathbb{T} = \mathbb{S}$ . If the time scale  $\mathbb{T}$  and its complement are unbounded sets in  $\mathbb{R}$  then as a direct consequence of Theorem 1 and 2, we get  $\partial_*\mathbb{T} = cl_*\mathbb{T} \setminus int_*(\mathbb{T}) = \mathbb{R}$ .

**Corollary 3.** *If  $\partial\mathbb{T} = \partial_*\mathbb{T}$ , then the time scale  $\mathbb{T}$  is bounded in  $\mathbb{R}$  and the set  $\mathbb{R} \setminus \mathbb{T}$  is dense in  $\mathbb{R}$ .*

**Example 4.** *Cantor set is a bounded time scale and its complement is dense. Then its boundary is the same in both  $\mathbb{R}$  and  $\mathbb{R}^*$*

**Theorem 4.** *The de Groot dual topology on a time scale is finer the subspace topology induced from the de Groot dual of usual topology of real numbers to the time scale.*

*Proof.* Let  $\mathbb{T}^*$  denotes the de Groot dual space of  $\mathbb{T}$  where  $\mathbb{T}$  inherits the usual topology of  $\mathbb{R}$  and  $\hat{\mathbb{T}}$  be a set with relative topology induced by  $\mathbb{R}^*$ . If  $A$  is a closed proper subset of  $\hat{\mathbb{T}}$ , then there exists a proper closed subset  $F$  in  $\mathbb{R}^*$  such that  $A = \mathbb{T} \cap F$ . Then  $F$  is compact in  $\mathbb{R}$ . Thus  $A = \mathbb{T} \cap F$  is compact in  $\mathbb{R}$ . This implies that  $A$  is closed in  $\mathbb{T}^*$ . On the other hand if  $A$  is closed in  $\mathbb{T}^*$  then it is compact, that is, closed in  $\mathbb{T}$ . There exists a proper closed subset  $F$  in  $\mathbb{R}$  such that  $A = \mathbb{T} \cap F$  and  $F$  is compact in  $\mathbb{T}^*$ . By the way  $A = \mathbb{T} \cap F$  is closed in  $\hat{\mathbb{T}}$ .

**Example 5.** *Let us consider the time scale  $\mathbb{Z} \subset \mathbb{R}$ . It is well known that the subspace topology on  $\mathbb{Z}$*

*is the discrete topology. Then the de Groot dual space  $\mathbb{Z}^*$  is the set of integers with co-finite topology.*

**Corollary 4.** *The de Groot dual topology on a bounded time scale coincides with the subspace topology induced from the de Groot dual of usual topology of real numbers to the time scale.*

**Theorem 5.** *If  $\sigma(\max\mathbb{T}) \neq \max\mathbb{T}$  or  $\rho(\min\mathbb{T}) \neq \min\mathbb{T}$  is satisfied for a time scale  $\mathbb{T}$  in  $\mathbb{R}$ , then  $\mathbb{T}$  is connected in  $\mathbb{R}^*$ .*

*Proof.* Let  $\sigma(\max\mathbb{T}) \neq \max\mathbb{T}$  or  $\rho(\min\mathbb{T}) \neq \min\mathbb{T}$  i.e.,  $\mathbb{T}$  be unbounded time scale in  $\mathbb{R}$ . Suppose that  $\mathbb{T}$  is not connected in  $\mathbb{R}^*$ . There are closed sets  $F$  and  $K$  in  $\mathbb{R}^*$  such that  $(F \cap \mathbb{T}) \cup (K \cap \mathbb{T}) = \mathbb{T}$ ,  $(F \cap \mathbb{T}) \neq \emptyset$  and  $(K \cap \mathbb{T}) \neq \emptyset$ . Since the sets  $F$  and  $K$  are closed proper subsets of  $\mathbb{R}^*$ , they are bounded sets in  $\mathbb{R}$ . Thus the sets  $(F \cap \mathbb{T})$  and  $(K \cap \mathbb{T})$  are bounded subsets  $\mathbb{R}$  and this contradicts with unboundedness of  $\mathbb{T}$ .

On the other hand if  $\mathbb{T}$  is an interval in  $\mathbb{R}$ , then it is connected  $\mathbb{R}$  and since the usual topology of  $\mathbb{R}$  is stronger than its de Groot dual is also connected in  $\mathbb{R}^*$ .

**Corollary 5.** *If a time scale  $\mathbb{T}$  is an interval or unbounded in  $\mathbb{R}$ , then it is connected in  $\mathbb{R}^*$*

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