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Research Article

On σ-c-subnormal subgroups of finite groups

JiaHui Liu, ShouHong Qiao*

School of Mathematics and Statistics, Guangdong University of Technology, Guangzhou 510520, P. R. China

Abstract

Let $\sigma = \{\sigma_i : i \in I\}$ be a partition of the set $\mathbb{P}$ of all primes. A finite group G is called σ-primary if the prime divisors of $|G|$, if any, all belong to the same member of σ. A finite group G is called σ-soluble if every chief factor of G is σ-primary. A subgroup H of a group G is called σ-subnormal in G if there is a chain of subgroups $H = H_0 \leq H_1 \leq \cdots \leq H_n = G$ such that either H_{i-1} is normal in H_i or $H_i/(H_{i-1})_{H_i}$ is σ-primary for all $i = 1, \ldots, n$; A subgroup H of a group G is called σ-c-subnormal in G if there is a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{\sigma G}$, where the subgroup $H_{\sigma G}$ is generated by all σ-subnormal subgroups of G contained in H. In this paper, we investigate the influence of σ-c-subnormality of some kinds of maximal subgroups on σ-solubility of finite groups, which generalizes some known results.

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1. Introduction

All groups considered in this paper will be finite.

The maximal subgroups play an important role in the study of finite groups. There are many classical results on deciding the structure of groups by the embedding properties of maximal subgroups. A group G is nilpotent if and only if every maximal subgroup of G is normal in G; A group G is supersoluble if and only if every maximal subgroup of G has prime index in G. In the paper [9], Y.M. Wang introduced the definition of c-normality, and proved that: a group G is soluble if and only if every maximal subgroup of G is c-normal in G.

In the paper [7], A. N. Skiba extended the concepts of nilpotency and solubility, and introduced σ-nilpotency and σ-solubility, where σ is some partition of the set $\mathbb{P}$ of all prime numbers.

Definition 1.1 ([7])**.** A group G is called σ-primary if the prime divisors of $|G|$, if any, all belong to the same member of σ; G is said to be σ-nilpotent if it is a direct product of some σ-primary groups; G is said to be σ-soluble if every chief factor of G is σ-primary.

*Corresponding Author.
Email addresses: 2112114033@mail2.gdut.edu.cn (J.H. Liu), qshqsh513@163.com(S.H. Qiao)
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A. N. Skiba also introduced the concept of σ-subnormality of subgroups in [7], and proved a parallel result to nilpotency of finite groups: a finite group G is σ-nilpotent if and only if every maximal subgroup of G is σ-subnormal.

Definition 1.2. A subgroup H of a group G is called σ-subnormal in G if there is a chain of subgroups $H = H_0 \leq H_1 \leq \cdots \leq H_n = G$ such that either $H_{i-1} \lhd H_i$ or $H_i/(H_{i-1})_{H_i}$ is σ-primary for any $i = 1, \ldots, n$.

We know that the nilpotency, the solubility and the σ-nilpotency of groups can be characterized by the embedding properties of the maximal subgroups. In order to characterize σ-solubility, the σ-c-normality of subgroups was introduced by Ning Su, Chenchen Cao and the second author of the present paper [8], which covers the c-normality and the σ-subnormality of subgroups.

Definition 1.3. A subgroup H of a group G is said to be σ-c-normal in G if there is a normal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{\sigma G}$, where the subgroup $H_{\sigma G}$ is generated by all σ-subnormal subgroups of G contained in H.

In [8], the authors presented two results on deciding the σ-solubility of finite groups by the σ-c-normality of maximal subgroups. With the aid of the detailed information of primitive groups, the normality of N is actually not necessary in the definition of σ-c-normality. In this paper, we introduce the definition of σ-c-subnormality of subgroups which relaxes the normality of T to subnormality.

Definition 1.4. A subgroup H of a group G is said to be σ-c-subnormal in G, if there is a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{\sigma G}$, where the subgroup $H_{\sigma G}$ is generated by all σ-subnormal subgroups of G contained in H.

In this paper, we give some criteria for σ-solubility of finite groups by the σ-c-subnormality of maximal subgroups, which improves or generalizes some related results. The first two results improve the main theorems in [8], which also improves the related results in [9].

Theorem 1.5. *A group G is σ-soluble if and only if every maximal subgroup of G is σ-c-subnormal in G.*

Theorem 1.6. *A group G is σ-soluble if and only if there exists a σ-soluble maximal subgroup of G that is σ-c-subnormal in G.*

With the help of Odd Order Theorem, we could obtain some results on σ-solubility of finite groups. Our third result is about σ-c-subnormality of maximal subgroups of a Sylow 2-subgroup.

Theorem 1.7. *Suppose that every maximal subgroup of a Sylow 2-subgroup of a group G is σ-c-subnormal in G. Then G is σ-soluble.*

For a more general case, we have the following result about a complete Hall σ-set.

Theorem 1.8. *Let G be a σ-group and $\{H_1, \cdots, H_t\}$ a complete Hall σ-set of G. If each maximal subgroup of H_i is σ-c-subnormal in G, then G is σ-soluble.*

Dually, we have the following result on cyclic subgroups of a Sylow 2-subgroup.

Theorem 1.9. *Let G be a finite group of even order, $P \in Syl_2(G)$. Suppose that every cyclic subgroup of P is σ-c-subnormal in G. Then G is σ-soluble.*

Let P be a cyclic Sylow p-subgroup of G. If p is the minimal divisor of $|G|$, then G is p-nilpotent. If p is not minimal, then we could not get more structural information on G. However, if we restrict the partition σ and assume the σ-c-subnormality of some nonidentity subgroup of P, then we can obtain the following result on p-supersoluble groups.

Theorem 1.10. *Let σ be a partition of all primes such that $|\sigma_i| \le 2$ for each $\sigma_i \in \sigma$. Let G be a finite group and p a prime that divides $|G|$. Suppose that a Sylow p-subgroup P of G is cyclic and that some nonidentity p-subgroup $P_0 \le P$ is σ-c-subnormal. Then G is p-supersoluble.*

Remark: The hypothesis that $|\sigma_i| \le 2$ in Theorem 1.10 is necessary. By [4, Theorem 4.9], every simple group has a cyclic Sylow subgroup. For any non-abelian group G, we choose suitable partition σ of all primes such that G is σ-primary. Then each subgroup of G is σ-subnormal in G. However, G is not p-supersoluble for any prime p dividing $|G|$.

2. Preliminaries

In this section, we give some lemmas that will be used in our proofs.

Lemma 2.1 ([7, Lemma 2.6])**.** *Let A, K and N be subgroups of G. Suppose that A is σ-subnormal in G and N is normal in G. Then the following statements hold:*

(1) $A \cap K$ is σ-subnormal in K.
(2) If K is σ-subnormal in A, then K is σ-subnormal in G.
(3) If K is σ-subnormal in G, then $A \cap K$ and $\langle A, K\rangle$ are σ-subnormal in G.
(4) AN/N is σ-subnormal in G/N.
(5) If $N \le K$ and K/N is σ-subnormal in G/N, then K is σ-subnormal in G.
(6) If G is a σ-group and A is σ-nilpotent, then A is contained in $F_\sigma(G)$, the σ-Fitting subgroup of G.

Lemma 2.2 ([7, Lemma 2.1])**.** *The class of all σ-soluble groups is closed under taking direct products, homomorphic images and subgroups. Moreover, any extension of a σ-soluble group by another σ-soluble group is a σ-soluble group as well.*

Lemma 2.3. *Let G be a group and $K \le M \le G$. If K is σ-c-subnormal in G, then K is σ-c-subnormal in M.*

Proof. By hypothesis, there is a subnormal subgroup T of G such that $G = KT$ and $K \cap T \le K_{\sigma G}$. Then $T \cap M \lhd\lhd M$, $M = K(T \cap M)$ and $K \cap (T \cap M) = (K \cap T) \cap M \le K_{\sigma G} \cap M \le K_{\sigma M}$ by Lemma 2.1(1). Thus K is σ-c-subnormal in M. □

Lemma 2.4. *Assume that N is a normal subgroup of G and K is a subgroup of G containing N. Then K/N is σ-c-subnormal in G/N if and only if K is σ-c-subnormal in G.*

Proof. Suppose that K is σ-c-subnormal in G. By definition, there is a subnormal subgroup T of G such that $G = KT$ and $K \cap T \le K_{\sigma G}$. By Lemma 2.1(4), $G/N = (K/N)(TN/N)$ and $(K/N) \cap (TN/N) = (K \cap TN)/N = (K \cap T)N/N \le K_{\sigma G}N/N \le (K/N)_{\sigma(G/N)}$. Thus K/N is σ-c-subnormal in G/N.

Conversely, suppose that K/N is σ-c-subnormal in G/N. By definition, there is a subnormal subgroup T of G containing N such that $G/N = (K/N)(T/N)$ and $(K/N) \cap (T/N) \le (K/N)_{\sigma(G/N)}$. Set $M/N := (K/N)_{\sigma(G/N)}$. Then $M \le K$ and M is σ-subnormal in G by Lemma 2.1(5). It follows that $G = KT$ and $K \cap T \le M \le K_{\sigma G}$, that is, K is σ-c-subnormal in G. □

Lemma 2.5. *Assume that N is a normal subgroup of G and K is a σ-c-subnormal subgroup of G. If $(|N|, |K|) = 1$, then KN/N is σ-c-subnormal in G/N.*

Proof. Since K is σ-c-subnormal, there is a subnormal subgroup T of G such that $G = KT$ and $K \cap T \le K_{\sigma G}$. Since $(|N|, |K|) = 1$, it follows that $N \le T$. By Lemma 2.1(4), $G/N = (KN/N)(T/N)$ and $(K \cap T)N/N \le K_{\sigma G}N/N \le (KN/N)_{\sigma(G/N)}$. This proves that KN/N is σ-c-subnormal in G/N. □

Lemma 2.6 ([8, Lemma 2.4])**.** *Let $N = N_1 \times N_2 \times \cdots \times N_t$ be a direct product of isomorphic non-abelian simple groups and suppose that N_1 is not σ-primary. Let K be a non-trivial subgroup of N. If K is σ-subnormal in N, then K is a direct product of some N_is.*

Lemma 2.7. *Let P be a σ-c-subnormal p-subgroup of a group G, where p is a prime. If P is not σ-subnormal in G, then there is a normal subgroup L of G such that $G = PL$ and $|G : L| = p$.*

Proof. Suppose that P is not σ-subnormal in G. Since P is σ-c-subnormal in G, there is a subnormal subgroup T of G such that $G = PT$ and $P \cap T \leq P_{\sigma G}$. This implies that $|G : T|$ is a p-power. It follows that $O^p(G) \leq T$ since T is subnormal in G. Let L be a maximal subgroup of G containing T. Clearly, $L/O^p(G)$ is maximal in $G/O^p(G)$, both subgroups are p-groups. Then $|G/O^p(G) : L/O^p(G)| = p$ and $L/O^p(G) \lhd G/O^p(G)$. Thus $|G : L| = p$ and $L \lhd G$. Also, we have $G = PT = PL$. □

Lemma 2.8 ([3, Chapter A, Lemma 14.3])**.** *If U is a subnormal subgroup of a finite group G, then the socle $Soc(G) \leq N_G(U)$.*

Lemma 2.9. *Suppose that every maximal subgroup of a group G has a subnormal complement in G. Then G is nilpotent.*

Proof. First, we prove that G is soluble. Clearly, G is not simple. Let N be a minimal normal subgroup of G. Let M/N be a maximal subgroup of G/N. Then M is maximal in G. By hypothesis, there is a subnormal subgroup T of G such that $G = MT$ and $M \cap T = 1$. Clearly, TN/N is subnormal in G/N, and $G/N = (M/N)(TN/N)$. By modular law, $M/N \cap TN/N = (M \cap T)N/N = 1$. Thus G/N satisfies the hypotheses of the lemma. By induction, G/N is soluble. Further, we may assume that N is non-abelian and the unique minimal normal subgroup of G.

Let P be a non-trivial Sylow subgroup of N. By Frattini's argument, $G = N_G(P)N$. Clearly, $N_G(P)$ is a proper subgroup of G since N is the unique minimal normal and is non-abelian. Then there exists a maximal subgroup M of G such that $N_G(P) \leq M$. It follows that $G = N_G(P)N = MN$ and $P \leq M \cap N$. By hypothesis, there is a subnormal subgroup T of G such that $G = MT$ and $M \cap T = 1$. This also implies that $N \cap T = 1$. By Lemma 2.8, $N \leq N_G(T)$. Thus $T \leq C_G(N) \leq N$ since N is unique, a contradiction. This contradiction shows that G is soluble.

Below we show that G is nilpotent. Let N be a minimal normal subgroup of G. Since G/N satisfies the hypotheses of the lemma, it follows that G/N is nilpotent. Suppose that $C_G(N) < G$. Let M be a maximal subgroup of G containing $C_G(N)$. By hypothesis, M has a subnormal complement T. Since $N \cap T \leq M \cap T = 1$, it follows that $T \leq C_G(N) \leq M$ by Lemma 2.8, a contradiction. This contradiction shows that $C_G(N) = G$, that is, $N \leq Z(G)$. This means that G is nilpotent, as desired. □

Lemma 2.10 (see [5, page 2032, KOROLLAR])**.** *Let G be a group satisfying that G has a unique non-abelian minimal normal subgroup N and a maximal subgroup M such that $M \cap N = 1$. Let S be a simple factor of N. Then S is isomorphic to a section of M.*

Lemma 2.11 ([6, Theorem 2.1])**.** *Suppose $K \lhd G$ with p a prime divisor of the orders of both K and G/K. If a Sylow p-subgroup of G is cyclic, then G is p-soluble.*

Lemma 2.12 ([1, Theorem 1 and Proposition 1, 335-342])**.** *Let $\mathfrak{F}$ be a saturated formation.*

(1) *Assume that G is a group such that G does not belong to $\mathfrak{F}$, but all its proper subgroups belong to $\mathfrak{F}$. Then $\frac{G^{\mathfrak{F}}\Phi(G)}{\Phi(G)}$ is the unique minimal normal subgroup of $\frac{G}{\Phi(G)}$. In addition, if the derived subgroup $(G^{\mathfrak{F}})'$ is a proper subgroup of $G^{\mathfrak{F}}$, then $G^{\mathfrak{F}}$ is a soluble group. Furthermore, if $G^{\mathfrak{F}}$ is soluble, then $G^{\mathfrak{F}}\Phi(G) = F(G)$, the Fitting subgroup of G, and $(G^{\mathfrak{F}})' = T \cap G^{\mathfrak{F}}$ for every $\mathfrak{F}$-critical maximal subgroup T of G.*

(2) *Assume that G is a group such that $G \notin \mathfrak{F}$ and there exists a maximal subgroup M of G such that $M \in \mathfrak{F}$ and $G = MF(G)$. Then $\frac{G^{\mathfrak{F}}}{(G^{\mathfrak{F}})'}$ is a chief factor of G, $G^{\mathfrak{F}}$ is a p-group for some prime p, $G^{\mathfrak{F}}$ has the exponent p if $p > 2$ and the exponent at most 4 if $p = 2$. Moreover, either $G^{\mathfrak{F}}$ is elementary abelian or $(G^{\mathfrak{F}})' = Z(G^{\mathfrak{F}}) = \Phi(G^{\mathfrak{F}})$ is an elementary abelian group.*

3. Proofs

In this section, we present the proofs of our theorems that are listed in the Introduction.

Proof of Theorem 1.5. By [8, Theorem A], we just need to prove the sufficiency of the theorem. Suppose that every maximal subgroup of G is σ-c-subnormal in G. We use induction on $|G|$ to show that G is σ-soluble.

If G is simple, then every maximal subgroup of G is σ-subnormal in G. It then follows from [7, Proposition 2.3] that G is σ-nilpotent, and hence σ-soluble. Therefore we may assume that G is not simple.

Let N be a minimal normal subgroup of G. Then N is a proper subgroup of G since G is not simple. By Lemma 2.1(4), G/N satisfies the hypothesis of the theorem, and thus G/N is σ-soluble by induction. Moreover, we may assume that N is the unique minimal normal subgroup of G and not σ-primary (otherwise we would already have that G is σ-soluble by Lemma 2.2). In particular, N is not soluble.

Let P be a non-trivial Sylow subgroup of N. By Frattini's argument, $G = N_G(P)N$. Clearly, $N_G(P)$ is a proper subgroup of G since N is the unique minimal normal subgroup and non-abelian. Then there exists a maximal subgroup M of G such that $N_G(P) \leq M$. It follows that $G = N_G(P)N = MN$ and $P \leq M \cap N$. By hypothesis, there is a subnormal subgroup K of G such that $G = MK$ and $M \cap K \leq M_{\sigma G}$.

By Lemma 2.8, $N \leq N_G(K)$. If $N \cap K = 1$, then $K \leq C_G(N) \leq N$ since N is unique, a contradiction. Thus $N \cap K \neq 1$. Write $N = N_1 \times N_2 \times \cdots \times N_t$, where N_is are isomorphic non-abelian simple groups. Without loss of generality, we suppose that $N \cap K = N_1 \times N_2 \times \cdots \times N_l$ for some integer $1 \leq l < t$. Since $1 \neq P \cap N_i \in Syl_p(N_i)$, it follows that $M \cap N_i \geq P \cap N_i > 1$, where $1 \leq i \leq l$. Since $G = MN$, it follows that M acts by conjugation on $\{N_1, N_2, \cdots, N_t\}$ transitively. Clearly, $M \cap (N_1 \times N_2 \times \cdots \times N_l) \leq M \cap K \leq M_{\sigma G}$. Thus $1 \neq M \cap (N_1 \times N_2 \times \cdots \times N_l) = (N_1 \times N_2 \times \cdots \times N_l) \cap M_{\sigma G}$ which is σ-subnormal in $N_1 \times N_2 \times \cdots \times N_l$ by Lemma 2.1(1). By Lemma 2.6, we may suppose that $N_1 \leq M$. Since M acts by conjugation on $\{N_1, N_2, \cdots, N_t\}$ transitively, it follows that $N \leq M$, a contradiction.

This finishes the proof. □

Proof of Theorem 1.6. The necessity part follows directly from Lemma 2.2 and Theorem 1.5. We now prove the sufficiency of the theorem.

Let M be a σ-soluble maximal subgroup of G which is σ-c-subnormal in G. We use induction on $|G|$ to show that G is σ-soluble. Clearly, G/M_G satisfies the hypothesis of the theorem. If $M_G \neq 1$, then G/M_G is σ-soluble by induction. Since M is σ-soluble, it then follows from Lemma 2.2 that G is σ-soluble. Hence we may assume that $M_G = 1$.

Since M is σ-c-subnormal in G, there is a subnormal subgroup K of G such that $G = MK$ and $M \cap K \leq M_{\sigma G}$. Assume that $K = G$. Then $M = M_{\sigma G}$ is σ-subnormal in G, thus either $M \lhd G$ (in this case G/M has prime order) or $G \cong G/M_G$ is a σ-primary group. In either case, we have G is σ-soluble by Lemma 2.2. Therefore we may assume that $K < G$.

Let N be a minimal normal subgroup of G. Since $M_G = 1$, we have $G = MN$. This yields that G/N is σ-soluble, and so N is unique.

By Lemma 2.8, $N \leq N_G(K)$. If $N \cap K = 1$, then $K \leq C_G(N) \leq N$ since N is unique, a contradiction. Thus $N \cap K \neq 1$. Write $N = N_1 \times N_2 \times \cdots \times N_t$, where N_is are isomorphic non-abelian simple groups. Without loss, we suppose that $N \cap K = N_1 \times N_2 \times \cdots \times N_l$ for some integer $1 \leq l < t$. Since $G = MN$, it follows that M acts by conjugation on $\{N_1, N_2, \cdots, N_t\}$ transitively. This leads to the fact that M acts by conjugation on $\{M \cap N_1, M \cap N_2, \cdots, M \cap N_t\}$ transitively.

Suppose that $M \cap N_1 \neq 1$. Then $1 \neq M \cap (N_1 \times N_2 \times \cdots \times N_l) = M \cap M_{\sigma G}$ which is σ-subnormal in $N_1 \times N_2 \times \cdots \times N_l$. By Lemma 2.6, we may suppose that $N_1 \leq M$. Since M acts by conjugation on $\{N_1, N_2, \cdots, N_t\}$ transitively, it follows that $N \leq M$, a contradiction. Thus we may suppose that $M \cap N_i = 1$ for any $i = 1, 2, \cdots, t$.

Consider the action of G by right multiplication on the right coset space $[G : M]$ of M in G. This action is faithful and primitive since M is maximal and core-free. Denote by α the coset M. The point stabilizers $G_\alpha = M$ and $N_\alpha = G_\alpha \cap N = M \cap N \neq 1$. Consider the projection π_i of N_α to N_i. If $\pi_i(N_\alpha) = \pi_i(N \cap M) = N_i$, then N_i is σ-soluble since M is σ-soluble. Thus we may suppose that $\pi_i(N_\alpha) = \pi_i(N \cap M) < N_i$. By [2, Theorem 4.6A] and its proof, we have that $N_\alpha = N \cap M = R_1 \times R_2 \times \cdots \times R_t$ with each $R_i < N_i$. Thus $R_i \leq M \cap N_i$, which is contrary to $M \cap N_i = 1$. This contradiction finishes the proof. □

Proof of Theorem 1.7. Let N be a normal subgroup of G. By Lemma 2.1(4), G/N satisfies the hypotheses of the theorem. By induction, G/N is σ-soluble. Thus we may suppose that N is unique and not σ-primary.

Let $P \in Syl_2(G)$ and let P_1 be a maximal subgroup of P. By hypothesis, there is a subnormal subgroup T of G such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{\sigma G}$. If $(P_1)_{\sigma G} \neq 1$, then $F_\sigma(G) \geq (P_1)_{\sigma G} > 1$. Thus $N \leq F_\sigma(G)$ since N is unique, contrary to the hypothesis that N is not σ-primary. Thus we may suppose that $(P_1)_{\sigma G} = 1$. In this case, the Sylow 2-subgroups of T have order 2. By Burnside's theorem, T is 2-nilpotent. Let T_1 be the normal 2-complement of T. Then T_1 is subnormal in G. We may suppose that $T_1 \neq 1$. By Lemma 2.8, $N \leq N_G(T_1)$. If $N \cap T_1 = 1$, then $T_1 \leq C_G(N)$, contrary to the uniqueness of N. Thus $N \cap T_1 \neq 1$. Since $N \cap T_1$ is subnormal in N, we have $N \cap T_1$ is soluble by Odd Order Theorem. Then N is soluble. This finishes the proof. □

Proof of Theorem 1.8. By Lemma 2.9, H_i is nilpotent. Let N be a normal subgroup of G. By Lemma 2.1(4), G/N satisfies the hypotheses of the lemma. By induction, G/N is σ-soluble. Thus we may suppose that N is unique and not σ-primary. Let M_i be a maximal subgroup of H_i. If $(M_i)_{\sigma G} \neq 1$ for some maximal subgroup M_i of H_i, then $F_\sigma(G) \geq (M_i)_{\sigma G} > 1$ by Lemma 2.1(6). Then $N \leq F_\sigma(G)$, contrary to the hypothesis that N is not σ-primary.

Below, we suppose that $(M_i)_{\sigma G} = 1$ for any maximal subgroup M_i of each H_i and for any $i = 1, \cdots, t$. In this case, there is a subnormal subgroup T_i of G such that $G = M_iT_i$ and $M_i \cap T_i \leq (M_i)_{\sigma G} = 1$.

If G is an odd order group, then G is soluble. Thus we may suppose that 2 divides $|G|$. Without loss of generality, we may suppose that 2 divides $|H_1|$. Let M be a maximal subgroup of H_1 of index 2. By hypothesis, there is a subnormal subgroup T of G such that $G = MT$ and $M \cap T \leq M_{\sigma G} = 1$. Then the Sylow 2-subgroups of T have order 2. By Burnside's theorem, T is 2-nilpotent. Let T_1 be the normal 2-complement of T. Then T_1 is subnormal in G. We may suppose that $T_1 \neq 1$. By Lemma 2.8, $N \leq N_G(T_1)$. If $N \cap T_1 = 1$, then $T_1 \leq C_G(N)$, contrary to the uniqueness of N. Thus $N \cap T_1 \neq 1$. Since $N \cap T_1$ is subnormal in N, we have $N \cap T_1$ is soluble by Odd Order Theorem. Then N is soluble. This finishes the proof. □

Proof of Theorem 1.9. Assume that the theorem is false and G is a counterexample with minimal order.

Step 1. G is a minimal non-σ-soluble group.

Let M be a maximal subgroup of G. Let C be a cyclic subgroup of M of order p or 4. By hypothesis, C is σ-c-subnormal in G. Thus C is σ-c-subnormal in M by Lemma

2.3. Hence M satisfies the hypotheses of the theorem. The minimal choice of G yields that M is σ-soluble. Hence we have G is not σ-soluble but every proper subgroup of G is σ-soluble.

Step 2. $G/\Phi(G)$ is a non-abelian simple group, and in particular, $G = G^{\mathfrak{S}_\sigma}$, the σ-soluble residual.

By Lemma 2.12, $G^{\mathfrak{S}_\sigma}\Phi(G)/\Phi(G)$ is the unique minimal normal subgroup of $G/\Phi(G)$. Clearly, $G^{\mathfrak{S}_\sigma}\Phi(G)/\Phi(G)$ is not σ-soluble. By Step 1,

$$G^{\mathfrak{S}_\sigma}\Phi(G)/\Phi(G) = G/\Phi(G),$$

and so $G/\Phi(G)$ is a non-abelian simple group.

Step 3. $O_{p'}(G) = 1$. Hence $F(G) = \Phi(G) = O_2(G)$.

It follows from Lemma 2.5 and Step 2.

Step 4. Each cyclic subgroup of P is σ-subnormal in G.

Let H be a cyclic subgroup of G. Suppose that H is not σ-subnormal in G. By Lemma 2.7, G has a normal subgroup of index 2, contrary to Step 2.

Step 5. The final contradiction.

If $O_2(G) = P$, then G is 2-closed, and so G is soluble. We may suppose that $O_2(G) \neq P$. Let $x \in P \setminus O_2(G)$. By Step 4, $\langle x\rangle$ is σ-subnormal in G. Then there is a subgroup chain $\langle x\rangle = H_0 \leq H_1 \leq \cdots \leq H_{n-1} \leq H_n = G$ such that either H_{i-1} is normal in H_i or $H_i/(H_{i-1})_{H_i}$ is σ-primary for any $i = 1, \ldots, n$. If $H_{n-1} \lhd H_n = G$, then $H_{n-1} = O_2(G)$, and so $x \in O_2(G)$, a contradiction. Suppose that H_{n-1} is not normal in G and $G/(H_{n-1})_G = H_n/(H_{i-1})_{H_n}$ is σ-primary. Since $G/O_2(G) = G/\Phi(G)$ is simple, we have $(H_{n-1})_G = 1$ or $(H_{n-1})_G = \Phi(G) = O_2(G)$. If $(H_{n-1})_G = 1$, then G is σ-primary, a contradiction. Suppose that $(H_{n-1})_G = O_2(G) = \Phi(G)$. Since $\pi(G) = \pi(G/\Phi(G))$, we also have that G is σ-primary, a final contradiction.

This finishes the proof of the theorem. □

Proof of Theorem 1.10. It suffices to prove that G is p-soluble since P is cyclic. If P_0 is not σ-subnormal in G, then there is a normal subgroup L of G such that $|G : L| = p$. Since p divides $|L|$ and $|G : L|$, by Lemma 2.11, G is p-soluble since P is cyclic, as desired. Thus we may suppose that P_0 is σ-subnormal in G.

Since P_0 is σ-subnormal in G, it follows that $P_0 \leq F_\sigma(G)$ by Lemma 2.1(6). If p divides $|G/F_\sigma(G)|$, by Lemma 2.11, G is p-soluble. Thus we may suppose that $F_\sigma(G) \geq P$.

Let N be a minimal normal subgroup of G contained in $F_\sigma(G)$. If N is a p'-subgroup, then G/N is p-soluble by Lemma 2.1(4) and induction, as desired. Thus we may suppose that p divides $|N|$. If p also divides $|G/N|$, then G is p-soluble by Lemma 2.11. Thus we may suppose that $P \leq N$. Since N is characteristically simple and P is cyclic, we have that N is simple. Since $N \leq F_\sigma$ is σ-primary and $|\sigma_i| \leq 2$, it follows that N is soluble. Thus $N = P$, and so G is p-soluble.

This finishes the proof. □

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