Turk. J. Math. Comput. Sci.
16(1)(2024) 154–161
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DOI : 10.47000/tjmcs.1344439

A Note on Hyper-Dual Numbers with the Leonardo-Alwyn Sequence

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Received: 16-08-2023 • Accepted: 13-03-2024

Abstract. We are interested in identifying hyper-dual numbers with the Leonardo-Alwyn sequence components. We investigate their homogeneous and non-homogeneous recurrence relations, the Binet's formula, and the generating function. With these algebraic properties, we are able to obtain some special cases of hyper-dual numbers with the Leonardo-Alwyn sequence according to p, q and c (multipliers).

2020 AMS Classification: 11B37, 11B39

Keywords: Leonardo sequence, recurrence relation, hyper-dual number.

1. Introduction

Sequences are important in number theory, a branch of mathematics. Additionally, sequences can be seen everywhere, including in our daily lives, such as the interest component of monthly payments made to pay off an item, economics, physics, cryptography, biology, engineering, and computer algorithms, in addition to mathematics. The Fibonacci sequence is the most probably famous of all sequences. For $n \geq 2$, the n-th Fibonacci number is defined by the following second order homogeneous recurrence relation

$$F_n = F_{n-1} + F_{n-2},$$

with initial conditions $F_0 = 0$ and $F_1 = 1$, [13]. Many extensions of the Fibonacci sequence are known, such as the Lucas and Leonardo sequences. The Fibonacci, Lucas, and Leonardo sequences, who are closely related to each other, are among the others the most studied sequences. For $n \geq 2$, the the n-th Lucas number is defined by the second order homogeneous recursive relation

$$L_n = L_{n-1} + L_{n-2},$$

with initial conditions $L_0 = L_1 = 1$, [13]. For $n \geq 2$, the n-th Leonardo number is defined by the following non-homogeneous recurrence relation

$$Le_n = Le_{n-1} + Le_{n-2} + 1,$$

with initial conditions $Le_0 = Le_1 = 1$ or for $n \geq 3$, the following the homogeneous recurrence relation

$$Le_{n+1} = 2Le_n - Le_{n-2},$$

with initial conditions $Le_0 = Le_1 = 1$ and $Le_2 = 3$, [1–3]. The Binet's formula of the Leonardo sequence is as follows:

$$Le_n = 2\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right) - 1,$$

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where the golden ratio $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are the roots of the quadratic polynomial $x^2 - x - 1 = 0$. Although the Leonardo sequence was recently defined, it has been discussed with different aspects. Many reserachers have subsequently developed many special Leonardo sequences. The Leonardo p-sequence, and incomplete Leonardo p-sequence are interested in [21]. The Leonardo k-numbers, and incomplete Leonardo k-numbers are studied in [14]. Y. Soykan characterized the modified p-Leonardo, p-Leonardo-Lucas, and p-Leonardo sequences as special cases of the generalized Leonardo sequence in [19].

Alwyn Francis (Horrie) generalized the Fibonacci sequence and introduced the Horadam sequence. For $n \geq 2$ and $a, b, p, q \in \mathbb{Z}$, the n-th Horadam number is satisfied the following second order homogeneous recurrence relation

$$W_n = pW_{n-1} + qW_{n-2},$$

with initial conditions $W_0 = a$, $W_1 = b$. The Binet's formula of the Horadam sequence is as follows:

$$W_n = \left(\frac{b - a\beta}{\alpha - \beta}\right)\alpha^n + \left(\frac{a\alpha - b}{\alpha - \beta}\right)\beta^n,$$

where $\alpha = \frac{p+\sqrt{p^2+4q}}{2}, \beta = \frac{p-\sqrt{p^2+4q}}{2}$, [7–9]. This sequence gives rise to some well known sequences such that Fibonacci, Lucas, generalized Fibonacci, generalized Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas, Tagiuri, Fermat, Fermat-Lucas and so on.

In the recent paper, H. Gökbaş defined a new family of the Leonardo sequence called the Leonardo-Alwyn sequence, [6]. The Leonardo-Alwyn sequence is dedicated to Leonardo Fibonacci (Leonardo of Pisa) and Alwyn Horadam. It is a generalization of the classical Leonardo sequence. Below we recall some of several important relations of the Leonardo-Alwyn sequence we need, for details see [6]. Let integers $p \geq 1$ and q, c be such that $p + q \neq 1$ and $p^2 + 4q \geq 1$. The n-th Leonardo-Alwyn number is given by

$$LA_n = pLA_{n-1} + qLA_{n-2} + c, \quad n \geq 2, \tag{1.1}$$

with initial conditions

$$LA_0 = \quad LA_1 = p = \begin{cases} W_0, & W_0 \neq 0, \\ W_1, & W_0 = 0, \end{cases}$$

where $W_0 = a$ and $W_1 = b$ are the first two Horadam numbers, $a, b \in \mathbb{Z}$ and $c = a + b$. For $n \geq 2$, the $(n + 1)$-th Leonardo-Alwyn number can be equivalently defined by

$$LA_{n+1} = (p + 1)LA_n + (q - p)LA_{n-1} - qLA_{n-2}, \tag{1.2}$$

with an additional initial condition $LA_2 = p^2 + pq + c$. For $n \geq 1$, the negative terms of the Leonardo-Alwyn sequence are given by

$$LA_{-n} = \frac{q - p}{q}LA_{-n+1} + \frac{p + 1}{q}LA_{-n+2} - \frac{1}{q}LA_{-n+3},$$

where $LA_0 = LA_1 = p$ and $LA_2 = p^2 + pq + c$. Several terms of the Leonardo-Alwyn sequence can be given as

$$\begin{cases} & \vdots \\ LA_{-2} = & \dfrac{qp - p^2 - p^3 - qc + pc}{q^2}, \\ LA_{-1} = & \dfrac{p - p^2 - c}{q}, \\ LA_0 = & p, \\ LA_1 = & p, \\ LA_2 = & p^2 + pq + c, \\ LA_3 = & p^3 + p^2q + pq + pc + c, \\ LA_4 = & p^4 + p^3q + 2p^2q + pq^2 + p^2c + pc + qc + c, \\ & \vdots \end{cases}$$

For $n \geq 0$, the Binet's formula of the n-th Leonardo-Alwyn number is

$$LA_n = \alpha t_1^n + \beta t_2^n + \gamma t_3^n, \tag{1.3}$$

where

$$\begin{cases} t_1 = & \dfrac{p+\sqrt{p^2+4q}}{2}, \\ t_2 = & \dfrac{p-\sqrt{p^2+4q}}{2}, \\ t_3 = & 1, \\ \alpha = & \dfrac{(p^2+pq-p+c)(p^2+4q-(p-2)\sqrt{p^2+4q})}{2(p^2+4q)(p+q-1)}, \\ \beta = & \dfrac{(p^2+pq-p+c)(p^2+4q+(p-2)\sqrt{p^2+4q})}{2(p^2+4q)(p+q-1)}, \\ \gamma = & \dfrac{c}{1-p-q}. \end{cases}$$

with $t_1+t_2+t_3 = p+1$, $t_1t_2t_3 = -q$ and $t_1t_2+t_1t_3+t_2t_3 = p-q$. In the sequel, we always assume $t_1, t_2, t_3, \alpha, \beta$ and γ are the functions in the Binet's formula of the n-th Leonardo-Alwyn number. Let l_n be defined by the following recurrence relation

$$l_n = pl_{n-1} + ql_{n-2}, \quad n \geq 2,$$

with initial conditions $l_1 = l_2 = p$. For $n \geq 0$, we have

$$LA_n = \left(1 + \frac{c}{p(p+q-1)}\right) l_{n+1} - \frac{c}{p+q-1}. \tag{1.4}$$

The generating function of the Leonardo-Alwyn sequence is

$$g(x) = \frac{LA_0 + [LA_1 - (p+1)LA_0]x + [LA_2 - (q-p)LA_0 - (p+1)LA_1]x^2}{1-(p+1)x-(q-p)x^2+qx^3},$$

where $1-(p+1)x-(q-p)x^2+qx^3 \neq 0$. For $n \geq 0$, the matrix form relation of the non-negative indices Leonardo-Alwyn numbers is

$$\begin{bmatrix} LA_{n+3} & LA_{n+2} & LA_{n+1} \\ LA_{n+2} & LA_{n+1} & LA_n \\ LA_{n+1} & LA_n & LA_{n-1} \end{bmatrix} = \begin{bmatrix} LA_3 & LA_2 & LA_1 \\ LA_2 & LA_1 & LA_0 \\ LA_1 & LA_0 & LA_{-1} \end{bmatrix} \begin{bmatrix} p+1 & 1 & 0 \\ q-p & 0 & 1 \\ -q & 0 & 0 \end{bmatrix}^n \tag{1.5}$$

and the matrix form relation of the negative indices Leonardo-Alwyn numbers is

$$\begin{bmatrix} LA_{-n+3} & LA_{-n+2} & LA_{-n+1} \\ LA_{-n+2} & LA_{-n+1} & LA_{-n} \\ LA_{-n+1} & LA_{-n} & LA_{-n-1} \end{bmatrix} = \begin{bmatrix} LA_3 & LA_2 & LA_1 \\ LA_2 & LA_1 & LA_0 \\ LA_1 & LA_0 & LA_{-1} \end{bmatrix} \begin{bmatrix} 0 & 0 & -q \\ 1 & 0 & p+1 \\ 0 & 1 & q-p \end{bmatrix}^n. \tag{1.6}$$

On the other respect, the complex unit ($i^2 = -1$) lead to construct other multi dimensional number systems, such as hyperbolic (double, split complex, perplex) numbers, dual numbers, generalized complex numbers, dual-generalized complex numbers, quaternions, and octonions over the years. Also nowadays, hyper-complex numbers are crucial for applied mathematics such as physics, computer graphics and computational intelligence. Dual numbers are one type of generalized complex numbers (see [10] for generalized complex numbers). Any dual number z can be given by

$$z_1 = a_1 + b_1\varepsilon,$$

where a_1, a_2 are real numbers and ε is a dual unit with $\varepsilon^2 = 0$, $\varepsilon \neq 0$, [17,20,22]. Dual numbers have one real part and one non-real part. Let $z_1 = a_1 + b_1\varepsilon$ and $z_2 = a_2 + b_2\varepsilon$ be two dual numbers. The addition (hence subtraction) and the multiplication of z_1 and z_2 are defined as $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)\varepsilon$ and $z_1z_2 = z_2z_1 = (a_1a_2) + (a_1b_2 + a_2b_1)\varepsilon$, respectively. Hyper-dual numbers are an extension of dual numbers. Any hyper-dual number can be given by

$$z_1 = a_1 + b_1\varepsilon_1 + c_1\varepsilon_2 + d_1\varepsilon_1\varepsilon_2,$$

where a_1, b_1, c_1, d_1 are real numbers and $\varepsilon_1, \varepsilon_2$ are dual units with ${\varepsilon_1}^2 = {\varepsilon_2}^2 = 0$, $\varepsilon_1\varepsilon_2 = \varepsilon_2\varepsilon_1$, [4, 5]. Hyper-dual numbers have one real part and three non-real part. Let $z_1 = a_1 + b_1\varepsilon_1 + c_1\varepsilon_2 + d_1\varepsilon_1\varepsilon_2$ and $z_2 = a_2 + b_2\varepsilon_1 + c_2\varepsilon_2 + d_2\varepsilon_1\varepsilon_2$ be two hyper-dual numbers. The addition (hence subtraction) and the multiplication of z_1 and z_2 are defined as $z_1 + z_2 =$

$(a_1 + a_2) + (b_1 + b_2)\varepsilon_1 + (c_1 + c_2)\varepsilon_2 + (d_1 + d_2)\varepsilon_1\varepsilon_2$ and $z_1 z_2 = (a_1 a_2) + (a_1 b_2 + a_2 b_1)\varepsilon_1 + (a_1 c_2 + a_2 c_1)\varepsilon_1 + (a_1 d_2 + a_2 d_1 + b_1 c_2 + b_2 c_1)\varepsilon_1\varepsilon_2$, respectively. A hyper-dual number z_1 can be given in terms of two dual numbers as

$$\begin{aligned} z_1 &= a_1 + b_1\varepsilon_1 + c_1\varepsilon_2 + d_1\varepsilon_1\varepsilon_2 \\ &= (a_1 + b_1\varepsilon_1) + (c_1 + d_1\varepsilon_1)\,\varepsilon_2 \\ &= z + z^*\varepsilon_2, \end{aligned}$$

where $\varepsilon_1\varepsilon_2 \neq 0$.

Inspired by the hyper-complex numbers, there has been a tendency to generalize sequences to the multi component context. The problem of generalization of Leonardo numbers by means of hyper-complex numbers has been considered by many authors. The hybrid numbers with the Leonardo sequence components are defined in [1]. The real, and complex generalizations of the Leonardo sequence and their special identities are discussed in [11,18]. The hyper-dual numbers with the Leonardo sequence components are characterised in [12]. The dual-quaternions with the Leonardo sequence components are interested in [15]. The Leonardo hybrid numbers by using the q-integers are introduced in [16]. The dual quaternions with the generalized Leonardo sequence components are studied in [23].

In this sequel to the above-mentioned developments, our main objective is to describe the hyper-dual Leonardo-Alwyn sequence as a new addition to the existing literature. It is an important issue to identify the hyper-dual Leonardo-Alwyn sequence. The hyper-dual Leonardo-Alwyn sequence is analogue to the hyper-dual Leonardo sequence for $p = q = c = 1$, to the hyper-dual John-Edouard sequence for $p = 2, q = 1, c = 4$, and to the hyper-dual Ernst sequence for $p = c = 1, q = 2$. More specialized hyper-dual Leonardo-Alwyn sequences can be defined according to values of p, q and c.

2. Hyper-Dual Leonardo-Alwyn Numbers

In this original section, the mathematical formulation of hyper-dual numbers with Leonardo-Alwyn components is presented and a variety of algebraic properties of them are obtained.

Definition 2.1. The n-th hyper-dual Leonardo-Alwyn number is defined by:

$$\mathcal{LA}_n = LA_n + LA_{n+1}\varepsilon_1 + LA_{n+2}\varepsilon_2 + LA_{n+3}\varepsilon_1\varepsilon_2, \tag{2.1}$$

where LA_n is the n-th Leonardo-Alwyn number and $\{\varepsilon_1, \varepsilon_2, \varepsilon_1\varepsilon_2\}$ are hyper-dual units.

Let $\mathcal{LA}_n$ and $\mathcal{LA}_m$ be two hyper-dual Leonardo-Alwyn numbers. Then, the equality, addition (hence subtraction), scalar multiplication, and hyper-dual multiplication of them are defined by, respectively:

$$\mathcal{LA}_n = \mathcal{LA}_m \Leftrightarrow LA_n = LA_m, LA_{n+1} = LA_{m+1}, LA_{n+2} = LA_{m+2}, LA_{n+3} = LA_{m+3}.$$

$$\begin{aligned} \mathcal{LA}_n + \mathcal{LA}_m &= (LA_n + LA_m) + (LA_{n+1} + LA_{m+1})\varepsilon_1 \\ &\quad + (LA_{n+2} + LA_{m+2})\varepsilon_2 + (LA_{n+3} + LA_{m+3})\varepsilon_1\varepsilon_2, \\ \lambda\mathcal{LA}_n &= (\lambda LA_n) + (\lambda LA_{n+1})\varepsilon_1 + (\lambda LA_{n+2})\varepsilon_2 + (\lambda LA_{n+3})\varepsilon_1\varepsilon_2,\ \lambda \in \mathbb{R}, \\ \mathcal{LA}_n\mathcal{LA}_m &= LA_n LA_m + (LA_{n+1}LA_m + LA_n LA_{m+1})\,\varepsilon_1 \\ &\quad + (LA_n LA_{m+2} + LA_{n+2}LA_m)\,\varepsilon_2 \\ &\quad + (LA_{n+1}LA_{m+2} + LA_n LA_{m+3} + LA_{n+3}LA_m + LA_{n+2}LA_{m+1})\,\varepsilon_1\varepsilon_2. \end{aligned}$$

Theorem 2.2. *For $n \geq 2$, the non-homogeneous recurrence relation of hyper-dual Leonardo-Alwyn numbers is*

$$\mathcal{LA}_n = p\mathcal{LA}_{n-1} + q\mathcal{LA}_{n-2} + \mathsf{C},$$

where $\mathsf{C} = c(1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_1\varepsilon_2) = c\,\mathsf{I}$ and LA_n is the n-th Leonardo-Alwyn number.

Proof. From Definition 2.1 of hyper-dual Leonardo-Alwyn numbers and the non-homogeneous recurrence relation of Leonardo-Alwyn numbers (1.1) for $n \geq 2$, we have

$$\begin{aligned} \mathcal{LA}_n &= LA_n + LA_{n+1}\varepsilon_1 + LA_{n+2}\varepsilon_2 + LA_{n+3}\varepsilon_1\varepsilon_2 \\ &= (pLA_{n-1} + qLA_{n-2} + c) + (pLA_n + qLA_{n-1} + c)\,\varepsilon_1 \\ &\quad + (pLA_{n+1} + qLA_n + c)\,\varepsilon_2 + (pLA_{n+2} + qLA_{n+1} + c)\,\varepsilon_1\varepsilon_2 \\ &= p\mathcal{LA}_{n-1} + q\mathcal{LA}_{n-2} + \mathsf{C}. \end{aligned}$$

Here, we take $C = c(1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_1\varepsilon_2) = c\,\mathsf{I}$. □

Throughout this section, $C = c(1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_1\varepsilon_2) = c\,\mathsf{I}$ is considered.

Theorem 2.3. *For $n \geq 2$, the homogeneous recurrence relation of hyper-dual Leonardo-Alwyn numbers is*

$$\mathcal{LA}_{n+1} = (p+1)\mathcal{LA}_n + (q-p)\mathcal{LA}_{n-1} - q\mathcal{LA}_{n-2}.$$

Proof. Taking Definition 2.1 of hyper-dual Leonardo-Alwyn numbers and the homogeneous recurrence relation of Leonardo-Alwyn numbers (1.2) for $n \geq 2$ into account, we get

$$\begin{aligned}\mathcal{LA}_{n+1} &= LA_{n+1} + LA_{n+2}\varepsilon_1 + LA_{n+3}\varepsilon_2 + LA_{n+4}\varepsilon_1\varepsilon_2 \\ &= ((p+1)LA_n + (q-p)LA_{n-1} - qLA_{n-2}) + ((p+1)LA_{n+1} + (q-p)LA_n - qLA_{n-1})\,\mathsf{i} \\ &+ ((p+1)LA_{n+2} + (q-p)LA_{n+1} - qLA_n)\,\mathsf{j} + ((p+1)LA_{n+3} + (q-p)LA_{n+2} - qLA_{n+1})\,\mathsf{k} \\ &= (p+1)\mathcal{LA}_n + (q-p)\mathcal{LA}_{n-1} - q\mathcal{LA}_{n-2}.\end{aligned}$$

□

In Table 1, we give some well-known special cases of the hyper-dual Leonardo-Alwyn sequence. It is clear that, one can define more specific hyper-dual Leonardo-Alwyn sequences according to p, q and c.

TABLE 1. Some special cases of the hyper-dual Leonardo-Alwyn sequence

p	q	c	Hyper-dual Leonardo-Alwyn numbers	$\mathcal{LA}_n = p\mathcal{LA}_{n-1} + q\mathcal{LA}_{n-2} + c\,\mathsf{I}$
1	1	1	Hyper-dual Leonardo numbers [12]	$\mathsf{Le}_n = \mathsf{Le}_{n-1} + \mathsf{Le}_{n-2} + \mathsf{I}$
2	1	4	Hyper-dual John-Edouard numbers	$\mathsf{JE}_n = 2\mathsf{JE}_{n-1} + \mathsf{JE}_{n-2} + 4\,\mathsf{I}$
1	2	1	Hyper-dual Ernst numbers	$\mathcal{LA}_n = \mathsf{ER}_{n-1} + 2\mathsf{ER}_{n-2} + \mathsf{I}$

Theorem 2.4. *For $n \geq 0$, the Binet's formula of $\mathcal{LA}_n$ has the following form*

$$\mathcal{LA}_n = \alpha t_1^n t_1^* + \beta t_2^n t_2^* + \gamma t_3^n t_3^*,$$

where $t_i^* = 1 + t_i\varepsilon_1 + t_i^2\varepsilon_2 + t_i^3\varepsilon_1\varepsilon_2$ for $i = 1, 2, 3$.

Proof. Considering the Binet's formula of Leonardo-Alwyn numbers in equation (1.3), we obtain:

$$\begin{aligned}\mathcal{LA}_{n+1} &= LA_{n+1} + LA_{n+2}\varepsilon_1 + LA_{n+3}\varepsilon_2 + LA_{n+4}\varepsilon_1\varepsilon_2 \\ &= \left(\alpha t_1^n + \beta t_2^n + \gamma t_3^n\right) + \left(\alpha t_1^{n+1} + \beta t_2^{n+1} + \gamma t_3^{n+1}\right)\varepsilon_1 + \left(\alpha t_1^{n+2} + \beta t_2^{n+2} + \gamma t_3^{n+2}\right)\varepsilon_2 + \left(\alpha t_1^{n+3} + \beta t_2^{n+3} + \gamma t_3^{n+3}\right)\varepsilon_1\varepsilon_2 \\ &= \alpha t_1^n\left(1 + t_1\varepsilon_1 + t_1^2\varepsilon_2 + t_1^3\varepsilon_1\varepsilon_2\right) + \beta t_2^n\left(1 + t_2\varepsilon_1 + t_2^2\varepsilon_2 + t_2^3\varepsilon_1\varepsilon_2\right) + \gamma t_3^n\left(1 + t_3\varepsilon_1 + t_3^2\varepsilon_2 + t_3^3\varepsilon_1\varepsilon_2\right) \\ &= \alpha t_1^n t_1^* + \beta t_2^n t_2^* + \gamma t_3^n t_3^*.\end{aligned}$$

Here, we take $t_i^* = 1 + t_i\varepsilon_1 + t_i^2\varepsilon_2 + t_i^3\varepsilon_1\varepsilon_2$ for $i = 1, 2, 3$. □

Theorem 2.5. *For $n \geq 0$, we have the following relation for the hyper-dual Leonardo-Alwyn numbers*

$$\mathcal{LA}_n = \left(1 + \frac{c}{p(p+q-1)}\right) l_{n+1}^* - \frac{c}{p+q-1} + \mathsf{I},$$

where $l_{n+1}^* = l_{n+1} + l_{n+2}\varepsilon_1 + l_{n+3}\varepsilon_2 + l_{n+4}\varepsilon_1\varepsilon_2$.

Proof. Taking Definition 2.1 of hyper-dual Leonardo-Alwyn numbers and equation (1.4) into account, we get:

$$\begin{aligned}
\mathcal{LA}_{n+1} &= LA_{n+1} + LA_{n+2}\varepsilon_1 + LA_{n+3}\varepsilon_2 + LA_{n+4}\varepsilon_1\varepsilon_2 \\
&= \left[\left(1+\frac{c}{p(p+q-1)}\right)l_{n+1} - \frac{c}{p+q-1}\right] + \left[\left(1+\frac{c}{p(p+q-1)}\right)l_{n+2} - \frac{c}{p+q-1}\right]\varepsilon_1 \\
&+ \left[\left(1+\frac{c}{p(p+q-1)}\right)l_{n+3} - \frac{c}{p+q-1}\right]\varepsilon_2 + \left[\left(1+\frac{c}{p(p+q-1)}\right)l_{n+3} - \frac{c}{p+q-1}\right]\varepsilon_1\varepsilon_2 \\
&= \left(1+\frac{c}{p(p+q-1)}\right)(l_{n+1} + l_{n+2}\varepsilon_1 + l_{n+3}\varepsilon_2 + l_{n+4}\varepsilon_1\varepsilon_2) - \frac{c}{p+q-1}(1+\varepsilon_1+\varepsilon_2+\varepsilon_1\varepsilon_2).
\end{aligned}$$

This completes the proof. □

Theorem 2.6. *The generating function of the hyper-dual Leonardo-Alwyn sequence is*

$$\mathsf{G}(x) = \frac{\mathcal{LA}_0 + [\mathcal{LA}_1 - (p+1)\mathcal{LA}_0]x + [\mathcal{LA}_2 - (q-p)\mathcal{LA}_0 - (p+1)\mathcal{LA}_1]x^2}{1-(p+1)x-(q-p)x^2+qx^3},$$

where $1-(p+1)x-(q-p)x^2+qx^3 \neq 0$.

Proof. The formal power series representation of the generating function for the hyper-dual Leonardo-Alwyn sequence is

$$\mathsf{G}(x) = \sum_{n=0}^{\infty} \mathcal{LA}_n x^n.$$

By multiplying $\mathsf{G}(x)$ by $1-(p+1)x-(q-p)x^2+qx^3$ gives:

$$\begin{aligned}
(1-(p+1)x-(q-p)x^2+qx^3)\mathsf{G}(x) =& \mathcal{LA}_0 + (\mathcal{LA}_1 - (p+1)\mathcal{LA}_0)x + (\mathcal{LA}_2 - (q-p)\mathcal{LA}_0 - (p+1)\mathcal{LA}_1)x^2 \\
&+ (\mathcal{LA}_3 - (p+1)\mathcal{LA}_2 - (q-p)\mathcal{LA}_1 + q\mathcal{LA}_0)x^3 + ... \\
&+ (\mathcal{LA}_{n+3} - (p+1)\mathcal{LA}_{n+2} - (q-p)\mathcal{LA}_{n+1} + q\mathcal{LA}_n)x^{n+3} + ...
\end{aligned}$$

Considering Theorem 2.2, the mathematical formulation of the generating function $\mathsf{G}(x)$ is clear. □

Example 2.7. The recurrence relation of the Leonardo-Alwyn sequence reduces to

$$\mathsf{X}_n = 3\mathsf{X}_{n-1} - \mathsf{X}_{n-2} + 2\,\mathsf{I} \quad \text{for} \quad p=3, q=-1, c=2$$

and to

$$\mathsf{Y}_n = \mathsf{Y}_{n-1} + 2\mathsf{Y}_{n-2} - 3\,\mathsf{I} \quad \text{for} \quad p=1, q=2, c=-3$$

(see several terms in Table 2).

Table 2. Several terms of special Leonardo-Alwyn sequences

n	**Case** $p=3, q=-1, c=2$	**Case** $p=1, q=2, c=-3$
-5	$443+168\varepsilon_1+63\varepsilon_2+23\varepsilon_1\varepsilon_2$	$\frac{53}{32}+\frac{21}{16}\varepsilon_1+\frac{13}{8}\varepsilon_2+\frac{5}{4}\varepsilon_1\varepsilon_2$
-4	$168+63\varepsilon_1+23\varepsilon_2+8\varepsilon_1\varepsilon_2$	$\frac{21}{16}+\frac{13}{8}\varepsilon_1+\frac{5}{4}\varepsilon_2+\frac{3}{2}\varepsilon_1\varepsilon_2$
-3	$63+23\varepsilon_1+8\varepsilon_2+3\varepsilon_1\varepsilon_2$	$\frac{13}{8}+\frac{5}{4}\varepsilon_1+\frac{3}{2}\varepsilon_2+\varepsilon_1\varepsilon_2$
-2	$23+8\varepsilon_1+3\varepsilon_2+3\varepsilon_1\varepsilon_2$	$\frac{5}{4}+\frac{3}{2}\varepsilon_1+\varepsilon_2+\varepsilon_1\varepsilon_2$
-1	$8+3\varepsilon_1+3\varepsilon_2+8\varepsilon_1\varepsilon_2$	$\frac{3}{2}+\varepsilon_1+\varepsilon_2$
0	$3+3\varepsilon_1+8\varepsilon_2+23\varepsilon_1\varepsilon_2$	$1+\varepsilon_1-\varepsilon_1\varepsilon_2$
1	$3+8\varepsilon_1+23\varepsilon_2+63\varepsilon_1\varepsilon_2$	$1-\varepsilon_2-4\varepsilon_1\varepsilon_2$
2	$8+23\varepsilon_1+63\varepsilon_2+168\varepsilon_1\varepsilon_2$	$-\varepsilon_1-4\varepsilon_2-9\varepsilon_1\varepsilon_2$
3	$23+63\varepsilon_1+168\varepsilon_2+443\varepsilon_1\varepsilon_2$	$-1-4\varepsilon_1-9\varepsilon_2-20\varepsilon_1\varepsilon_2$
4	$63+168\varepsilon_1+443\varepsilon_2+1163\varepsilon_1\varepsilon_2$	$-4-9\varepsilon_1-20\varepsilon_2-41\varepsilon_1\varepsilon_2$
5	$168+443\varepsilon_1+1163\varepsilon_2+3048\varepsilon_1\varepsilon_2$	$-9-20\varepsilon_1-41\varepsilon_2-84\varepsilon_1\varepsilon_2$

The Binet's formula of X_n is

$$\begin{aligned}\mathsf{X}_n = & \left(\tfrac{5-\sqrt{5}}{2}\right)\left(\tfrac{3+\sqrt{5}}{2}\right)^n\left(1+\left(\tfrac{3+\sqrt{5}}{2}\right)\varepsilon_1+\left(\tfrac{3+\sqrt{5}}{2}\right)^2\varepsilon_2+\left(\tfrac{3+\sqrt{5}}{2}\right)^3\varepsilon_1\varepsilon_2\right)\\ & +\left(\tfrac{5+\sqrt{5}}{2}\right)\left(\tfrac{3-\sqrt{5}}{2}\right)^n\left(1+\left(\tfrac{3-\sqrt{5}}{2}\right)\varepsilon_1+\left(\tfrac{3-\sqrt{5}}{2}\right)^2\varepsilon_2+\left(\tfrac{3-\sqrt{5}}{2}\right)^3\varepsilon_1\varepsilon_2\right)\\ & -2\left(1+\varepsilon_1+\varepsilon_2+\varepsilon_1\varepsilon_2\right),\end{aligned}$$

and the Binet's formula of Y_n is

$$\mathsf{Y}_n = -\tfrac{2^n}{3}\left(1+2\varepsilon_1+4\varepsilon_2+8\varepsilon_1\varepsilon_2\right)-\tfrac{(-1)^n}{6}\left(1-\varepsilon_1+\varepsilon_2-\varepsilon_1\varepsilon_2\right)+\tfrac{3}{2}\left(1+\varepsilon_1+\varepsilon_2+\varepsilon_1\varepsilon_2\right).$$

The generating function of X_n is

$$\mathsf{G}_{\mathsf{X}_n}(x) = \frac{(3+3\varepsilon_1+8\varepsilon_2+23\varepsilon_1\varepsilon_2)-(9+4\varepsilon_1+9\varepsilon_2+29\varepsilon_1\varepsilon_2)\,x+(8+3\varepsilon_1+3\varepsilon_2+8\varepsilon_1\varepsilon_2)\,x^2}{1-4x+4x^2-x^3},$$

where $1-4x+4x^2-x^3 \neq 0$, and the generating function of Y_n is

$$\mathsf{G}_{\mathsf{Y}_n}(x) = \frac{(1+\varepsilon_1-\varepsilon_1\varepsilon_2)-(1+2\varepsilon_1+\varepsilon_2+2\varepsilon_1\varepsilon_2)\,x-(3+2\varepsilon_1+2\varepsilon_2)\,x^2}{1-2x-x^2+2x^3},$$

where $1-2x-x^2+2x^3 \neq 0$.

Theorem 2.8. *For $n \geq 0$, the matrix form relation of non-negative indices hyper-dual Leonardo-Alwyn numbers is*

$$\begin{bmatrix}\mathcal{LA}_{n+3} & \mathcal{LA}_{n+2} & \mathcal{LA}_{n+1}\\ \mathcal{LA}_{n+2} & \mathcal{LA}_{n+1} & \mathcal{LA}_{n}\\ \mathcal{LA}_{n+1} & \mathcal{LA}_{n} & \mathcal{LA}_{n-1}\end{bmatrix} = \begin{bmatrix}\mathcal{LA}_{3} & \mathcal{LA}_{2} & \mathcal{LA}_{1}\\ \mathcal{LA}_{2} & \mathcal{LA}_{1} & \mathcal{LA}_{0}\\ \mathcal{LA}_{1} & \mathcal{LA}_{0} & \mathcal{LA}_{-1}\end{bmatrix}\begin{bmatrix}p+1 & 1 & 0\\ q-p & 0 & 1\\ -q & 0 & 0\end{bmatrix}^n,$$

and the matrix form relation of negative indices hyper-dual Leonardo-Alwyn numbers is

$$\begin{bmatrix}\mathcal{LA}_{-n+3} & \mathcal{LA}_{-n+2} & \mathcal{LA}_{-n+1}\\ \mathcal{LA}_{-n+2} & \mathcal{LA}_{-n+1} & \mathcal{LA}_{-n}\\ \mathcal{LA}_{-n+1} & \mathcal{LA}_{-n} & \mathcal{LA}_{-n-1}\end{bmatrix} = \begin{bmatrix}\mathcal{LA}_{3} & \mathcal{LA}_{2} & \mathcal{LA}_{1}\\ \mathcal{LA}_{2} & \mathcal{LA}_{1} & \mathcal{LA}_{0}\\ \mathcal{LA}_{1} & \mathcal{LA}_{0} & \mathcal{LA}_{-1}\end{bmatrix}\begin{bmatrix}0 & 0 & -q\\ 1 & 0 & p+1\\ 0 & 1 & q-p\end{bmatrix}^n.$$

Proof. The proof is just calculation with the matrix relations (1.5) and (1.6). □

3. Conclusions

In this paper, hyper-dual numbers with the Leonardo-Alwyn sequence as component have been examined. Firstly, the hyper-dual Leonardo-Alwyn sequence is presented. Then, the recurrence relations, the Binet's formula and the generating function are calculated. With this paper, several special cases of hyper-dual numbers with the Leonardo-Alwyn sequence components according to p, q, c can be easily obtained.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The authors have read and agreed to the published version of the manuscript.

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