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Research Article

Further research on separated degrees of M-fuzzifying convex spaces

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Abstract

In this paper, we redefine the concepts of join spaces and product spaces of M-fuzzifying convex spaces. Then we further investigate the S_i ($i = 0, 1, 2$) separated degrees of an M-fuzzifying convex space in a logical viewpoint. Finally, we study the S_i ($i = 0, 1, 2$) separated degrees of an M-fuzzifying convex space from the aspect of convergence structures.

Mathematics Subject Classification (2020). 54A20

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1. Introduction

Convexity plays an important role in many mathematical environments, such as vector spaces, metric spaces, lattices, graphs, matroids and so on. Combining with the axiomatic approach, the concept of convex structures [20] is introduced by abstracting the common properties of convex sets in different mathematical structures. In an axiomatic viewpoint, convex structures provide a more general framework of studying convexity.

Since Zadeh [32] introduced fuzzy sets, many mathematical structures have been combined with fuzzy set theory, such as fuzzy topology [1, 6, 19, 31], fuzzy order [29, 30], fuzzy convergence [3, 4, 33–36] and so on. Convex structures have also been generalized to the fuzzy case. Rosa [14] first introduced the concept of fuzzy convexities with the real unit interval $[0, 1]$ as the lattice background. Later, Maruyama [9] extended $[0, 1]$ to a completely distributive lattice L and proposed the notion of L-fuzzy convexities. Adopting the terminology of fuzzy topology, these two fuzzy convexities are both called L-convex structures now. From a logical aspect, Shi and Xiu [16] introduced the concept of M-fuzzifying convex structures, where M also denotes a completely distributive lattice. Recently, Shi and Xiu [17] proposed the notion of (L, M)-fuzzy convex structures, which can include L-convex structures and M-fuzzifying convex structures as special cases. Up to now, fuzzy convex structures have deserved more and more attention and have been studied from different aspects, including closure operators [11, 12, 15, 37], interval operators [13, 18, 27], geometrical properties [22–24] and topological properties [26, 28].

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Shi and Xiu [16] defined product spaces of M-fuzzifying convex spaces via subbases. Adopting the axiomatic approach, Xiu and Pang [26] introduced axiomatic convexity bases in the framework of M-fuzzifying convex spaces, which provided a foundation of defining product spaces. Following this trend, our first aim of this paper is to redefine join spaces and product spaces of M-fuzzifying convex spaces from the aspect of axiomatic convexity subbases.

Zhou and Shi [38] first defined separation axioms in L-convex spaces and investigated their hereditary and productivity. Later, Zhou and Shi [39] introduced the sum space of L-convex spaces and studied additivity of S_i $(i = 1, 2, 3, 4)$ separation aixoms. Liang and Li [7] first defined S_i $(i = 0, 1, 2)$ separated degrees of an M-fuzzifying convex space, which describes the degree to which an M-fuzzifying convex space is S_i $(i = 0, 1, 2)$ separated. Further, Liang and Li [8] defined S_i $(i = 3, 4)$ separated degrees of an M-fuzzifying convex space and studied their productivity. Dong and Shi [2] proposed the concept of disjoint sums of M-fuzzifying convex spaces and discussed the additivity of S_i $(i = 1, 2, 3, 4)$ separated degrees. Considering the productivity of S_i $(i = 0, 1, 2)$ separated degrees, Pang [10] introduced M-fuzzifying convergence structures and defined S_i $(i = 0, 1, 2)$ separated degrees of an M-fuzzifying convex space via its induced M-fuzzifying convergence structure. Notice that the S_i $(i = 0, 1, 2)$ separated degrees in the sense of Pang has some advantages compared with that in the the sense of Liang and Li, especially on the productivity of separated degrees. In a degree viewpoint, Xiu and Pang [25] also defined M-fuzzifying convexity-preserving (M-CP in short) and M-fuzzifying convex-to-convex (M-CC in short) degrees of a mapping between M-fuzzifying convex spaces, which can be used to characterize the degrees to which a mapping between M-fuzzifying convex spaces is M-CP and M-CC, respectively.

In the classical case, there are close relationships between separation properties and CP and CC mappings. Up to now, these concepts have all been defined with some degrees. By this motivation, our second aim of this paper is to investigate their relationships in a degree approach.

2. Preliminaries

We consider in this paper a completely distributive lattice M, i.e., a complete lattice M satisfies

$$\bigvee_{i\in I}\bigwedge_{j\in J_i} a_{ij} = \bigwedge_{f\in\prod_{i\in I} J_i}\bigvee_{i\in I} a_{if(i)}$$

or

$$\bigwedge_{i\in I}\bigvee_{j\in J_i} a_{ij} = \bigvee_{f\in\prod_{i\in I} J_i}\bigwedge_{i\in I} a_{if(i)}$$

for all $X_i = \{a_{ij} \mid j \in J_i\} \subseteq 2^M$ $(i \in I)$. The bottom (resp. top) element of M is denoted by $\perp$ (resp. $\top$). For $a, b \in M$, we say that a is wedge below b in M (in symbols, $a \prec b$) if for every subset $D \subseteq M$, $\bigvee D \geqslant b$ implies $d \geqslant a$ for some $d \in M$. A complete lattice M is completely distributive if and only if $b = \bigvee\{a \in M \mid a \prec b\}$ for each $b \in M$ [21]. We can then define a *residual implication* on M by

$$a \to b = \bigvee\{c \in M \mid a \wedge c \leq b\}.$$

In particular, we denote $a \to \perp$ by $\neg a$ for each $a \in M$.

We will often use, without explicitly mentioning, the following properties of *residual implication* on M.

Lemma 2.1 ([5]). *Let M be a completely distributive lattice. Then the following statements hold:*

(1) $\top \to a = a$.

(2) $a \leq b$ *if and only if* $a \to b = \top$.

(3) $(a \to b) \to (c \to b) \geq c \to a$.
(4) $(a \vee b) \to (c \vee d) \geq (a \to c) \wedge (b \to d)$.
(5) $(a \wedge b) \to (c \wedge d) \geq (a \to c) \wedge (b \to d)$.
(6) $(\neg a \vee \neg b) \to (\neg c \vee \neg d) \geq (c \to a) \wedge (d \to b)$.
(7) $\bigvee_{j \in J} a_j \to \bigvee_{j \in J} b_j \geq \bigwedge_{j \in J}(a_j \to b_j)$.
(8) $\bigwedge_{j \in J} a_j \to \bigwedge_{j \in J} b_j \geq \bigwedge_{j \in J}(a_j \to b_j)$.
(9) $a \to \bigwedge_{j \in J} a_j = \bigwedge_{j \in J}(a \to a_j)$, *hence* $a \to b \leq a \to c$ *whenever* $b \leq c$.
(10) $\bigvee_{j \in J} a_j \to b = \bigwedge_{j \in J}(a_j \to b)$, *hence* $a \to c \geq b \to c$ *whenever* $a \leq b$.

For a nonempty set X, 2^X denotes the powerset of X and M^X denotes the set of all M-subsets on X. For each $A \in 2^X$, let χ_A denote the characteristic function of A. For $\{A_j\}_{j \in J} \subseteq 2^X$, we say $\{A_j\}_{j \in J}$ is a directed subset of 2^X provided that for each $B, C \in \{A_j\}_{j \in J}$, there exists $D \in \{A_j\}_{j \in J}$ such that $B \subseteq D$ and $C \subseteq D$, which is denoted by $\{A_j\}_{j \in J} \overset{dir}{\subseteq} 2^X$. Dually, let $\{A_j\}_{j \in J} \overset{cdir}{\subseteq} 2^X$ denote that $\{A_j\}_{j \in J}$ is a codirected subset of 2^X, which means that for each $B, C \in \{A_j\}_{j \in J}$, there exists $D \in \{A_j\}_{j \in J}$ such that $D \subseteq B$ and $D \subseteq C$.

Let $f : X \longrightarrow Y$ be a mapping. Define $f^{\to} : 2^X \longrightarrow 2^Y$ by $f^{\to}(A) = \{f(x) \mid x \in A\}$ for each $A \in 2^X$ and $f^{\leftarrow} : 2^Y \longrightarrow 2^X$ by $f^{\leftarrow}(B) = \{x \mid f(x) \in B\}$ for each $B \in 2^Y$.

Definition 2.2 ([3]). A fuzzy inclusion order on M^X is a mapping $\mathcal{S} : M^X \times M^X \longrightarrow M$ which is defined by

$$\forall U,\ V \in M^X,\ \ \mathcal{S}(U, V) = \bigwedge_{x \in X} \Big(U(x) \to V(x)\Big).$$

Definition 2.3 ([16]). A mapping $\mathscr{C} : 2^X \longrightarrow M$ is called an M-fuzzifying convex structure on X if it satisfies the following conditions:
(MYC1) $\mathscr{C}(\varnothing) = \mathscr{C}(X) = \top$;
(MYC2) $\mathscr{C}(\bigcap_{k \in K} A_k) \geq \bigwedge_{k \in K} \mathscr{C}(A_k)$;
(MYC3) $\mathscr{C}(\bigcup_{j \in J} A_j) \geq \bigwedge_{j \in J} \mathscr{C}(A_j)$ for each $\{A_j\}_{j \in J} \overset{dir}{\subseteq} 2^X$.
For an M-fuzzifying convex structure $\mathscr{C}$ on X, the pair $(X, \mathscr{C})$ is called an M-fuzzifying convex space.

In [10], Pang introduced the concept of M-fuzzifying convergence structures via M-fuzzifying convex filters in the framework of M-fuzzifying convex spaces.

Definition 2.4 ([10]). A mapping $\mathcal{F} : 2^X \longrightarrow M$ is called an M-fuzzifying convex filter on X if it satisfies:
(MF1) $\mathcal{F}(\varnothing) = \bot$, $\mathcal{F}(X) = \top$;
(MF2) $\mathcal{F}(\bigcap_{j \in J} A_j) = \bigwedge_{j \in J} \mathcal{F}(A_j)$ for each $\{A_j\}_{j \in J} \overset{cdir}{\subseteq} 2^X$.
The family of all M-fuzzifying convex filters on X is denoted by $\mathcal{F}_M(X)$.

Example 2.5 ([10]). For each $x \in X$, define $[x] : 2^X \longrightarrow M$ by

$$\forall A \in 2^X,\ \ [x](A) = \chi_A(x).$$

Then $[x] \in \mathcal{F}_M(X)$, which is called point M-fuzzifying convex filter of x.

Since each M-fuzzifying convex filter is an M-subset on 2^X, there exits a natural fuzzy inclusion order on $\mathcal{F}_M(X)$, which is denoted by $\mathcal{S}_{\mathcal{F}}(\cdot, \cdot) : \mathcal{F}_M(X) \times \mathcal{F}_M(X) \longrightarrow M$. Explicitly,

$$\forall \mathcal{F}, \mathcal{G} \in \mathcal{F}_M(X),\ \mathcal{S}_{\mathcal{F}}(\mathcal{F}, \mathcal{G}) = \bigwedge_{A \in 2^X} (\mathcal{F}(A) \to \mathcal{G}(A)).$$

Definition 2.6 ([10]). An M-fuzzifying convergence structure on X is a mapping $\lim : \mathcal{F}_M(X) \longrightarrow M^X$ which satisfies:
(MC1) $\lim([x])(x) = \top$;

(MC2) $\mathcal{S}_{\mathcal{F}}(\mathcal{F},\mathcal{G}) \leq \mathcal{S}(\lim(\mathcal{F}),\lim(\mathcal{G}))$.
For an M-fuzzifying convergence structure lim on X, the pair $(X,\lim)$ is called an M-fuzzifying convergence space.

Pang [10] showed every M-fuzzifying convex space can induce an M-fuzzifying convergence space.

Proposition 2.7 ([10]). *Let $(X,\mathscr{C})$ be an M-fuzzifying convex space and define $\lim^{\mathscr{C}} : \mathcal{F}_M(X) \longrightarrow M^X$ as follows:*

$$\forall \mathcal{F} \in \mathcal{F}_M(X), \forall x \in X,\ \lim^{\mathscr{C}}(\mathcal{F})(x) = \bigwedge_{x\in A}\Big(\mathscr{C}(X-A) \to \mathcal{F}(A)\Big).$$

Then $\lim^{\mathscr{C}}$ is an M-fuzzifying convergence structure on X.

3. Join space and product space of M-fuzzifying convex spaces

In [16], Shi and Xiu introduced the concepts of join spaces and product spaces of M-fuzzifying convex spaces. Here we will redefine these two concepts by using axiomatic subbases in M-fuzzifying convex spaces and make some further research on their properties.

Definition 3.1 ([26]). A mapping $\varphi : 2^X \longrightarrow M$ is called an M-fuzzifying convexity subbase of some M-fuzzifying convex space provided that φ satisfies
(MYSB1) $\bigvee_{\bigcap_{i\in\Omega} A_i=\varnothing} \bigwedge_{i\in\Omega}\varphi(A_i) = \top$,
(MYSB2) $\bigvee_{\bigcup^{dir}_{j\in J} A_j=X} \bigwedge_{j\in J} \bigvee_{\bigcap_{i\in I_j} A_{ji}=A_j} \bigwedge_{i\in I_j}\varphi(A_{ji}) = \top$.

In [26], Xiu and Pang gave the formula of generating an M-fuzzifying convex structure $\mathscr{C}$ by means of an M-fuzzifying convexity subbase φ as follows:

$$\mathscr{C}(A) = \bigvee_{\bigcup^{dir}_{j\in J} A_j=A}\ \bigwedge_{j\in J}\ \bigvee_{\bigcap_{k\in K_j} A_{jk}=A_j}\ \bigwedge_{k\in K_j}\varphi(A_{jk}).$$

Lemma 3.2. *Suppose that $\{\mathscr{C}_i\}_{i\in I}$ is a family of M-fuzzifying convex structures on X. Then the mapping $\bigvee_{i\in I}\mathscr{C}_i : 2^X \longrightarrow M$ defined by*

$$\forall A \in 2^X, (\bigvee_{i\in I}\mathscr{C}_i)(A) = \bigvee_{i\in I}\mathscr{C}_i(A)$$

is an M-fuzzifying convexity subbase.

Proof. It is easy to check that $\mathscr{C}_i$ satisfies (MYSB1) and (MYSB2) for each $i \in I$. Then it follows immediately that $\bigvee_{i\in I}\mathscr{C}_i$ satisfies (MYSB1) and (MYSB2), as desired. □

By Lemma 3.2, we can obtain an M-fuzzifying convex structure which is generated by the M-fuzzifying convexity subbase $\bigvee_{i\in I}\mathscr{C}_i$. From this aspect, we propose the definition of join spaces.

Definition 3.3. Suppose that $\{\mathscr{C}_i\}_{i\in I}$ is a family of M-fuzzifying convex structures on X. The M-fuzzifying convex structure generated by the M-fuzzifying convexity subbase $\bigvee_{i\in I}\mathscr{C}_i$ is called the join structure of $\{\mathscr{C}_i\}_{i\in I}$, which is denoted by $\bigsqcup_{i\in I}\mathscr{C}_i$. The pair $(X,\bigsqcup_{i\in I}\mathscr{C}_i)$ is called the join space of $\{(X,\mathscr{C}_i)\}_{i\in I}$.

By means of join spaces, we will give the definition of product space of M-fuzzifying convex spaces. To this end, we first present the following lemma.

Lemma 3.4. *Suppose that $\{(X_i,\mathscr{C}_i)\}_{i\in I}$ is a family of M-fuzzifying convex spaces, $X = \prod_{i\in I} X_i$, and $\{p_i : X \longrightarrow X_i\}_{i\in I}$ is the family of projection mappings. Then the mapping $p_i^{-1}(\mathscr{C}_i) : 2^X \longrightarrow M$ defined by*

$$\forall A \in 2^X, p_i^{-1}(\mathscr{C}_i)(A) = \bigvee_{p_i^{\leftarrow}(A_i)=A}\mathscr{C}_i(A_i)$$

is an M-fuzzifying convex structure on X for each $i \in I$.

Proof. It suffices to show that $p_i^{-1}(\mathscr{C}_i)$ satisfies (MYC1)–(MYC3).

(MYC1) It is straightforward.

(MYC2) Take any $\alpha \in M$ such that

$$\alpha \prec \bigwedge_{k\in K} p_i^{-1}(\mathscr{C}_i)(A_k) = \bigwedge_{k\in K} \bigvee_{p_i^{\leftarrow}(A_{ki})=A_k} \mathscr{C}_i(A_{ki}).$$

Then for each $k \in K$, there exists B_{ki} such that $p_i^{\leftarrow}(B_{ki}) = A_k$ and $\alpha \le \mathscr{C}_i(B_{ki})$. Let $B_i = \bigcap_{k\in K} B_{ki}$. Then it follows that

$$p_i^{\leftarrow}(B_i) = \bigcap_{k\in K} p_i^{\leftarrow}(B_{ki}) = \bigcap_{k\in K} A_k$$

and

$$\alpha \le \bigwedge_{k\in K} \mathscr{C}_i(B_{ki}) \le \mathscr{C}_i(\bigcap_{k\in K} B_{ki}) = \mathscr{C}_i(B_i) \le \bigvee_{p_i^{\leftarrow}(C_i)=\bigcap_{k\in K} A_k} \mathscr{C}_i(C_i) = p_i^{-1}(\mathscr{C}_i)(\bigcap_{k\in K} A_k).$$

By the arbitrariness of α, we obtain

$$\bigwedge_{k\in K} p_i^{-1}(\mathscr{C}_i)(A_k) \le p_i^{-1}(\mathscr{C}_i)(\bigcap_{k\in K} A_k).$$

(MYC3) For each $\{A_j\}_{j\in J} \overset{dir}{\subseteq} 2^X$, take any $\alpha \in M$ such that

$$\alpha \prec \bigwedge_{j\in J} p_i^{-1}(\mathscr{C}_i)(A_j) = \bigwedge_{j\in J} \bigvee_{p_i^{\leftarrow}(A_{ji})=A_j} \mathscr{C}_i(A_{ji}).$$

Then for each $j \in J$, there exists B_{ji} such that $p_i^{\leftarrow}(B_{ji}) = A_j$ and $\alpha \le \mathscr{C}_i(B_{ji})$. Let $B_i = \bigcup_{j\in J} B_{ji}$. Then it follows that

$$p_i^{\leftarrow}(B_i) = \bigcup_{j\in J} p_i^{\leftarrow}(B_{ji}) = \bigcup_{j\in J} A_j.$$

Since p_i is surjective, it follows that $B_{ji} = p_i^{\rightarrow}(p_i^{\leftarrow}(B_{ji})) = p_i^{\rightarrow}(A_j)$. This implies that $\{B_{ji} \mid j \in J\}$ is directed. Then we have

$$\alpha \le \bigwedge_{j\in J} \mathscr{C}_i(B_{ji}) \le \mathscr{C}_i(\bigcup_{j\in J} B_{ji}) = \mathscr{C}_i(B_i) \le \bigvee_{p_i^{\leftarrow}(C_i)=\bigcup_{j\in J} A_j} \mathscr{C}_i(C_i) = p_i^{-1}(\mathscr{C}_i)(\bigcup_{j\in J} A_j).$$

By the arbitrariness of α, we obtain

$$\bigwedge_{j\in J} p_i^{-1}(\mathscr{C}_i)(A_j) \le p_i^{-1}(\mathscr{C}_i)(\bigcup_{j\in J} A_j).$$

□

By Lemma 3.4, for a family of M-fuzzifying convex spaces $\{(X_i, \mathscr{C}_i)\}_{i\in I}$, we can obtain a family of M-fuzzifying convex structures $\{p_i^{-1}(\mathscr{C}_i)\}_{i\in I}$ on the product set $\prod_{i\in I} X_i$. Then we can propose the definition of product spaces of $\{(X_i, \mathscr{C}_i)\}_{i\in I}$ by means of the join space of $\{(\prod_{i\in I} X_i, p_i^{-1}(\mathscr{C}_i))\}_{i\in I}$.

Definition 3.5. Suppose that $\{(X_i, \mathscr{C}_i)\}_{i\in I}$ is a family of M-fuzzifying convex spaces, $X = \prod_{i\in I} X_i$, and $\{p_i : X \longrightarrow X_i\}_{i\in I}$ is the family of projection mappings. Then the join space of $\{(X, p_i^{-1}(\mathscr{C}_i))\}_{i\in I}$ is called the product space of $\{(X_i, \mathscr{C}_i)\}_{i\in I}$.

Usually, we denote the product space of $\{(X_i, \mathscr{C}_i)\}_{i\in I}$ by $\{(\prod_{i\in I} X_i, \prod_{i\in I} \mathscr{C}_i)\}$. For a finite family of M-fuzzifying convex spaces $\{(X_i, \mathscr{C}_i) \mid i = 1, 2, \cdots, n\}$, we use $\mathscr{C}_1 \times \mathscr{C}_2 \cdots \times \mathscr{C}_n$ to denote their product.

In the classical case, each projection mapping is a CP and CC mapping. Next we present its fuzzy counterpart in the framework of M-fuzzifying convex spaces.

Proposition 3.6. *Suppose that* $\{(X_i,\mathscr{C}_i)\}_{i\in I}$ *is a family of* M*-fuzzifying convex spaces and* $\{p_i : \prod_{j\in I} X_j \longrightarrow X_i\}_{i\in I}$ *is the family of projection mappings. Then for each* $i \in I$, $p_i : (\prod_{j\in I} X_j, \prod_{j\in I}\mathscr{C}_j) \longrightarrow (X_i,\mathscr{C}_i)$ *is an* M*-CP mapping.*

Proof. For convenience, let $X = \prod_{i\in I} X_i$ and $\mathscr{C} = \prod_{i\in I}\mathscr{C}_i$. By Definition 3.5, we have for each $A \in 2^X$,

$$\mathscr{C}(A) = \bigvee_{\bigcup_{j\in J}^{dir} A_j = A}\ \bigwedge_{j\in J}\ \bigvee_{\bigcap_{k\in K_j} A_{jk}=A_j}\ \bigwedge_{k\in K_j}\ \bigvee_{i\in I}\ \bigvee_{p_i^{\leftarrow}(B_i)=A_{jk}} \mathscr{C}_i(B_i).$$

Then it suffices to show that $\mathscr{C}(p_{i_0}^{\leftarrow}(A_{i_0})) \ge \mathscr{C}_{i_0}(A_{i_0})$ for each $i_0 \in I$ and $A_{i_0} \in 2^{X_{i_0}}$. By the definition of $\mathscr{C}$, we have

$$\begin{aligned}
\mathscr{C}(p_{i_0}^{\leftarrow}(A_{i_0})) &= \bigvee_{\bigcup_{j\in J}^{dir} A_j = p_{i_0}^{\leftarrow}(A_{i_0})}\ \bigwedge_{j\in J}\ \bigvee_{\bigcap_{k\in K_j} A_{jk}=A_j}\ \bigwedge_{k\in K_j}\ \bigvee_{i\in I}\ \bigvee_{p_i^{\leftarrow}(B_i)=A_{jk}} \mathscr{C}_i(B_i) \\
&\ge \bigvee_{\bigcap_{k\in K_j} A_{jk}=p_{i_0}^{\leftarrow}(A_{i_0})}\ \bigwedge_{k\in K_j}\ \bigvee_{i\in I}\ \bigvee_{p_i^{\leftarrow}(B_i)=A_{jk}} \mathscr{C}_i(B_i) \\
&\ge \bigvee_{i\in I}\ \bigvee_{p_i^{\leftarrow}(B_i)=p_{i_0}^{\leftarrow}(A_{i_0})} \mathscr{C}_i(B_i) \\
&\ge \mathscr{C}_{i_0}(A_{i_0}).
\end{aligned}$$

By the arbitrariness of i_0, we obtain $p_i : (\prod_{j\in I} X_j, \prod_{j\in I}\mathscr{C}_j) \longrightarrow (X_i,\mathscr{C}_i)$ is an M-CP mapping for each $i \in I$. □

Proposition 3.7. *Suppose that* $\{(X_i,\mathscr{C}_i)\}_{i\in I}$ *is a family of* M*-fuzzifying convex spaces and* $\{p_i : \prod_{j\in I} X_j \longrightarrow X_i\}_{i\in I}$ *is the family of projection mappings. Then for each* $i \in I$, $p_i : (\prod_{j\in I} X_j, \prod_{j\in I}\mathscr{C}_j) \longrightarrow (X_i,\mathscr{C}_i)$ *is an* M*-CC mapping.*

Proof. For convenience, let $X = \prod_{i\in I} X_i$ and $\mathscr{C} = \prod_{i\in I}\mathscr{C}_i$. By Definition 3.5, we have for each $A \in 2^X$,

$$\mathscr{C}(A) = \bigvee_{\bigcup_{j\in J}^{dir} A_j = A}\ \bigwedge_{j\in J}\ \bigvee_{\bigcap_{k\in K_j} A_{jk}=A_j}\ \bigwedge_{k\in K_j}\ \bigvee_{i\in I}\ \bigvee_{p_i^{-1}(B_i)=A_{jk}} \mathscr{C}_i(B_i).$$

Then it suffices to show that for each $i_0 \in I$, $\mathscr{C}(A) \le \mathscr{C}_{i_0}(p_{i_0}^{\rightarrow}(A))$.

Take each $\alpha \in M$ such that $\alpha \prec \mathscr{C}(A)$. Then there exists a directed set $\{A_j\}_{j\in J} \subseteq 2^X$ such that $\bigcup_{j\in J}^{dir} A_j = A$ and for each $j \in J$, there exists $\{A_{jk}\}_{k\in K_j} \subseteq 2^X$ such that $\bigcap_{k\in K_j} A_{jk} = A_j$ and for each $k \in K_j$, there exists $i_{jk} \in I$ and $B_{i_{jk}} \in 2^{X_{i_{jk}}}$ such that $p_i^{-1}(B_{i_{jk}}) = A_{jk}$ and $\alpha \le \mathscr{C}_{i_{jk}}(B_{i_{jk}})$. Thus, we get

$$\bigcup_{j\in J}^{dir}\bigcap_{k\in K_j} p_i^{\leftarrow}(B_{i_{jk}}) = \bigcup_{j\in J}^{dir}\bigcap_{k\in K_j} A_{jk} = \bigcup_{j\in J}^{dir} A_j = A$$

and $p_{i_0}^{\rightarrow}(A) = \bigcup_{j\in J}^{dir} p_{i_0}^{\rightarrow}(\bigcap_{k\in K_j} p_i^{\leftarrow}(B_{i_{jk}}))$. For each $j \in J$, it follows that

$$p_{i_0}^{\rightarrow}(\bigcap_{k\in K_j} p_i^{\leftarrow}(B_{i_{jk}})) = \begin{cases} B_{i_{jk_0}} (i_0 = i_{jk_0}), & i_0 \in \{i_{jk} | k \in K_j\}; \\ X_{i_0}, & i_0 \notin \{i_{jk} | k \in K_j\}. \end{cases}$$

Then

$$\mathscr{C}_{i_0}(p_{i_0}^{\rightarrow}(\bigcap_{k\in K_j} p_i^{\leftarrow}(B_{i_{jk}}))) = \begin{cases} \mathscr{C}_{i_0}(B_{i_{jk_0}}) = \mathscr{C}_{i_{jk_0}}(B_{i_{jk_0}}), & i_0 \in \{i_{jk} | k \in K_j\}; \\ \mathscr{C}_{i_0}(X_{i_0}) = \top, & i_0 \notin \{i_{jk} | k \in K_j\}. \end{cases}$$

This implies that

$$\mathscr{C}_{i_0}(p_{i_0}^{\rightarrow}(A)) \ge \bigwedge_{j\in J} \mathscr{C}_{i_0}(p_{i_0}^{\rightarrow}(\bigcap_{k\in K_j} p_i^{\leftarrow}(B_{i_{jk}}))) \ge \alpha.$$

By the arbitrariness of α, we obtain $\mathscr{C}(A) \le \mathscr{C}_{i_0}(p_{i_0}^{\rightarrow}(A))$, as desired. □

4. Degrees of separation axioms in M-fuzzifying convex spaces

In [7], Liang and Li proposed S_0, S_1 and S_2 separated degrees of M-fuzzifying convex spaces and discussed their properties. In this section, we will give some further investigations on their properties and discuss their connections with M-CP degrees and M-CC degrees of a mapping between M-fuzzifying convex spaces.

Firstly, let us recall the S_0, S_1 and S_2 separated degrees of M-fuzzifying convex spaces in the sense of Liang and Li.

Definition 4.1 ([7])**.** For an M-fuzzifying convex space $(X, \mathscr{C})$, define the degree to which $(X, \mathscr{C})$ is S_0 separated as follows:

$$S_0(X, \mathscr{C}) = \bigwedge_{x \neq y} \Big(\bigvee_{x \notin A \ni y} \mathscr{C}(A) \vee \bigvee_{y \notin B \ni x} \mathscr{C}(B) \Big).$$

Definition 4.2 ([7])**.** For an M-fuzzifying convex space $(X, \mathscr{C})$, define the degree to which $(X, \mathscr{C})$ is S_1 separated as follows:

$$S_1(X, \mathscr{C}) = \bigwedge_{x \neq y} \bigvee_{y \notin A \ni x} \mathscr{C}(A).$$

The degree to which $(X, \mathscr{C})$ is S_1 separated can also be characterized as follows:

Proposition 4.3 ([7])**.** *Let $(X, \mathscr{C})$ be an M-fuzzifying convex space. Then*

$$S_1(X, \mathscr{C}) = \bigwedge_{x \in X} \mathscr{C}(\{x\}).$$

Definition 4.4 ([7])**.** For an M-fuzzifying convex space $(X, \mathscr{C})$, define the degree to which $(X, \mathscr{C})$ is S_2 separated as follows:

$$S_2(X, \mathscr{C}) = \bigwedge_{x \neq y} \bigvee_{x \in A, y \notin A} \mathscr{C}(A) \wedge \mathscr{C}(X - A).$$

For the productivity of M-fuzzifying convex spaces, Liang and Li presented the following proposition.

Proposition 4.5 ([7])**.** *Suppose that $\{(X_i, \mathscr{C}_i)\}_{i \in I}$ is a family of M-fuzzifying convex spaces. Then*

(1) $\bigwedge_{i \in I} S_0(X_i, \mathscr{C}_i) \leq S_0(\prod_{i \in I} X_i, \prod_{i \in I} \mathscr{C}_i)$,
(2) $\bigwedge_{i \in I} S_1(X_i, \mathscr{C}_i) \leq S_1(\prod_{i \in I} X_i, \prod_{i \in I} \mathscr{C}_i)$,
(3) $\bigwedge_{i \in I} S_2(X_i, \mathscr{C}_i) \leq S_2(\prod_{i \in I} X_i, \prod_{i \in I} \mathscr{C}_i)$.

From a logic aspect, the above proposition gives the degree characterization of the conclusion that if a family of convex spaces is S_i $(i = 0, 1, 2)$, then their product space is S_i $(i = 0, 1, 2)$. Actually, in the classical case, a family of convex spaces is S_1 if and only if their product space is S_1. So in a degree sense, the inverse of the inequality in Proposition 4.5 (2) also holds. To this end, we present the following proposition.

Proposition 4.6. *Let $\{(X_i, \mathscr{C}_i)\}_{i \in I}$ be a family of M-fuzzifying convex spaces and let $(\prod_{i \in I} X_i, \prod_{i \in I} \mathscr{C}_i)$ be the product space. Then*

$$S_1\Big(\prod_{i \in I} X_i, \prod_{i \in I} \mathscr{C}_i\Big) = \bigwedge_{i \in I} S_1(X_i, \mathscr{C}_i).$$

Proof. For convenience, let $X = \prod_{i \in I} X_i$ and $\mathscr{C} = \prod_{i \in I} \mathscr{C}_i$. In Proposition 4.5, it was proved that $\bigwedge_{i \in I} S_1(X_i, \mathscr{C}_i) \leq S_1(X, \mathscr{C})$. Now it remains to verify that

$$S_1(X, \mathscr{C}) \leq \bigwedge_{i \in I} S_1(X_i, \mathscr{C}_i).$$

Take any $\alpha \in M$ such that

$$\alpha \leq S_1(X, \mathscr{C}) = \bigwedge_{x \in X} \mathscr{C}(\{x\}).$$

Then it follows that $\alpha \leq \mathscr{C}(\{x\})$ for each $x \in X$. For each $i_0 \in I$ and $x_{i_0} \in X_{i_0}$, take $x \in X$ such that $p_{i_0}(x) = x_{i_0}$. Since $p_{i_0} : (X, \mathscr{C}) \longrightarrow (X_{i_0}, \mathscr{C}_{i_0})$ is a CC mapping, it follows that

$$\mathscr{C}_{i_0}(\{x_{i_0}\}) = \mathscr{C}_{i_0}(p_{i_0}^{\rightarrow}(\{x\})) \geq \mathscr{C}(\{x\}) \geq \alpha.$$

By the arbitrariness of i_0 and x_{i_0}, we have

$$\alpha \leq \bigwedge_{i_0 \in I} \bigwedge_{x_{i_0} \in X_{i_0}} \mathscr{C}_{i_0}(\{x_{i_0}\}) = \bigwedge_{i \in I} S_1(X_i, \mathscr{C}_i).$$

By the arbitrariness of α, we obtain

$$S_1(X, \mathscr{C}) \leq \bigwedge_{i \in I} S_1(X_i, \mathscr{C}_i),$$

as desired. □

In the classical convex spaces, there are close relationships between separation axioms and CP and CC mappings. Now these concepts have been generalized with some degrees. So we will consider their relationships in a degree sense. To this end, we first recall the definitions of M-CP degrees and M-CC degrees between M-fuzzifying convex spaces.

Definition 4.7 ([25]). Let $(X, \mathscr{C}_X)$ and $(X, \mathscr{C}_Y)$ be M-fuzzifying convex spaces, and let $f : X \longrightarrow Y$ be a mapping. Then $D_{cp}(f)$ defined by

$$D_{cp}(f) = \bigwedge_{B \in 2^Y} \Big(\mathscr{C}_Y(B) \to \mathscr{C}_X(f^{\leftarrow}(B))\Big)$$

is called the M-CP degree of f.

Definition 4.8 ([25]). Let $(X, \mathscr{C}_X)$ and $(X, \mathscr{C}_Y)$ be M-fuzzifying convex spaces, and let $f : X \longrightarrow Y$ be a mapping. Then $D_{cc}(f)$ defined by

$$D_{cc}(f) = \bigwedge_{A \in 2^X} \Big(\mathscr{C}_X(A) \to \mathscr{C}_Y(f^{\rightarrow}(A))\Big)$$

is called the M-CC degree of f.

For a bijective and CP mapping $f : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ between classical convex spaces, if $(Y, \mathcal{C}_Y)$ is S_i $(i = 0, 1)$ separated, then so is $(X, \mathcal{C}_X)$. Now, we will give the degree characterization of this conclusion.

Proposition 4.9. *Let $(X, \mathscr{C}_X)$ and $(X, \mathscr{C}_Y)$ be M-fuzzifying convex spaces, and let $f : X \longrightarrow Y$ be a bijective mapping. Then*
(1) $D_{cp}(f) \leq S_0(Y, \mathscr{C}_Y) \to S_0(X, \mathscr{C}_X)$.
(2) $D_{cp}(f) \leq S_1(Y, \mathscr{C}_Y) \to S_1(X, \mathscr{C}_X)$.
(3) $D_{cp}(f) \leq S_2(Y, \mathscr{C}_Y) \to S_2(X, \mathscr{C}_X)$.

Proof. (1) and (2) can be verified in a similar way. We only verify (1).

(1) By the definitions of S_0 and D_{cp}, we have

$$
\begin{aligned}
& S_0(Y,\mathscr{C}_Y) \to S_0(X,\mathscr{C}_X) \\
= & \bigwedge_{y_1\neq y_2}\Big(\bigvee_{y_1\notin B_1\ni y_2}\mathscr{C}_Y(B_1)\vee\bigvee_{y_2\notin B_2\ni y_1}\mathscr{C}_Y(B_2)\Big)\to\bigwedge_{x_1\neq x_2}\Big(\bigvee_{x_1\notin A_1\ni x_2}\mathscr{C}_X(A_1)\vee\bigvee_{x_2\notin A_2\ni x_1}\mathscr{C}_X(A_2)\Big) \\
= & \bigwedge_{y_1\neq y_2}\Big(\bigvee_{y_1\notin B_1\ni y_2}\mathscr{C}_Y(B_1)\vee\bigvee_{y_2\notin B_2\ni y_1}\mathscr{C}_Y(B_2)\Big) \\
& \qquad \to \bigwedge_{f(x_1)\neq f(x_2)}\Big(\bigvee_{f(x_1)\notin f(A_1)\ni f(x_2)}\mathscr{C}_X(A_1)\vee\bigvee_{f(x_2)\notin f(A_2)\ni f(x_1)}\mathscr{C}_X(A_2)\Big) \\
\geq & \bigwedge_{y_1\neq y_2}\Big(\big(\bigvee_{y_1\notin B_1\ni y_2}\mathscr{C}_Y(B_1)\vee\bigvee_{y_2\notin B_2\ni y_1}\mathscr{C}_Y(B_2)\big) \\
& \qquad \to \bigwedge_{y_1\neq y_2}\big(\bigvee_{y_1\notin f(A_1)\ni y_2}\mathscr{C}_X(A_1)\vee\bigvee_{y_2\notin f(A_2)\ni y_1}\mathscr{C}_X(A_2)\big)\Big) \\
\geq & \bigwedge_{y_1\neq y_2}\Big(\big(\bigvee_{y_1\notin B_1\ni y_2}\mathscr{C}_Y(B_1)\vee\bigvee_{y_2\notin B_2\ni y_1}\mathscr{C}_Y(B_2)\big) \\
& \qquad \to \big(\bigvee_{y_1\notin B_1\ni y_2}\mathscr{C}_X(f^{\leftarrow}(B_1))\vee\bigvee_{y_2\notin B_2\ni y_1}\mathscr{C}_X(f^{\leftarrow}(B_2))\big)\Big) \\
\geq & \bigwedge_{y_1\neq y_2}\Big(\big(\bigvee_{y_1\notin B_1\ni y_2}\mathscr{C}_Y(B_1)\to\bigvee_{y_1\notin B_1\ni y_2}\mathscr{C}_X(f^{\leftarrow}(B_1))\big) \\
& \qquad \wedge\big(\bigvee_{y_2\notin B_2\ni y_1}\mathscr{C}_Y(B_2)\to\bigvee_{y_2\notin B_2\ni y_1}\mathscr{C}_X(f^{\leftarrow}(B_2))\big)\Big) \\
\geq & \bigwedge_{y_1\neq y_2}\Big(\big(\bigwedge_{y_1\notin B_1\ni y_2}(\mathscr{C}_Y(B_1)\to\mathscr{C}_X(f^{\leftarrow}(B_1)))\big)\wedge\big(\bigwedge_{y_2\notin B_2\ni y_1}(\mathscr{C}_Y(B_2)\to\mathscr{C}_X(f^{\leftarrow}(B_2)))\big)\Big) \\
\geq & \bigwedge_{B\in 2^X}\Big(\mathscr{C}_Y(B)\to\mathscr{C}_X(f^{\leftarrow}(B))\Big) \\
= & D_{cp}(f).
\end{aligned}
$$

(3) By the definitions of S_2 and D_{cp}, we have

$$
\begin{aligned}
& S_2(Y,\mathscr{C}_Y) \to S_2(X,\mathscr{C}_X) \\
= & \bigwedge_{y_1\neq y_2}\bigvee_{y_1\in B,y_2\notin B}\big(\mathscr{C}_Y(B)\wedge\mathscr{C}_Y(Y-B)\big)\to\bigwedge_{x_1\neq x_2}\bigvee_{x_1\in A,x_2\notin A}\big(\mathscr{C}_X(A)\wedge\mathscr{C}_X(X-A)\big) \\
\geq & \bigwedge_{y_1\neq y_2}\bigvee_{y_1\in B,y_2\notin B}\big(\mathscr{C}_Y(B)\wedge\mathscr{C}_Y(Y-B)\big) \\
& \qquad \to \bigwedge_{f(x_1)\neq f(x_2)}\bigvee_{f(x_1)\in f^{\rightarrow}(A),f(x_2)\notin f^{\rightarrow}(A)}\big(\mathscr{C}_X(A)\wedge\mathscr{C}_X(X-A)\big) \\
\geq & \bigwedge_{y_1\neq y_2}\bigvee_{y_1\in B,y_2\notin B}\big(\mathscr{C}_Y(B)\wedge\mathscr{C}_Y(Y-B)\big)\to\bigwedge_{y_1\neq y_2}\bigvee_{y_1\in f^{\rightarrow}(A),y_2\notin f^{\rightarrow}(A)}\big(\mathscr{C}_X(A)\wedge\mathscr{C}_X(X-A)\big) \\
\geq & \bigwedge_{y_1\neq y_2}\Big(\bigvee_{y_1\in B,y_2\notin B}\big(\mathscr{C}_Y(B)\wedge\mathscr{C}_Y(Y-B)\big)\to\bigvee_{y_1\in f^{\rightarrow}(A),y_2\notin f^{\rightarrow}(A)}\big(\mathscr{C}_X(A)\wedge\mathscr{C}_X(X-A)\big)\Big) \\
\geq & \bigwedge_{y_1\neq y_2}\Big(\bigvee_{y_1\in B,y_2\notin B}\big(\mathscr{C}_Y(B)\wedge\mathscr{C}_Y(Y-B)\big)\to\bigvee_{y_1\in B,y_2\notin B}\big(\mathscr{C}_X(f^{\leftarrow}(B))\wedge\mathscr{C}_X(X-f^{\leftarrow}(B))\big)\Big) \\
\geq & \bigwedge_{y_1\neq y_2}\bigwedge_{y_1\in B,y_2\notin B}\Big(\big(\mathscr{C}_Y(B)\wedge\mathscr{C}_Y(Y-B)\big)\to\big(\mathscr{C}_X(f^{\leftarrow}(B))\wedge\mathscr{C}_X(X-f^{\leftarrow}(B))\big)\Big) \\
\geq & \bigwedge_{y_1\neq y_2}\bigwedge_{y_1\in B,y_2\notin B}\big(\mathscr{C}_Y(B)\to\mathscr{C}_X(f^{\leftarrow}(B))\big)\wedge\big(\mathscr{C}_Y(Y-B)\to\mathscr{C}_X(f^{\leftarrow}(Y-B))\big) \\
\geq & \bigwedge_{B\in 2^Y}\big(\mathscr{C}_Y(B)\to\mathscr{C}_X(f^{\leftarrow}(B))\big)\wedge\big(\mathscr{C}_Y(Y-B)\to\mathscr{C}_X(f^{\leftarrow}(Y-B))\big)
\end{aligned}
$$

$$= \bigwedge_{C\in 2^X} \big(\mathscr{C}_Y(C) \to \mathscr{C}_X(f^{\leftarrow}(C))\big)$$
$$= D_{cp}(f).$$

□

Corollary 4.10. *Let $(X, \mathscr{C}_X)$ and $(X, \mathscr{C}_Y)$ be M-fuzzifying convex spaces, and let $f : X \longrightarrow Y$ be a bijective mapping. Then*
(1) $D_{cp}(f) \wedge S_0(Y, \mathscr{C}_Y) \leq S_0(X, \mathscr{C}_X)$.
(2) $D_{cp}(f) \wedge S_1(Y, \mathscr{C}_Y) \leq S_1(X, \mathscr{C}_X)$.
(3) $D_{cp}(f) \wedge S_2(Y, \mathscr{C}_Y) \leq S_2(X, \mathscr{C}_X)$.

Corollary 4.10 exactly demonstrates the degree characterization of the conclusion that for a bijective and CP mapping $f : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$, if $(Y, \mathcal{C}_Y)$ is S_i $(i = 0, 1, 2)$ separated, then so is $(X, \mathcal{C}_X)$.

For a bijective and CC mapping $f : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ between convex spaces, if $(X, \mathcal{C}_X)$ is S_i $(i = 0, 1, 2)$ separated, then so is $(Y, \mathcal{C}_Y)$. Now, we will give the degree characterization of this conclusion.

Proposition 4.11. *Let $(X, \mathscr{C}_X)$ and $(X, \mathscr{C}_Y)$ be M-fuzzifying convex spaces, and let $f : X \longrightarrow Y$ be a bijective mapping. Then*
(1) $D_{cc}(f) \leq S_0(X, \mathscr{C}_X) \to S_0(Y, \mathscr{C}_Y)$.
(2) $D_{cc}(f) \leq S_1(X, \mathscr{C}_X) \to S_1(Y, \mathscr{C}_Y)$.
(3) $D_{cc}(f) \leq S_2(X, \mathscr{C}_X) \to S_2(Y, \mathscr{C}_Y)$.

Proof. (1) and (2) can be verified in a similar way. We only verify (1).

(1) By the definitions of S_0 and D_{cc}, we have

$$\begin{aligned}
& S_0(X, \mathscr{C}_X) \to S_0(Y, \mathscr{C}_Y) \\
= & \bigwedge_{x_1\neq x_2} \Big(\bigvee_{x_1\notin A_1\ni x_2} \mathscr{C}_X(A_1) \vee \bigvee_{x_2\notin A_2\ni x_1} \mathscr{C}_X(A_2)\Big) \to \bigwedge_{y_1\neq y_2} \Big(\bigvee_{y_1\notin B_1\ni y_2} \mathscr{C}_Y(B_1) \vee \bigvee_{y_2\notin B_2\ni y_1} \mathscr{C}_Y(B_2)\Big) \\
= & \bigwedge_{x_1\neq x_2} \Big(\bigvee_{x_1\notin A_1\ni x_2} \mathscr{C}_X(A_1) \vee \bigvee_{x_2\notin A_2\ni x_1} \mathscr{C}_X(A_2)\Big) \\
& \qquad \to \bigwedge_{x_1\neq x_2} \Big(\bigvee_{x_1\notin f^{\leftarrow}(B_1)\ni x_2} \mathscr{C}_Y(B_1) \vee \bigvee_{x_2\notin f^{\leftarrow}(B_2)\ni x_1} \mathscr{C}_Y(B_2)\Big) \\
\geq & \bigwedge_{x_1\neq x_2} \Big(\big(\bigvee_{x_1\notin A_1\ni x_2} \mathscr{C}_X(A_1) \vee \bigvee_{x_2\notin A_2\ni x_1} \mathscr{C}_X(A_2)\big) \\
& \qquad \to \big(\bigvee_{x_1\notin A_1\ni x_2} \mathscr{C}_Y(f^{\rightarrow}(A_1)) \vee \bigvee_{x_2\notin A_2\ni x_1} \mathscr{C}_Y(f^{\rightarrow}(A_2))\big)\Big) \\
\geq & \bigwedge_{x_1\neq x_2} \Big(\big(\bigvee_{x_1\notin A_1\ni x_2} \mathscr{C}_X(A_1) \to \bigvee_{x_1\notin A_1\ni x_2} \mathscr{C}_Y(f^{\rightarrow}(A_1))\big) \\
& \qquad \wedge\big(\bigvee_{x_2\notin A_2\ni x_1} \mathscr{C}_X(A_2) \to \bigvee_{x_2\notin A_2\ni x_1} \mathscr{C}_Y(f^{\rightarrow}(A_2))\big)\Big) \\
\geq & \bigwedge_{x_1\neq x_2} \Big(\bigwedge_{x_1\notin A_1\ni x_2} \big(\mathscr{C}_X(A_1) \to \mathscr{C}_Y(f^{\rightarrow}(A_1))\big) \wedge \bigwedge_{x_2\notin A_2\ni x_1} \big(\mathscr{C}_X(A_2) \to \mathscr{C}_Y(f^{\rightarrow}(A_2))\big)\Big) \\
\geq & \bigwedge_{A\in 2^X} \big(\mathscr{C}_X(A) \to \mathscr{C}_Y(f^{\rightarrow}(A))\big) \\
= & D_{cc}(f).
\end{aligned}$$

(3) By the definitions of S_2 and D_{cc}, we have

$$\begin{aligned}
& S_2(X, \mathscr{C}_X) \to S_2(Y, \mathscr{C}_Y) \\
= & \bigwedge_{x_1\neq x_2} \bigvee_{x_1\in A, x_2\notin A} \mathscr{C}_X(A) \wedge \mathscr{C}_X(X - A) \to \bigwedge_{y_1\neq y_2} \bigvee_{y_1\in B, y_2\notin B} \mathscr{C}_Y(B) \wedge \mathscr{C}_Y(Y - B)
\end{aligned}$$

$$
\begin{aligned}
&\geq \bigwedge_{x_1\neq x_2}\bigvee_{x_1\in A, x_2\notin A}\mathscr{C}_X(A)\wedge\mathscr{C}_X(X-A)\to\bigwedge_{f(x_1)\neq f(x_2)}\bigvee_{f(x_1)\in B, f(x_2)\notin B}\mathscr{C}_Y(B)\wedge\mathscr{C}_Y(Y-B)\\
&\geq \bigwedge_{x_1\neq x_2}\bigvee_{x_1\in A, x_2\notin A}\mathscr{C}_X(A)\wedge\mathscr{C}_X(X-A)\to\bigwedge_{x_1\neq x_2}\bigvee_{x_1\in A, x_2\notin A}\mathscr{C}_Y(f^{\to}(A))\wedge\mathscr{C}_Y(Y-f^{\to}(A))\\
&\geq \bigwedge_{x_1\neq x_2}\Big(\bigvee_{x_1\in A, x_2\notin A}\mathscr{C}_X(A)\wedge\mathscr{C}_X(X-A)\to\bigvee_{x_1\in A, x_2\notin A}\mathscr{C}_Y(f^{\to}(A))\wedge\mathscr{C}_Y(Y-f^{\to}(A))\Big)\\
&\geq \bigwedge_{x_1\neq x_2}\bigwedge_{x_1\in A, x_2\notin A}(\mathscr{C}_X(A)\wedge\mathscr{C}_X(X-A)\to\mathscr{C}_Y(f^{\to}(A))\wedge\mathscr{C}_Y(Y-f^{\to}(A)))\\
&\geq \bigwedge_{x_1\neq x_2}\bigwedge_{x_1\in A, x_2\notin A}(\mathscr{C}_X(A)\to\mathscr{C}_Y(f^{\to}(A)))\wedge(\mathscr{C}_X(X-A)\to\mathscr{C}_Y(Y-f^{\to}(A))))\\
&\geq \bigwedge_{x_1\neq x_2}\bigwedge_{x_1\in A, x_2\notin A}(\mathscr{C}_X(A)\to\mathscr{C}_Y(f^{\to}(A)))\wedge(\mathscr{C}_X(X-A)\to\mathscr{C}_Y(f^{\to}(X-A))))\\
&\geq \bigwedge_{A\in 2^X}(\mathscr{C}_X(A)\to\mathscr{C}_Y(f^{\to}(A)))\\
&= D_{cc}(f).
\end{aligned}
$$

□

Corollary 4.12. *Let $(X,\mathscr{C}_X)$ and $(X,\mathscr{C}_Y)$ be M-fuzzifying convex spaces, and let $f: X\longrightarrow Y$ be a bijective mapping. Then*
(1) $D_{cc}(f)\wedge S_0(X,\mathscr{C}_X)\leq S_0(Y,\mathscr{C}_Y)$.
(2) $D_{cc}(f)\wedge S_1(X,\mathscr{C}_X)\leq S_1(Y,\mathscr{C}_Y)$.
(3) $D_{cc}(f)\wedge S_2(X,\mathscr{C}_X)\leq S_2(Y,\mathscr{C}_Y)$.

Corollary 4.12 exactly demonstrates the degree characterization of the conclusion that for a bijective and CC mapping $f:(X,\mathcal{C}_X)\longrightarrow(Y,\mathcal{C}_Y)$, if $(X,\mathcal{C}_X)$ is S_i $(i=0,1,2)$ separated, then so is $(Y,\mathcal{C}_Y)$.

5. Separated degrees of M-fuzzifying convex spaces by means of M-fuzzifying convergence structures

In [10], Pang introduced convergence structures in the framework of M-fuzzifying convex spaces, which are called M-fuzzifying convergence structures. As an application of M-fuzzifying convergence structures, Pang defined separated degrees of M-fuzzifying convex spaces by means of M-fuzzifying convergence structures. Notice that the separated degrees of M-fuzzifying convex spaces via M-fuzzifying convergence structures in [10] have some advantages compared with that in [7], especially on the productive properties.

In this section, we will go on investigating the separated degrees of M-fuzzifying convex spaces by means of M-fuzzifying convergence structures. In order to distinguish the separated degrees of M-fuzzifying convex spaces via M-fuzzifying convergence structures from that in M-fuzzifying convex spaces, we denote the separated degrees of M-fuzzifying convex spaces via M-fuzzifying convergence structures by S_i^* $(i=0,1,2)$.

Definition 5.1 ([10])**.** Let $(X,\lim)$ be an M-fuzzifying convergence space and define $S_0^c(X,\lim)$ by

$$S_0^c(X,\lim)=\bigwedge_{x\neq y}\big(\neg\lim([x])(y)\vee\neg\lim([y])(x)\big).$$

Then $S_0^c(X,\lim)$ is called the degree to which $(X,\lim)$ is S_0-separated.

Definition 5.2 ([10])**.** Let $(X,\lim)$ be an M-fuzzifying convergence space and define $S_1^c(X,\lim)$ by

$$S_1^c(X,\lim)=\bigwedge_{x\neq y}\big(\neg\lim([x])(y)\wedge\neg\lim([y])(x)\big).$$

Then $S_1^c(X,\lim)$ is called the degree to which $(X,\lim)$ is S_1-separated.

Definition 5.3 ([10]). Let $(X, \lim)$ be an M-fuzzifying convergence space and define $S_2^c(X, \lim)$ by

$$S_2^c(X,\lim) = \bigwedge_{x \neq y} \bigwedge_{\mathcal{F} \in \mathcal{F}_M(X)} \big(\neg \lim(\mathcal{F})(x) \vee \neg \lim(\mathcal{F})(y)\big).$$

Then $S_2^c(X, \lim)$ is called the degree to which $(X, \lim)$ is S_2-separated.

The separated degree $S_i^c(X, \lim)$ $(i = 0, 1, 2)$ of an M-fuzzifying convergence space $(X, \lim)$ describes the degree to which $(X, \lim)$ is S_i $(i = 0, 1, 2)$ separated. Then the separated degrees of an M-fuzzifying convex space $(X, \mathscr{C})$ can be defined by means of its induced M-fuzzifying convergence space $(X, \lim^{\mathscr{C}})$.

Definition 5.4 ([10]). Suppose that $(X, \mathscr{C})$ is an M-fuzzifying convex space. Then

$$S_i^*(X,\mathscr{C}) = S_i^c(X, \lim^{\mathscr{C}}) \ (i = 0, 1)$$

is called the degree to which $(X, \mathscr{C})$ is weakly S_i-separated.

Definition 5.5 ([10]). Suppose that $(X, \mathscr{C})$ is an M-fuzzifying convex space. Then

$$S_2^*(X,\mathscr{C}) = S_2^c(X, \lim^{\mathscr{C}})$$

is called the degree to which $(X, \mathscr{C})$ is S_2^*-separated.

Note that for an M-fuzzifying convex space, there are two ways to define its S_i-separated degrees $(i = 0, 1, 2)$. In the following, we will show the the relations between two kinds of separated degrees.

Proposition 5.6. *Suppose that $(X, \mathscr{C})$ is an M-fuzzifying convex space. Then*
(1) $S_0^*(X, \mathscr{C}) \geq S_0(X, \mathscr{C})$.
(2) $S_1^*(X, \mathscr{C}) \geq S_1(X, \mathscr{C})$.

Proof. (1) By the definitions of $S_0^*(X, \mathscr{C})$ and $S_0(X, \mathscr{C})$, we have

$$\begin{aligned}
S_0^*(X,\mathscr{C}) &= S_0^c(X, \lim^{\mathscr{C}}) \\
&= \bigwedge_{x \neq y} \big(\neg \lim^{\mathscr{C}}([x])(y) \vee \neg \lim^{\mathscr{C}}([y])(x)\big) \\
&= \bigwedge_{x \neq y} \Big(\big(\big(\bigwedge_{y \in A} \big(\mathscr{C}(X - A) \to [x](A)\big)\big) \to \bot\big) \vee \\
&\qquad \big(\big(\bigwedge_{x \in B} \big(\mathscr{C}(X - B) \to [y](B)\big)\big) \to \bot\big)\Big) \\
&= \bigwedge_{x \neq y} \Big(\big(\big(\bigwedge_{x \notin A \ni y} \big(\mathscr{C}(X - A) \to \bot\big)\big) \to \bot\big) \vee \\
&\qquad \big(\big(\bigwedge_{y \notin B \ni x} \big(\mathscr{C}(X - B) \to \bot\big)\big) \to \bot\big)\Big) \\
&= \bigwedge_{x \neq y} \Big(\big(\big(\bigwedge_{y \notin B \ni x} \big(\mathscr{C}(B) \to \bot\big)\big) \to \bot\big) \vee \\
&\qquad \big(\big(\bigwedge_{x \notin A \ni y} \big(\mathscr{C}(A) \to \bot\big)\big) \to \bot\big)\Big) \\
&\geq \bigwedge_{x \neq y} \Big(\bigvee_{x \notin A \ni y} \mathscr{C}(A) \vee \bigvee_{y \notin B \ni x} \mathscr{C}(B)\Big) \\
&= S_0(X, \mathscr{C}).
\end{aligned}$$

(2) It can be proved in a similar way. □

By the above proposition, we know that if an M-fuzzifying convex space is S_i $(i = 0, 1)$ separated in the sense of Definition 4.1, then it is S_i $(i = 0, 1)$ separated in the sense of Definition 5.4. That's why we call weakly S_i $(i = 0, 1)$ separated in Definition 5.4.

When $M = \{0,1\}$, S_2-separated degrees and S_2^*-separated degrees will reduce to the classical S_2 and S_2^* separation axioms in convex spaces. It is easy to see that classical S_2 and S_2^* axioms are independent. This means that the S_2-separated degrees in Definition 4.4 and S_2^* separated degrees in Definition 5.5 are incomparable.

In the sequel, we will further investigate the properties of separated degrees of an M-fuzzifying convex space in the sense of Definitions 5.4 and 5.5.

From the aspect of convergence structures, an M-CP mapping between M-fuzzifying convergence spaces is defined, which can be used to characterize the M-CP mapping between M-fuzzifying convex spaces. In order to equip the M-CP mapping between M-fuzzifying convergence spaces with some degrees, we first recall the definition of an M-CP mapping between M-fuzzifying convergence spaces.

Definition 5.7 ([10])**.** A mapping $f : (X, \lim_X) \longrightarrow (Y, \lim_Y)$ between M-fuzzifying convergence spaces is called M-fuzzifying convexity-preserving (M-CP, in short) provided that

$$\lim_X(\mathcal{F})(x) \le \lim_Y(f^{\Rightarrow}(\mathcal{F}))(f(x))$$

for each $\mathcal{F} \in \mathcal{F}_M(X)$ and $x \in X$.

Using the degree approach, we can define the degree to which a mapping between M-fuzzifying convergence spaces is M-CP.

Definition 5.8. Let $(X, \lim_X)$ and $(Y, \lim_Y)$ be M-fuzzifying convergence spaces, and let $f : X \longrightarrow Y$ be a mapping. Then $D^c_{cp}(f)$ defined by

$$D^c_{cp}(f) = \bigwedge_{x\in X} \bigwedge_{\mathcal{F}\in\mathcal{F}_M(X)} \Big(\lim_X(\mathcal{F})(x) \to \lim_Y(f^{\Rightarrow}(\mathcal{F}))(f(x))\Big)$$

is called the M-CP degree of f.

Actually, Definition 5.8 provides a degree approach to M-CP mappings from the aspect of M-fuzzifying convergence structures. This can also be used to give a new definition of M-CP degrees of a mapping between M-fuzzifying convex spaces.

Definition 5.9. Let $(X, \mathscr{C}_X)$ and $(Y, \mathscr{C}_Y)$ be M-fuzzifying convex spaces, and let $f : X \longrightarrow Y$ be a mapping. Then $D^*_{cp}(f)$ defined by

$$D^*_{cp}(f) = D^c_{cp}(f) = \bigwedge_{x\in X} \bigwedge_{\mathcal{F}\in\mathcal{F}_M(X)} \Big(\lim^{\mathscr{C}_X}(\mathcal{F})(x) \to \lim^{\mathscr{C}_Y}(f^{\Rightarrow}(\mathcal{F}))(f(x))\Big)$$

is called the weak M-CP degree of f.

Proposition 5.10. *Let $(X, \mathscr{C}_X)$ and $(Y, \mathscr{C}_Y)$ be M-fuzzifying convex spaces, and let $f : X \longrightarrow Y$ be a mapping. Then*

$$D_{cp}(f) \le D^*_{cp}(f).$$

Proof. By the definition of $D_{cp}(f)$ and $D^*_{cp}(f)$, we have

$$\begin{aligned}
& D^*_{cp}(f) \\
= & \bigwedge_{x\in X} \bigwedge_{\mathcal{F}\in\mathcal{F}_M(X)} (\lim^{\mathscr{C}_X}(\mathcal{F})(x) \to \lim^{\mathscr{C}_Y}(f^{\Rightarrow}(\mathcal{F}))(f(x))) \\
= & \bigwedge_{x\in X} \bigwedge_{\mathcal{F}\in\mathcal{F}_M(X)} \Big(\bigwedge_{x\in A} \big(\mathscr{C}_X(X-A) \to \mathcal{F}^{\to}(A)\big) \to \bigwedge_{f(x)\in B} \big(\mathscr{C}_Y(Y-B) \to f^{\Rightarrow}(\mathcal{F})(B)\big)\Big) \\
= & \bigwedge_{x\in X} \bigwedge_{\mathcal{F}\in\mathcal{F}_M(X)} \bigwedge_{f(x)\in B} \Big(\bigwedge_{x\in A} \big(\mathscr{C}_X(X-A) \to \mathcal{F}^{\to}(A)\big) \to \big(\mathscr{C}_Y(Y-B) \to f^{\Rightarrow}(\mathcal{F})(B)\big)\Big) \\
= & \bigwedge_{x\in X} \bigwedge_{\mathcal{F}\in\mathcal{F}_M(X)} \bigwedge_{x\in f^{\leftarrow}(B)} \Big(\bigwedge_{x\in A} \big(\mathscr{C}_X(X-A) \to \mathcal{F}^{\to}(A)\big) \to \big(\mathscr{C}_Y(Y-B) \to \mathcal{F}(f^{\leftarrow}(B))\big)\Big)
\end{aligned}$$

$$
\begin{aligned}
&\geq \bigwedge_{x\in X}\bigwedge_{\mathcal{F}\in\mathcal{F}_M(X)}\bigwedge_{x\in f^{\leftarrow}(B)}\Big(\big(\mathscr{C}_X(X-f^{\leftarrow}(B))\to\mathcal{F}(f^{\leftarrow}(B))\big)\to\big(\mathscr{C}_Y(Y-B)\to\mathcal{F}(f^{\leftarrow}(B))\big)\Big)\\
&\geq \bigwedge_{x\in X}\bigwedge_{\mathcal{F}\in\mathcal{F}_M(X)}\bigwedge_{x\in f^{\leftarrow}(B)}\Big(\mathscr{C}_Y(Y-B)\to\mathscr{C}_X(X-f^{\leftarrow}(B))\Big)\\
&= \bigwedge_{x\in X}\bigwedge_{\mathcal{F}\in\mathcal{F}_M(X)}\bigwedge_{x\in f^{\leftarrow}(B)}\Big(\mathscr{C}_Y(Y-B)\to\mathscr{C}_X(f^{\leftarrow}(Y-B))\Big)\\
&\geq \bigwedge_{C\in 2^X}\Big(\mathscr{C}_Y(C)\to\mathscr{C}_X(f^{\leftarrow}(C))\Big)\\
&= D_{cp}(f),
\end{aligned}
$$

as desired. □

In Definitions 4.7 and 5.9, we provided two different ways to define the M-CP degree of a mapping between two M-fuzzifying convex spaces. By Proposition 5.10, we know that if a mapping $f:(X,\mathscr{C}_X)\longrightarrow(Y,\mathscr{C}_Y)$ is M-CP in the sense of Definition 4.7, then it is M-CP in the sense of Definition 5.9. That's why we call $D^*_{cp}(f)$ in Definition 5.9 weak M-CP degree.

In the following proposition, we present the relationships between separation axioms and M-CP mappings with respect to M-fuzzifying convergence spaces in a degree sense.

Proposition 5.11. *Let $(X,\mathscr{C}_X)$ and $(X,\mathscr{C}_Y)$ be M-fuzzifying convergence spaces, and let $f:X\longrightarrow Y$ be an injective mapping. Then*
(1) $D^c_{cp}(f)\leq S^c_0(Y,\lim_Y)\to S^c_0(X,\lim_X)$.
(2) $D^c_{cp}(f)\leq S^c_1(Y,\lim_Y)\to S^c_1(X,\lim_X)$.
(3) $D^c_{cp}(f)\leq S^c_2(Y,\lim_Y)\to S^c_2(X,\lim_X)$.

Proof. (1) and (2) can be verified in a similar way. We only verify (1).

(1) By the definitions of S^c_0 and D^c_{cp}, we have

$$
\begin{aligned}
&S^c_0(Y,\lim_Y)\to S^c_0(X,\lim_X)\\
&= \bigwedge_{y_1\neq y_2}\big(\neg\lim_Y([y_1])(y_2)\vee\neg\lim_Y([y_2])(y_1)\big)\\
&\qquad\to\bigwedge_{x_1\neq x_2}\big(\neg\lim_X([x_1])(x_2)\vee\neg\lim_X([x_2])(x_1)\big)\\
&\geq \bigwedge_{f(x_1)\neq f(x_2)}\big(\neg\lim_Y([f(x_1)])(f(x_2))\vee\neg\lim_Y([f(x_2)])(f(x_1))\big)\\
&\qquad\to\bigwedge_{x_1\neq x_2}\big(\neg\lim_X([x_1])(x_2)\vee\neg\lim_X([x_2])(x_1)\big)\\
&= \bigwedge_{x_1\neq x_2}\big(\neg\lim_Y([f(x_1)])(f(x_2))\vee\neg\lim_Y([f(x_2)])(f(x_1))\big)\\
&\qquad\to\bigwedge_{x_1\neq x_2}\big(\neg\lim_X([x_1])(x_2)\vee\neg\lim_X([x_2])(x_1)\big)\\
&\geq \bigwedge_{x_1\neq x_2}\Big(\big(\neg\lim_Y([f(x_1)])(f(x_2))\vee\neg\lim_Y([f(x_2)])(f(x_1))\big)\\
&\qquad\to\big(\neg\lim_X([x_1])(x_2)\vee\neg\lim_X([x_2])(x_1)\big)\Big)\\
&\geq \bigwedge_{x_1\neq x_2}\Big(\big(\lim_X([x_1])(x_2)\to\lim_Y([f(x_1)])(f(x_2))\big)\\
&\qquad\wedge\big(\lim_X([x_2])(x_1)\to\lim_Y([f(x_2)])(f(x_1))\big)\Big)\\
&\geq \bigwedge_{x\in X}\bigwedge_{\mathcal{F}\in\mathcal{F}_M(X)}\Big(\lim_X(\mathcal{F})(x)\to\lim_Y(f^{\Rightarrow}(\mathcal{F}))(f(x))\Big)\\
&= D^c_{cp}(f).
\end{aligned}
$$

(3) By the definitions of S_2^c and D_{cp}^c, we have

$$\begin{aligned}
& S_2^c(Y,\lim_Y) \to S_2^c(X,\lim_X) \\
= & \bigwedge_{y_1\neq y_2}\bigwedge_{\mathcal{F}\in\mathcal{F}_M(Y)} \big(\neg\lim_Y(\mathcal{F})(y_2)\vee\neg\lim_Y(\mathcal{F})(y_1)\big) \\
& \qquad \to \bigwedge_{x_1\neq x_2}\bigwedge_{\mathcal{G}\in\mathcal{F}_M(X)} \big(\neg\lim_X(G)(x_2)\vee\neg\lim_X(G)(x_1)\big) \\
\geq & \bigwedge_{f(x_1)\neq f(x_2)}\bigwedge_{\mathcal{F}\in\mathcal{F}_M(Y)} \big(\neg\lim_Y(\mathcal{F})(f(x_2))\vee\neg\lim_Y(\mathcal{F})(f(x_1))\big) \\
& \qquad \to \bigwedge_{x_1\neq x_2}\bigwedge_{\mathcal{G}\in\mathcal{F}_M(X)} \big(\neg\lim_X(G)(x_2)\vee\neg\lim_X(G)(x_1)\big) \\
= & \bigwedge_{x_1\neq x_2}\bigwedge_{\mathcal{F}\in\mathcal{F}_M(Y)} \big(\neg\lim_Y(\mathcal{F})(f(x_2))\vee\neg\lim_Y(\mathcal{F})(f(x_1))\big) \\
& \qquad \to \bigwedge_{x_1\neq x_2}\bigwedge_{\mathcal{G}\in\mathcal{F}_M(X)} \big(\neg\lim_X(G)(x_2)\vee\neg\lim_X(G)(x_1)\big) \\
\geq & \bigwedge_{x_1\neq x_2}\bigwedge_{\mathcal{G}\in\mathcal{F}_M(X)} \big(\neg\lim_Y(f^{\Rightarrow}(\mathcal{G}))(f(x_2))\vee\neg\lim_Y(f^{\Rightarrow}(\mathcal{G}))(f(x_1))\big) \\
& \qquad \to \bigwedge_{x_1\neq x_2}\bigwedge_{\mathcal{G}\in\mathcal{F}_M(X)} \big(\neg\lim_X(\mathcal{G})(x_2)\vee\neg\lim_X(\mathcal{G})(x_1)\big) \\
\geq & \bigwedge_{x_1\neq x_2}\bigwedge_{\mathcal{G}\in\mathcal{F}_M(X)} \Big(\big(\neg\lim_Y(f^{\Rightarrow}(\mathcal{G}))(f(x_2))\vee\neg\lim_Y(f^{\Rightarrow}(\mathcal{G}))(f(x_1))\big) \\
& \qquad \to \big(\neg\lim_X(\mathcal{G})(x_2)\vee\neg\lim_X(\mathcal{G})(x_1)\big)\Big) \\
\geq & \bigwedge_{x_1\neq x_2}\bigwedge_{\mathcal{G}\in\mathcal{F}_M(X)} \Big(\big(\lim_X(\mathcal{G})(x_2)\to\lim_Y(f^{\Rightarrow}(\mathcal{G}))(f(x_2))\big) \\
& \qquad \wedge\big(\lim_X(\mathcal{G})(x_1)\to\lim_Y(f^{\Rightarrow}(\mathcal{G}))(f(x_1))\big)\Big) \\
\geq & \bigwedge_{x\in X}\bigwedge_{\mathcal{G}\in\mathcal{F}_M(X)} \Big(\lim_X(\mathcal{G})(x)\to(\lim_Y(f^{\Rightarrow}(\mathcal{G}))(f(x)))\Big) \\
= & D_{cp}^c(f).
\end{aligned}$$

□

Corollary 5.12. *Let $(X,\mathscr{C}_X)$ and $(X,\mathscr{C}_Y)$ be M-fuzzifying convergence spaces, and let $f: X\longrightarrow Y$ be an injective mapping. Then*
(1) $D_{cp}^c(f)\wedge S_0^c(Y,\lim_Y)\leq S_0^c(X,\lim_X)$.
(2) $D_{cp}^c(f)\wedge S_1^c(Y,\lim_Y)\leq S_1^c(X,\lim_X)$.
(3) $D_{cp}^c(f)\wedge S_2^c(Y,\lim_Y)\leq S_2^c(X,\lim_X)$.

Proposition 5.13. *Let $(X,\mathscr{C}_X)$ and $(X,\mathscr{C}_Y)$ be M-fuzzifying convex spaces, and let $f: X\longrightarrow Y$ be an injective mapping. Then*
(1) $D_{cp}(f)\leq S_0^*(Y,\mathscr{C}_Y)\to S_0^*(X,\mathscr{C}_X)$.
(2) $D_{cp}(f)\leq S_1^*(Y,\mathscr{C}_Y)\to S_1^*(X,\mathscr{C}_X)$.
(3) $D_{cp}(f)\leq S_2^*(Y,\mathscr{C}_Y)\to S_2^*(X,\mathscr{C}_X)$.

Proof. By Propositions 5.10 and 5.11, for each $i=0,1,2$, we have

$$\begin{aligned}
D_{cp}(f) &\leq D_{cp}^*(f)=D_{cp}^c(f) \\
&\leq S_i^c(Y,\lim^{\mathscr{C}_Y})\to S_i^c(X,\lim^{\mathscr{C}_X}) \\
&= S_i^*(Y,\mathscr{C}_Y)\to S_i^*(X,\mathscr{C}_X),
\end{aligned}$$

as desired. □

Corollary 5.14. *Let* $(X, \mathscr{C}_X)$ *and* $(X, \mathscr{C}_Y)$ *be* M*-fuzzifying convex spaces, and let* $f : X \longrightarrow Y$ *be an injective mapping. Then*
(1) $D_{cp}(f) \wedge S_0^*(Y, \mathscr{C}_Y) \leq S_0^*(X, \mathscr{C}_X)$.
(2) $D_{cp}(f) \wedge S_1^*(Y, \mathscr{C}_Y) \leq S_1^*(X, \mathscr{C}_X)$.
(3) $D_{cp}(f) \wedge S_2^*(Y, \mathscr{C}_Y) \leq S_2^*(X, \mathscr{C}_X)$.

In Sections 4 and 5, we discussed the relationships between separated degrees and M-CP degrees of mappings between M-fuzzifying convex spaces. From Propositions 4.9 and 5.13, it is observed that using M-fuzzifying convergence structures to define separated degrees has more advantages.

6. Conclusions

In this paper, we mainly made further research on S_i $(i = 0, 1, 2)$ separated degrees of an M-fuzzifying convex space from two aspects. Section 4 focused on separated degrees of an M-fuzzifying convex space in the sense of Liang and Li [7]. Section 5 focused on separated degrees of an M-fuzzifying convex space in the sense of Pang [10]. Based on the relationships between separated degrees and M-CP degrees, we can see that separated degrees of an M-fuzzifying convex space via convergence structures have more advantages. Following this paper, we will consider the following problems as the future work:

- Defining S_3 and S_4 separated degrees of an M-fuzzifying convex space by means of its induced M-fuzzifying convergence structure and study their productivity.
- Defining M-CC degrees between M-fuzzifying convex spaces by means of M-fuzzifying convergence structures and study their relationships with M-CC degrees in Definition 4.8.
- Investigating the relationships between S_i $(i = 3, 4)$ separated degrees of an M-fuzzifying convex space and M-CP degrees between M-fuzzifying convex spaces.

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