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Classification of Vector Fields and Soliton Structures on a Tangent Bundle with a Ricci Quarter-Symmetric Metric Connection

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ABSTRACT

Consider TM as the tangent bundle of a (pseudo-)Riemannian manifold M, equipped with a Ricci quarter-symmetric metric connection $\overline{\nabla}$. This research article aims to accomplish two primary objectives. Firstly, the paper undertakes the classification of specific types of vector fields, including incompressible vector fields, harmonic vector fields, concurrent vector fields, conformal vector fields, projective vector fields, and $\widetilde{\varphi}(Ric)$ vector fields within the framework of $\overline{\nabla}$ on $T\dot{M}$. Secondly, the paper establishes the necessary and sufficient conditions for the tangent bundle TM to become as a Riemannian soliton and a generalized Ricci-Yamabe soliton with regard to the connection $\overline{\nabla}$.

Keywords: *Complete lift metric, Ricci quarter-symmetric metric connection, tangent bundle, vector field, Riemannian soliton, generalized Ricci-Yamabe soliton.*

AMS Subject Classification (2020): *Primary 53C07; Secondary 53A45.*

1. Introduction

Following the introduction of the concept of a semi-symmetric linear connection on a differentiable manifold by Friedmann and Schouten in [2], Yano [15] extended this notion to the realm of Riemannian manifolds by introducing what is known as a semi-symmetric metric connection, abbreviated as SSMC. Yano's inspiration for this extension stemmed from Hayden's work on torsional metric connections, as documented in [8]. Yano's contribution includes the proof that for a Riemannian manifold to possess a semi-symmetric metric connection with a vanishing curvature tensor, it is necessary and sufficient that the manifold be conformally flat.

Subsequently, Golab introduced the concept of a quarter-symmetric connection on differentiable manifolds in [5]. However, the most comprehensive form of quarter-symmetric metric connections on Riemannian, Hermitian, and Kaehlerian manifolds was presented by Yano and Imai [17]. A linear connection is classified as a quarter-symmetric connection if the torsion tensor T of any connection takes on the specific form:

$$T(\xi_1, \xi_2) = u(\xi_2)\phi\xi_1 - u(\xi_1)\phi\xi_2. \tag{1.1}$$

Here, u represents a non-zero 1-form, ϕ stands for a $(1,1)-$ tensor, and ξ_i $(i = 1, 2)$ denote vector fields. Notably, when the tensor ϕ coincides with the identity tensor ($\phi = id$), the quarter-symmetric connection simplifies to the semi-symmetric connection. This reveals that the notion of a quarter-symmetric connection serves as a broader framework encompassing the concept of a semi-symmetric connection. It is evident that a quarter-symmetric metric connection aligns with Hayden's connection characterized by the given torsion tensor (1.1). This implies a connection between these different geometric structures, shedding light on the intricate relationships among them.

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If we define the tensor ϕ as a $(1,1)$-type Ricci tensor, which is specified as:

$$g(\phi\xi_1, \xi_2) = R(\xi_1, \xi_2),$$

where R is the Ricci tensor of a Riemannian manifold, then the resulting connection, which maintains quarter-symmetry, is referred to as a Ricci quarter-symmetric connection. In the case where a Ricci quarter-symmetric connection ∇ is established on a Riemannian manifold and satisfies the condition:

$$(\nabla_{\xi_1} g)(\xi_2, \xi_3) = 0,$$

∇ is denoted as a Ricci quarter-symmetric metric connection, often abbreviated as RQSMC and this characterization holds true for all vector fields ξ_1,ξ_2,ξ_3 on M. The concept of RQSMC was introduced and extensively studied by Kamilya and De [10]. They also succeeded in identifying the necessary and sufficient conditions for the symmetry of the Ricci tensor associated with an RQSMC, providing valuable insights into this geometric structure. Furthermore, the paper [11] derives inequalities for submanifolds in real space forms with a Ricci quarter-symmetric metric connection. These inequalities establish the interrelation among Ricci curvature, scalar curvature, and mean curvature under the influence of the Ricci quarter-symmetric metric connection.

The Poincaré conjecture, a famous unresolved problem in 20th-century mathematics, was first proposed by Henri Poincaré in the early 1900s. It was based on the idea that a closed, simply connected 3-dimensional manifold should be isomorphic to a sphere (S^3). In the 1970s, William Thurston made significant contributions to this problem, and in 1982, he introduced the Geometrization Conjecture, a broader hypothesis suggesting that all 3-dimensional Riemannian manifolds can be classified in a similar manner. The concept of Ricci flow gained prominence following Grigori Perelman's proof of the Geometrization Conjecture and Richard Brendle and Simon Schoen's proof of the Differentiable Sphere Theorem. Ricci flow is a mathematical process that transforms the metric (essentially, the shape) of a manifold M into a metric with a constant curvature, often resembling a sphere, while being proportional to the Ricci tensor. Ricci flow and other geometric flows have diverse applications across various fields, including topology, geometry, and physics. They have been utilized in areas such as string theory, thermodynamics, the general theory of relativity and cosmology, quantum field theory, and the uniformization theorem, among others.

Hamilton introduced the concept of the Yamabe flow to address the Yamabe problem, which aims to find a metric on a given compact Riemannian manifold that conforms to the original metric g while maintaining a constant scalar curvature. It's worth noting that while the Ricci and Yamabe flows are equivalent in two dimensions, they exhibit fundamental differences in higher dimensions. In higher dimensions, the Yamabe soliton preserves the conformal class of the metric, whereas the Ricci soliton typically does not. Hamilton's work on the Ricci and Yamabe flows inspired further exploration by mathematicians like Udrişte, who delved into the realm of Riemannian flows. Udrişte extended the concept of the Ricci flow in a natural manner to create a nonlinear partial differential equation (PDE) that involves the Riemannian curvature tensor. In this framework, the metric g is interpreted as the solution to this aforementioned PDE. The notion of a Ricci soliton is substituted with that of a Riemannian soliton in Udrişte's approach. A Riemannian soliton can be viewed as a kinematic solution within the context of Riemannian flow, and its profile offers a broader perspective that encompasses spaces of constant sectional curvature, as described in [19].

The foundation of the concept of the tangent bundle TM is rooted in Sasaki's seminal paper from 1958, which holds a foundational role in this field [12]. In this pioneering work, Sasaki constructed a metric, denoted as $\widetilde{g}$ on TM by leveraging the Riemannian metric g dinitially defined on a differentiable manifold M. This metric $\widetilde{g}$ established on the tangent bundle TM is commonly referred to as the Sasaki metric. Various important metrics on TM have been devised through classical lifting techniques, all relying on the Riemannian metric g defined on M. These metrics include:

1. The Sasaki metric (or the metric $I + III$);
2. The metric $II + III$;
3. The complete lift metric (or the metric II);
4. The metric $I + II$; where $I = g_{ij}dx^i dx^j$, $II = 2g_{ij}dx^i\delta y^j$, $III = g_{ij}\delta y^i \delta y^j$ are all quadratic differential forms defined on TM [18]. These metrics play a pivotal role in the analysis and understanding of tangent bundles and their geometric properties.

In our research article [4], we introduce an original concept: a RQSMC $\overline{\nabla}$ on the tangent bundle TM, complemented by the complete lift metric ${}^{C}g$, defined over a Riemannian manifold M. In our investigation, we meticulously compute all curvature tensors associated with $\overline{\nabla}$ and conduct a comprehensive analysis of its inherent properties. Furthermore, we define the mean connection stemming from $\overline{\nabla}$, which contributes to a deeper understanding of the connection's behavior and characteristics. Our investigation extends to the realms of Ricci and gradient Ricci solitons, which have been subjects of extensive recent research. Within this context, we establish necessary and sufficient conditions that determine when the tangent bundle TM transforms into a Ricci soliton or a gradient Ricci soliton with respect to the connection $\overline{\nabla}$. Finally, our study delves into the quest for conditions under which the tangent bundle TM achieves local conformal flatness in relation to $\overline{\nabla}$. This exploration enhances our grasp of the local geometric properties of the tangent bundle in the context of the introduced RQSMC and provides valuable insights into the manifold's behavior under this specific connection.

The primary objective of this research article is to categorize specific types of vector fields—namely, incompressible, harmonic, concurrent, conformal, projective, and $\widetilde{\varphi}(Ric)$ vector fields based on their behavior with respect to the connection $\overline{\nabla}$ defined on the tangent bundle TM. Following this classification, our investigation delves into the establishment of conditions that govern vector fields $\widetilde{V}$ on TM, rendering them compatible with the structure $({}^{C}g, \widetilde{V}, \lambda)$, where ${}^{C}g$ represents the complete lift metric and λ denotes a constant. Specifically, we explore the criteria under which this structure qualifies as a Riemannian soliton and a generalized Ricci-Yamabe soliton. In essence, our study contributes to a deeper understanding of how various vector fields interact with the tangent bundle under the influence of the connection $\overline{\nabla}$ and sheds light on the conditions under which certain geometric structures, such as solitons, manifest themselves in this context.

2. Preliminaries

Consider an $n-$dimensional differentiable manifold M. The tangent bundle TM is a construction based on the assembly of disjoint tangent spaces, each corresponding to distinct points across M. For a given local coordinate system $\{U, x^h\}$ within M and using Cartesian coordinates (y^h) in each tangent space $T_P M$ at a point $P \in M$, established through the natural basis $\left\{\frac{\partial}{\partial x^h} \mid_P\right\}$, we can define a local coordinate system in TM denoted as $\{\pi^{-1}(U), x^h, y^h\}$. Here, π signifies the natural projection function defined as $\pi : TM \longmapsto M$ and P represents an arbitrary point belonging to U. Furthermore, the coordinate system (x^h, y^h) is referred to as the induced coordinates on $\pi^{-1}(U)$, originating from the initial coordinate system $\{U, x^h\}$ within M.

Suppose we have a vector field ξ defined within the open subset U of M, and it can be locally expressed as $\xi = \xi^h \frac{\partial}{\partial x^h}$. Additionally, let ∇ be a torsion-free linear connection on M. In this context, we can respectively define the vertical lift ${}^{V}\xi$, the horizontal lift ${}^{H}\xi$ and the complete lift ${}^{C}\xi$ of ξ by [18]

$$ {}^{V}\xi = \xi^h \partial_{\overline{h}}, $$

$$ {}^{H}\xi = \xi^h \partial_h - y^s \Gamma^h_{sk} \xi^k \partial_{\overline{h}} $$

and

$$ {}^{C}\xi = \xi^h \partial_h + y^s \partial_s \xi^h \partial_{\overline{h}} $$

according to the (x^h, y^h), where $\partial_h = \frac{\partial}{\partial x^h}, \partial_{\overline{h}} = \frac{\partial}{\partial y^h}$ and Γ^h_{jk} are the components of the connection ∇.

Suppose that a (p, q) tensor S is given on M by

$$ S = S^{j_1 \ldots j_p}_{l i_2 \ldots i_q} \partial_{\overline{j_1}} \otimes \ldots \otimes \partial_{\overline{j_p}} \otimes dx^{i_2} \otimes \ldots \otimes dx^{i_q}. $$

Then, we can define a $(p, q-1)$ tensor γS on TM within $\pi^{-1}(U)$. The expression for γS is as follows:

$$ \gamma S = (y^{\overline{l}} S^{j_1 \ldots j_p}_{l i_2 \ldots i_q}) \partial_{\overline{j_1}} \otimes \ldots \otimes \partial_{\overline{j_p}} \otimes dx^{i_2} \otimes \ldots \otimes dx^{i_q} $$

according to (x^i, y^i) (for details, see [18]).

To facilitate tensor calculus, an adapted frame denoted as $\{E_\beta\}$ can be constructed over each induced coordinate neighborhood $\pi^{-1}(U) \subset TM$. The adapted frame $\{E_\beta\}$ on $\pi^{-1}(U)$ consists of $2n$-dimensional linearly independent vector fields E_j and $E_{\overline{j}}$, organized as follows:

$$ \begin{aligned} E_j &= \partial_j - y^s \Gamma^h_{sj} \partial_{\overline{h}}, \\ E_{\overline{j}} &= \partial_{\overline{j}}. \end{aligned} $$

The adapted frame can also be represented as $\{E_\beta\} = \left\{E_j, E_{\bar{j}}\right\}$. In the context of this adapted frame $\{E_\beta\}$, the vertical lift ${}^V\xi$, the horizontal lift ${}^H\xi$ and the complete lift ${}^C\xi$ of ξ are respectively expressed by [18]

$$\begin{aligned} {}^V\xi &= \xi^j E_{\bar{j}}, \\ {}^H\xi &= \xi^j E_j, \\ {}^C\xi &= \xi^j E_j + y^s \nabla_s \xi^j E_{\bar{j}}. \end{aligned} \tag{2.1}$$

The complete lift metric ${}^Cg$ on TM over a Riemannian manifold (M, g) takes on the following structure:

$$\begin{aligned} {}^Cg\left({}^H\xi, {}^H\vartheta\right) &= 0, \\ {}^Cg\left({}^H\xi, {}^V\vartheta\right) &= {}^Cg\left({}^V\xi, {}^H\vartheta\right) = g(\xi, \vartheta), \\ {}^Cg\left({}^V\xi, {}^V\vartheta\right) &= 0 \end{aligned}$$

for all vector fields ξ and ϑ on M [18]. In the adapted local frame on TM, the covariant and contravariant components of ${}^Cg$ are expressed as follows:

$${}^Cg_{\alpha\beta} = \begin{pmatrix} 0 & g_{ij} \\ g_{ij} & 0 \end{pmatrix}$$

and

$${}^Cg^{\alpha\beta} = \begin{pmatrix} 0 & g^{ij} \\ g^{ij} & 0 \end{pmatrix}.$$

The RQSMC $\overline{\nabla}$ on TM according to the ${}^Cg$ is given as follows.

Proposition 2.1. *[4] Given a tangent bundle TM endowed with the complete lift metric ${}^Cg$ derived from a (pseudo-)Riemannian manifold (M, g). Then The RQSMC $\overline{\nabla}$ on TM is defined as follows:*

$$\begin{cases} \overline{\nabla}_{E_i} E_j = \Gamma_{ij}^k E_k + \{y^s R_{sij}{}^k + y_j R_i{}^k - y^k R_{ij}\} E_{\bar{k}}, \\ \overline{\nabla}_{E_i} E_{\bar{j}} = \Gamma_{ij}^k E_{\bar{k}}, \\ \overline{\nabla}_{E_{\bar{i}}} E_j = 0, \ \overline{\nabla}_{E_{\bar{i}}} E_{\bar{j}} = 0 \end{cases} \tag{2.2}$$

according to the adapted frame $\{E_\beta\}$, where Γ_{ij}^h and $R_{hji}{}^s$ respectively represent the components of the Riemannian connection ∇ and the Riemannian curvature tensor field R corresponding to the pseudo-Riemannian metric g on M.

Proposition 2.2. *[4] Given a tangent bundle TM endowed with the complete lift metric ${}^Cg$ derived from a (pseudo-)Riemannian manifold (M, g). The curvature tensor $\overline{R}$ of the RQSMC $\overline{\nabla}$ of $(TM, {}^C g)$ is given as follows:*

$$\begin{aligned} \overline{R}(E_i, E_j)E_k &= R_{ijk}{}^l E_l + \{y^s \nabla_s R_{ijk}{}^l\} E_{\bar{l}}, \\ \overline{R}(E_i, E_j)E_{\bar{k}} &= R_{ijk}{}^l E_{\bar{l}}, \\ \overline{R}(E_i, E_{\bar{j}})E_k &= \{R_{ijk}{}^l + R_{ik}\delta_j^l - g_{jk} R_i{}^l\} E_{\bar{l}}, \\ \overline{R}(E_{\bar{i}}, E_j)E_k &= \{R_{ijk}{}^l + g_{ik} R_j{}^l - R_{jk}\delta_i^l\} E_{\bar{l}}, \\ \overline{R}(E_{\bar{i}}, E_{\bar{j}})E_k &= 0, \ \overline{R}(E_{\bar{i}}, E_j)E_{\bar{k}} = 0, \\ \overline{R}(E_i, E_{\bar{j}})E_{\bar{k}} &= 0, \ \overline{R}(E_{\bar{i}}, E_{\bar{j}})E_{\bar{k}} = 0 \end{aligned} \tag{2.3}$$

according to the adapted frame $\{E_\beta\}$.

Proposition 2.3. *[4] Let $\overline{K}_{\alpha\beta} = \overline{R}_{\sigma\alpha\beta}{}^\sigma$ denote the Ricci tensor of the RQSMC $\overline{\nabla}$. Then, the tensor $\overline{K}$ is as follows:*

$$\begin{aligned} \overline{K}_{jk} &= (3-n) R_{jk}, \\ \overline{K}_{j\bar{k}} &= 0, \\ \overline{K}_{\bar{j}k} &= 0, \\ \overline{K}_{\bar{j}\bar{k}} &= 0. \end{aligned} \tag{2.4}$$

3. Special vector fields on TM in the context of the RQSMC $\overline{\nabla}$

In this section, our initial focus lies on exploring the harmonic and incompressible properties of the elevated vector fields. Subsequently, we proceed to deduce the overarching expressions for concurrent, conformal, projective, and $\widetilde{\varphi}(Ric)$ vector fields according to the RQSMC $\overline{\nabla}$ on TM. Through this analysis, we derive noteworthy insights concerning these vector fields and their essential characteristics.

3.1. Lifting vector field being incompressible and harmonic

Definition 3.1. Consider TM as the tangent bundle of a (pseudo-)Riemannian manifold M equipped with the RQSMC $\overline{\nabla}$. A vector field $\widetilde{V}$ on TM is termed incompressible in accordance with the RQSMC $\overline{\nabla}$ if it meets the following criterion:

$$tr(\overline{\nabla}\widetilde{V}) = \overline{\nabla}_\alpha \widetilde{V}^\alpha = E_\alpha \widetilde{V}^\alpha + \overline{\Gamma}^\alpha_{\alpha\beta}\widetilde{V}^\beta = 0,$$

where $\widetilde{V}$ is expressed as $\widetilde{V} = v^h E_h + v^{\overline{h}} E_{\overline{h}}$.

Proposition 3.1. *Consider TM as the tangent bundle of a (pseudo-)Riemannian manifold M equipped with the RQSMC $\overline{\nabla}$. Then, for any vector field V on M the following statements hold:*
i) ${}^V V$ is classified as an incompressible vector field in accordance with the RQSMC $\overline{\nabla}$;
ii) ${}^H V$ or ${}^C V$ qualifies as an incompressible vector field on TM in accordance with the RQSMC $\overline{\nabla}$ if and only if V stands as an incompressible vector field on M in accordance with the Riemannian connection ∇.

Proof. By making use (2.1) and (2.2), we achieve the following:

$$\begin{aligned}
tr(\overline{\nabla}\,{}^V V) &= \overline{\nabla}_\alpha\,{}^V V^\alpha = \overline{\nabla}_{\overline{h}} v^{\overline{h}} = 0;\\
tr(\overline{\nabla}\,{}^H V) &= \overline{\nabla}_\alpha\,{}^H V^\alpha = \overline{\nabla}_h v^h\\
&= \left(\partial_h - y^s \Gamma^m_{sh}\partial_{\overline{m}}\right) v^h + \overline{\Gamma}^{\ \ h}_{hm}\, v^m\\
&= \nabla_h v^h = tr(\nabla V);\\
tr(\overline{\nabla}\,{}^C V) &= \overline{\nabla}_\alpha\,{}^C V^\alpha = \overline{\nabla}_h v^h + \overline{\nabla}_{\overline{h}}\, v^{\overline{h}}\\
&= \left(\partial_h - y^s \Gamma^m_{sh}\partial_{\overline{m}}\right) v^h + \overline{\Gamma}^{\ \ h}_{hm}\, v^m + \partial_{\overline{h}}\left(y^s \nabla_s v^h\right)\\
&= 2\nabla_h v^h = 2tr(\nabla V),
\end{aligned}$$

from which it is easy to see that the results (i) and (ii). □

Definition 3.2. Consider TM as the tangent bundle of a (pseudo-)Riemannian manifold M equipped with the RQSMC $\overline{\nabla}$. A vector field $\widetilde{V}$ on TM is termed a harmonic vector field according to the RQSMC $\overline{\nabla}$ if it fulfills the condition:

$$\left(\overline{\nabla}_i \widetilde{V}^\epsilon\right)\widetilde{g}_{\epsilon j} - \left(\overline{\nabla}_j \widetilde{V}^\epsilon\right)\widetilde{g}_{\epsilon i} = 0$$

according to the adapted frame. In here, $\widetilde{g}_{ij}$ represents the components of the complete lift metric on TM and $\widetilde{V} = v^h E_h + v^{\overline{h}} E_{\overline{h}}$.

The following lemma is derived directly from straightforward calculations.

Lemma 3.1. *Consider TM as the tangent bundle of a (pseudo-)Riemannian manifold M equipped with the RQSMC $\overline{\nabla}$. Then*
i) For ${}^V V$, we obtain

$$\left(\overline{\nabla}_\alpha\,{}^V V^\epsilon\right)\widetilde{g}_{\epsilon\beta} - \left(\overline{\nabla}_\beta\,{}^V V^\epsilon\right)\widetilde{g}_{\epsilon\alpha} = \begin{pmatrix} \nabla_i v_j - \nabla_j v_i & 0\\ 0 & 0\end{pmatrix}; \tag{3.1}$$

ii) For ${}^H V$, we obtain

$$\begin{aligned}
&\left(\overline{\nabla}_\alpha\,{}^H V^\epsilon\right)\widetilde{g}_{\epsilon\beta} - \left(\overline{\nabla}_\beta\,{}^H V^\epsilon\right)\widetilde{g}_{\epsilon\alpha} \qquad (3.2)\\
&= \begin{pmatrix} y^s\left[R_{siaj} - R_{sjai} + g_{si}R_{ja} - g_{sj}R_{ia}\right]v^a & \nabla_i v_j\\ -\nabla_j v_i & 0\end{pmatrix};
\end{aligned}$$

iii) For ${}^{C}V$, we obtain

$$\left(\overline{\nabla}_{\alpha}\,{}^{C}V^{\epsilon}\right)\widetilde{g}_{\epsilon\beta}-\left(\overline{\nabla}_{\beta}\,{}^{C}V^{\epsilon}\right)\widetilde{g}_{\epsilon\alpha} = \begin{pmatrix} y^{s}\left[\nabla_{s}\left(\nabla_{i}v_{j}-\nabla_{j}v_{i}\right)+\left(R_{siaj}-R_{sjai}+g_{si}R_{ja}-g_{sj}R_{ia}\right)v^{a}\right] & \nabla_{i}v_{j}-\nabla_{j}v_{i} \\ \nabla_{i}v_{j}-\nabla_{j}v_{i} & 0 \end{pmatrix}. \tag{3.3}$$

Therefore, as a consequence of Lemma 3.1, we arrive at the subsequent result.

Proposition 3.2. *Consider TM as the tangent bundle of a (pseudo-)Riemannian manifold M equipped with the RQSMC $\overline{\nabla}$. Then, for any vector field V on M we establish the followings:*

i) ${}^{V}V$ is a harmonic vector field on TM in accordance with the RQSMC $\overline{\nabla}$ if and only if V qualifies as a harmonic vector field in accordance with the Riemannian connection ∇;

ii) ${}^{C}V$ is a harmonic vector field on TM in accordance with the RQSMC $\overline{\nabla}$ if and only if V is a harmonic vector field according to the Riemannian connection ∇ and satisfies the condition $R_{siaj}-R_{sjai}+g_{si}R_{ja}-g_{sj}R_{ia}=0$;

iii) ${}^{H}V$ is a harmonic vector field on TM in accordance with the RQSMC $\overline{\nabla}$ if and only if V is parallel in accordance with the Riemannian connection ∇ and satisfies the condition $R_{siaj}-R_{sjai}+g_{si}R_{ja}-g_{sj}R_{ia}=0$.

3.2. *Concurrent vector field*

Consider a vector field $\widetilde{V}=v^{h}E_{h}+v^{\overline{h}}E_{\overline{h}}$ on TM. If v^{h} depend solely on the base manifold's coordinates $\left(x^{h}\right)$, $\widetilde{V}$ is termed a fiber-preserving vector field on TM.

Definition 3.3. A vector field $\widetilde{V}$ on TM is referred to as a concurrent vector field in accordance with the RQSMC $\overline{\nabla}$ if it adheres to the condition:

$$\overline{\nabla}_{\beta}\widetilde{V}^{\epsilon}=\overline{\nabla}_{E_{\beta}}\widetilde{V}^{\epsilon}=\widetilde{k}\delta_{\beta}^{\epsilon}, \tag{3.4}$$

where $\widetilde{k}$ represents a function defined on TM, $\delta_{\beta}^{\epsilon}$ denotes the Kronecker delta symbol and $\widetilde{V}$ is expressed as $\widetilde{V}=v^{h}E_{h}+v^{\overline{h}}E_{\overline{h}}$.

Proposition 3.3. *Consider TM as the tangent bundle of a (pseudo-)Riemannian manifold M equipped with the RQSMC $\overline{\nabla}$. The vector field $\widetilde{V}$ on TM is identified as a fiber-preserving concurrent vector field in accordance with the RQSMC $\overline{\nabla}$ if and only if it takes the form:*

$$\widetilde{V}=\begin{pmatrix} v^{h} \\ \frac{1}{n}\left[tr\left(\nabla V\right)\right]y^{h} \end{pmatrix}$$

and satisfies the subsequent condition:

$$\frac{1}{n}\left[\nabla_{j}\left(tr\left(\nabla V\right)\right)y^{h}\right]+\left(y^{s}R_{sja}^{\ \ \ h}+y_{a}R_{j}^{h}-y^{h}R_{ja}\right)v^{a}=0.$$

Proof. Initially, by considering $\epsilon=h,\beta=\overline{j}$ in (3.4) within the adapted frame, we obtain:

$$\begin{aligned}\overline{\nabla}_{\overline{j}}v^{h} &= E_{\overline{j}}v^{h}+\overline{\Gamma}_{\overline{j}a}^{h}v^{a}+\overline{\Gamma}_{\overline{j}\overline{a}}^{h}v^{\overline{a}}=\widetilde{k}\delta_{\overline{j}}^{h}\\ &\Rightarrow\ \partial_{\overline{j}}v^{h}=0\\ &\Rightarrow\ v^{h}=v^{h}\left(x^{h}\right).\end{aligned}$$

Similarly, when considering $\epsilon=h,\beta=j$ and $\epsilon=\overline{h},\beta=\overline{j}$, we respectively acquire the following:

$$\begin{aligned}\overline{\nabla}_{j}v^{h} &= E_{j}v^{h}+\overline{\Gamma}_{ja}^{h}v^{a}+\overline{\Gamma}_{j\overline{a}}^{h}v^{\overline{a}}=\widetilde{k}\delta_{j}^{h}\\ &\Rightarrow\ \partial_{j}v^{h}+\Gamma_{ja}^{h}v^{a}=\widetilde{k}\delta_{j}^{h}\\ &\Rightarrow\ \nabla_{j}v^{h}=\widetilde{k}\delta_{j}^{h}\qquad (h\rightarrow j)\\ &\Rightarrow\ \frac{1}{n}\nabla_{j}v^{j}=\widetilde{k}\end{aligned}$$

and

$$\begin{aligned}
\overline{\nabla}_{\overline{j}} v^{\overline{h}} &= E_{\overline{j}} v^{\overline{h}} + \overline{\Gamma}^{\overline{h}}_{\overline{j}a} v^a + \overline{\Gamma}^{\overline{h}}_{\overline{j}\overline{a}} v^{\overline{a}} = \widetilde{k} \delta^{\overline{h}}_{\overline{j}} \\
&\Rightarrow \; \partial_{\overline{j}} v^{\overline{h}} = \frac{1}{n} \nabla_j v^j \delta^{\overline{h}}_{\overline{j}} \\
&\Rightarrow \; \partial_{\overline{j}} v^{\overline{h}} = \frac{1}{n} \left[tr\left(\nabla V\right) \delta^{\overline{h}}_{\overline{j}} \right] \\
&\Rightarrow \; \partial_{\overline{j}} v^{\overline{h}} = \frac{1}{n} \left[tr\left(\nabla V\right) \left(\partial_{\overline{j}} y^h\right) \right] \\
&\Rightarrow \; \partial_{\overline{j}} v^{\overline{h}} = \partial_{\overline{j}} \left[\frac{1}{n} tr\left(\nabla V\right) y^h \right] \\
&\Rightarrow \; v^{\overline{h}} = \frac{1}{n} \left[tr\left(\nabla V\right) \right] y^h .
\end{aligned}$$

Finally, by putting $\epsilon = \overline{h}, \beta = j$, we arrive at:

$$\begin{aligned}
\overline{\nabla}_j v^{\overline{h}} &= E_j v^{\overline{h}} + \overline{\Gamma}^{\overline{h}}_{ja} v^a + \overline{\Gamma}^{\overline{h}}_{j\overline{a}} v^{\overline{a}} = \widetilde{k} \delta^{\overline{h}}_j \\
&\Rightarrow \; \begin{array}{c} E_j \left[\frac{1}{n} \left[tr\left(\nabla V\right) \right] y^h \right] + \left(y^s R_{sja}^{\;\;\;\;h} + y_a R^h_j - y^h R_{ja} \right) v^a \\ + \Gamma^h_{ja} \left[\frac{1}{n} \left[tr\left(\nabla V\right) \right] y^a \right] = 0 \end{array} \\
&\Rightarrow \; \begin{array}{c} \left(\partial_j - y^s \Gamma^m_{sj} \partial_{\overline{m}} \right) \left[\frac{1}{n} \left[tr\left(\nabla V\right) \right] y^h \right] \\ + \left(y^s R_{sja}^{\;\;\;\;h} + y_a R^h_j - y^h R_{ja} \right) v^a + y^a \Gamma^h_{ja} \left[\frac{1}{n} \left[tr\left(\nabla V\right) \right] \right] = 0 \end{array} \\
&\Rightarrow \; \begin{array}{c} \frac{1}{n} \left[\partial_j \left(tr\left(\nabla V\right) \right) y^h \right] - y^s \Gamma^h_{sj} \left[\frac{1}{n} \left[tr\left(\nabla V\right) \right] \right] \\ + \left(y^s R_{sja}^{\;\;\;\;h} + y_a R^h_j - y^h R_{ja} \right) v^a + y^a \Gamma^h_{ja} \left[\frac{1}{n} \left[tr\left(\nabla V\right) \right] \right] = 0 \end{array} \\
&\Rightarrow \; \frac{1}{n} \left[\partial_j \left(tr\left(\nabla V\right) \right) y^h \right] + \left(y^s R_{sja}^{\;\;\;\;h} + y_a R^h_j - y^h R_{ja} \right) v^a = 0 \\
&\Rightarrow \; \frac{1}{n} \left[\nabla_j \left(tr\left(\nabla V\right) \right) y^h \right] + \left(y^s R_{sja}^{\;\;\;\;h} + y_a R^h_j - y^h R_{ja} \right) v^a = 0.
\end{aligned}$$

□

3.3. *Conformal vector fields*

This section focuses on the study of fiber-preserving conformal vector fields on the tangent bundle TM over a (pseudo-)Riemannian manifold (M, g) using the the RQSMC $\overline{\nabla}$. These vector fields are a special class of transformations defined on a Riemannian manifold (M, g). A conformal transformation is a vector field V on M such that its local flows ϕ_t preserve the conformal class of the metric g for every t. Moreover, when the tangent bundle TM is equipped with a Riemannian metric g, a conformal transformation V of g is considered fiber-preserving if its local flows also preserve the fibers of TM. This section provides a characterization of infinitesimal fiber-preserving conformal transformations, often referred to as conformal vector fields, on the tangent bundle TM. It establishes a necessary and sufficient condition for a vector field V to be an infinitesimal fiber-preserving conformal transformation concerning the RQSMC $\overline{\nabla}$. This condition is expressed through a set of relationships involving specific tensor fields on M with type $(1, 0)$ and (1.1). It's worth noting that for fiber-preserving conformal vector fields on TM concerning the complete lift metric ${}^C g$ and the synectic lift metric, we can refer to the papers [3] and [13].

Definition 3.4. Consider TM as the tangent bundle of a (pseudo-)Riemannian manifold M equipped with the RQSMC $\overline{\nabla}$. A vector field $\widetilde{V} = v^h E_h + v^{\overline{h}} E_{\overline{h}}$ on TM is a fiber-preserving conformal vector field in accordance with the RQSMC $\overline{\nabla}$ if it satisfies

$$L_{\widetilde{V}} \widetilde{g}_{\alpha\beta} = (\overline{\nabla}_\alpha \widetilde{V}^\epsilon) \widetilde{g}_{\epsilon\beta} + (\overline{\nabla}_\beta \widetilde{V}^\epsilon) \widetilde{g}_{\epsilon\alpha} = 2\widetilde{\Omega} \widetilde{g}_{\alpha\beta}. \tag{3.5}$$

In here, $L_{\widetilde{V}} \widetilde{g}$ is the Lie derivative of the Riemannian metric $\widetilde{g}$ according to $\widetilde{V}$ and $\widetilde{\Omega}$ is a scalar function on TM.

When we consider the indices $(\alpha,\beta)=(i,\bar{j})$, $(\bar{i},j)$ and (i,j) in the previous equation (3.5), the following relationships can be written:

$$\begin{cases} i)\ \left(\nabla_i v^h\right) g_{hj} + \left(E_{\bar{j}} v^{\bar{h}}\right) g_{hi} = 2\widetilde{\Omega} g_{ij}, \\ ii)\ \left(E_{\bar{i}} v^{\bar{h}}\right) g_{hj} + \left(\nabla_j v^h\right) g_{hi} = 2\widetilde{\Omega} g_{ij}, \\ iii)\ \begin{matrix} \left[E_i v^{\bar{h}} + \left(y^s R_{sia}{}^h + y_a R_i^h - y^h R_{ia}\right) v^a + \Gamma_{ia}^h v^{\bar{a}}\right] g_{hj} \\ + \left[E_j v^{\bar{h}} + \left(y^s R_{sja}{}^h + y_a R_j^h - y^h R_{ja}\right) v^a + \Gamma_{ja}^h v^{\bar{a}}\right] g_{hi} \end{matrix} = 0. \end{cases} \tag{3.6}$$

Proposition 3.4. *The scalar function $\widetilde{\Omega}$ on TM depends only on the variables $\left(x^h\right)$ according to $\left(x^h, y^h\right)$.*

Proof. Applying the operator $E_{\bar{k}}$ to each side of the equation *(ii)* in (3.6), we arrive at:

$$g_{hj} E_{\bar{k}} E_{\bar{i}} v^{\bar{h}} = 2E_{\bar{k}}\left(\overline{\Omega}\right) g_{ij}.$$

From this, we deduce:

$$E_{\bar{k}}\left(\overline{\Omega}\right) g_{ij} = E_{\bar{i}}\left(\overline{\Omega}\right) g_{kj},$$

which further leads to:

$$(n-1)E_{\bar{k}}\left(\overline{\Omega}\right) = 0.$$

This conclusion implies that the scalar function $\widetilde{\Omega}$ on TM is exclusively dependent on the variables $\left(x^h\right)$ in terms of the coordinates $\left(x^h, y^h\right)$. As a consequence, we can treat $\widetilde{\Omega}$ as a function on M. For the sake of clarity, we will denote $\widetilde{\Omega}$ by ρ in the subsequent discussions. □

Drawing upon equation (3.6), Proposition 3.4, and the observation that $E_{\bar{i}}\left(v^{\bar{h}}\right)$ only depends on the variables $\left(x^h\right)$, we can express the following:

$$v^{\bar{h}} = y^a A_a^h + B^h, \tag{3.7}$$

where A_a^h and B^h are certain functions which depend only on the variable $\left(x^h\right)$.

In the context of a (pseudo-)Riemannian manifold (M,g), a vector field V is referred to as a Killing vector field if it satisfies:

$$L_V g_{ij} = \nabla_i v_j + \nabla_j v_i = 0.$$

This condition implies that the Lie derivative of the metric tensor g with respect to the vector field V vanishes, indicating that V generates an isometry, i.e., it preserves the metric structure of the manifold under its flow.

Proposition 3.5. *If we consider*

$$B = B^h \frac{\partial}{\partial x^h},$$

then the vector field B is a Killing vector field in accordance with the Riemannian connection ∇ on M.

Proof. Substituting (3.7) into the equation *(iii)* in (3.6) we get

$$\nabla_i B_j + \nabla_j B_i = 0 \tag{3.8}$$

and

$$v^a (R_{siaj} + R_{sjai} + g_{sa} R_{ij} - g_{sj} R_{ia} + g_{sa} R_{ji} - g_{si} R_{ja}) + \nabla_i A_{sj} + \nabla_j A_{si} = 0, \tag{3.9}$$

where $B_i = g_{im} B^m$ and $A_{sj} = g_{hj} A_s^h$. Hence, by (3.8) it follows

$$L_B g_{ij} = \nabla_i B_j + \nabla_j B_i = 0.$$

This demonstrates that the vector field B is a Killing vector field on M according to the Riemannian connection ∇. □

Substituting (3.7) into the equation *(ii)* in (3.6), we have

$$\begin{aligned} & E_{\bar{i}}\left(v^{\bar{h}}\right) g_{hj}+\left(\nabla_j v^h\right) g_{hi}=2\rho g_{ij} \\ & \Rightarrow \partial_{\bar{i}}\left(y^s A_s^h+B^h\right) g_{hj}+\left(\nabla_j v^h\right) g_{hi}=2\rho g_{ij} \\ & \Rightarrow A_i^h g_{hj}+\left(\nabla_j v^h\right) g_{hi}=2\rho g_{ij} \\ & \Rightarrow g_{hj} A_i^h=2\rho g_{ij}-g_{hi}\left(\nabla_j v^h\right). \end{aligned} \tag{3.10}$$

Given a linear connection ∇ on a manifold M, a vector field V on M is classified as a projective vector field if there exists a 1-form θ such that for any vector fields X and Y on M, the following condition holds:

$$(L_V\nabla)(X,Y)=\theta(X)Y+\theta(Y)X.$$

In this context, θ is referred to as the associated 1-form of V. Locally, it can be expressed as:

$$L_V\Gamma_{ij}^h=\theta_i\delta_j^h+\theta_j\delta_i^h.$$

Here, Γ_{ij}^h represents the Christoffel symbols associated with the connection ∇, and δ_i^h is the Kronecker delta.

Proposition 3.6. *The vector field V, represented by its components as $\left(v^h\right)$, is a projective vector field on M in accordance with the Riemannian connection ∇ if the following equation holds: $2\delta_a^h R_{ij}-R_{ia}\delta_j^h-R_{ja}\delta_i^h=0$.*

Proof. Taking the covariant derivative ∇_k of both sides of equation (3.10), we obtain

$$\begin{aligned} g_{hj}\nabla_k A_i^h &= \nabla_k\left[2\rho g_{ij}-g_{hi}\left(\nabla_j v^h\right)\right] \\ &= 2\left(\nabla_k\rho\right) g_{ij}-g_{hi}\nabla_k\nabla_j v^h \\ &= 2\rho_k g_{ij}-g_{hi}\left(L_V\Gamma_{kj}^{\ h}-R_{akj}^{\ \ \ h}v^a\right) \\ \nabla_k A_{ij} &= 2\rho_k g_{ij}-L_V\Gamma_{kj}^{\ h}g_{hi}-R_{akij}v^a. \end{aligned} \tag{3.11}$$

When we substitute equation (3.11) into equation (3.9), we obtain:

$$v^a(R_{siaj}+R_{sjai}+g_{sa}R_{ij}-g_{sj}R_{ia}+g_{sa}R_{ji}-g_{si}R_{ja})+\nabla_i A_{sj}+\nabla_j A_{si}=0$$

$$\Rightarrow \begin{array}{c} v^a(R_{siaj}+R_{sjai}+g_{sa}R_{ij}-g_{sj}R_{ia}+g_{sa}R_{ji}-g_{si}R_{ja}) \\ +2\rho_i g_{sj}-L_V\Gamma_{ij}^{\ h}g_{hs}-R_{aisj}v^a+2\rho_j g_{si}-L_V\Gamma_{ji}^{\ h}g_{hs}-R_{ajsi}v^a \end{array}=0$$

$$\Rightarrow v^a\left(g_{sa}R_{ij}-g_{sj}R_{ia}+g_{sa}R_{ji}-g_{si}R_{ja}\right)+2\left(\rho_i g_{sj}+\rho_j g_{si}\right)=2L_V\Gamma_{ij}^{\ h}g_{hs}$$

$$\Rightarrow L_V\Gamma_{ij}^{\ h}=\rho_i\delta_j^h+\rho_j\delta_i^h+\frac{1}{2}v^a\left(2\delta_a^h R_{ij}-R_{ia}\delta_j^h-R_{ja}\delta_i^h\right),$$

where $\rho_i=\nabla_i\rho$. Hence, V is a projective vector field on M in accordance with the Riemannian connection ∇ if $2\delta_a^h R_{ij}-R_{ia}\delta_j^h-R_{ja}\delta_i^h=0$. □

Now, let us examine the converse problem. Suppose that the manifold M admits a projective vector field $V=v^h\frac{\partial}{\partial x^h}$ in accordance with the Riemannian connection ∇. In this context, we can state the following proposition:

Proposition 3.7. *The vector field $\widetilde{V}$ on TM, defined as*

$$\widetilde{V}=v^h E_h+\left(y^s A_s^h+B^h\right) E_{\bar{h}}$$

constitutes a fiber-preserving conformal vector field on TM in accordance with the RQSMC $\overline{\nabla}$, where $A_i^h=g^{ha}A_{ai}$, $A_{ij}=2\rho g_{ij}+\nabla_i v_j-L_V g_{ij}$, and $g_{ji}B^j=B_i$, $\nabla_i B_j=L_B g_{ij}-\nabla_j B_i$.

Proof. Given B_h, v^h and A_i^h that satisfy the conditions outlined above, it is evident that the vector field $\widetilde{V}=v^h E_h+\left(y^s A_s^h+B^h\right) E_{\bar{h}}$ qualifies as a fiber-preserving conformal vector field on TM in accordance with the RQSMC $\overline{\nabla}$. The detailed calculations supporting this assertion are omitted for brevity. □

3.4. Projective vector fields

In [14], a comprehensive classification of fiber-preserving projective vector fields on TM was presented based on the complete lift metric ^{C}g by Yamauchi. Additionally, Hasegawa and Yamauchi, [7], provided a classification of projective vector fields considering the lift connections. In the upcoming section, we will delve into the examination of fiber-preserving projective vector fields on classified fiber-preserving projective vector fields on TM according to the RQSMC $\overline{\nabla}$. To begin, let us introduce the following lemma, which will be essential for our subsequent analysis:

Lemma 3.2. *The Lie derivations of the adapted frame in accordance with the fiber-preserving vector field $\widetilde{V} = v^h E_h + v^{\overline{h}} E_{\overline{h}}$ are given as follows [13]:*

$$L_{\widetilde{V}} E_h = -(\partial_h v^a) E_a + \left\{ y^b v^c R_{hcb}^{a} - v^{\overline{b}} \Gamma_{bh}^{a} - (E_h v^{\overline{a}} \right\} E_{\overline{a}},$$

$$L_{\widetilde{V}} E_{\overline{h}} = \left\{ v^b \Gamma_{bh}^{a} - (E_{\overline{h}} v^{\overline{a}}) \right\} E_{\overline{a}}.$$

Definition 3.5. Consider TM as the tangent bundle of a (pseudo-)Riemannian manifold M equipped with the RQSMC $\overline{\nabla}$. A vector field $\widetilde{V} = v^h E_h + v^{\overline{h}} E_{\overline{h}}$ on TM is a fiber-preserving projective vector field in accordance with the RQSMC $\overline{\nabla}$ if and only if there exist a 1-form $\widetilde{\theta} = \left(\widetilde{\theta}_i, \widetilde{\theta}_{\overline{i}}\right)$ on TM such that

$$\begin{aligned} \left(L_{\widetilde{V}} \overline{\nabla}\right)(\widetilde{Y}, \widetilde{Z}) &= L_{\widetilde{V}}(\overline{\nabla}_{\widetilde{Y}} \widetilde{Z}) - \overline{\nabla}_{\widetilde{Y}}(L_{\widetilde{V}} \widetilde{Z}) - \overline{\nabla}_{(L_{\widetilde{V}} \widetilde{Y})} \widetilde{Z} \\ &= \widetilde{\theta}(\widetilde{Y}) \widetilde{Z} + \widetilde{\theta}(\widetilde{Z}) \widetilde{Y} \end{aligned} \tag{3.12}$$

for any vector fields $\widetilde{Y}$ and $\widetilde{Z}$ on TM.

In accordance with the RQSMC $\overline{\nabla}$, the general expression for fiber-preserving projective vector fields on TM can be stated as follows:

Theorem 3.1. *Consider TM as the tangent bundle of a (pseudo-)Riemannian manifold M equipped with the RQSMC $\overline{\nabla}$. A vector field $\widetilde{V} = v^h E_h + v^{\overline{h}} E_{\overline{h}}$ on TM is a fiber-preserving projective vector field with the associated 1-form $\overline{\theta}$ on TM in accordance with the RQSMC $\overline{\nabla}$ if and only if $\widetilde{V}$ takes the form*

$$\widetilde{V} =^{H} V +^{V} B + \gamma A,$$

where the vector fields $V = (v^h)$, $B = (B^h)$, the $(1,1)$ −tensor field $A = (A_i^h)$ and 1-form $\overline{\theta}$ satisfy the following conditions

$$(i)\ \overline{\theta} = \theta_i dx^i,$$
$$(ii)\ \nabla_i \theta_j = 0,$$
$$(iii)\ \nabla_j A_i^h = \theta_j \delta_i^h - v^c R_{cji}{}^{h},$$
$$(iv)\ R_{aij}{}^{h} B^a = B^h R_{ij} - B_j R_i^h,$$
$$(v) L_V \Gamma_{ij}^h = \theta_i \delta_j^h + \theta_j \delta_i^h\ .$$

Proof. With help of (3.12) , we can write the following system

$$\begin{aligned} \left(L_{\widetilde{V}} \overline{\nabla}\right)(E_{\overline{i}}, E_{\overline{j}}) &= L_{\widetilde{V}}(\overline{\nabla}_{E_{\overline{i}}} E_{\overline{j}}) - \overline{\nabla}_{E_{\overline{i}}}(L_{\widetilde{V}} E_{\overline{j}}) - \overline{\nabla}_{(L_{\widetilde{V}} E_{\overline{i}})} E_{\overline{j}} \\ &= \widetilde{\theta}(E_{\overline{i}}) E_{\overline{j}} + \widetilde{\theta}(E_{\overline{j}}) E_{\overline{i}}, \end{aligned} \tag{3.13}$$

$$\begin{aligned} \left(L_{\widetilde{V}} \overline{\nabla}\right)(E_{\overline{i}}, E_j) &= L_{\widetilde{V}}(\overline{\nabla}_{E_{\overline{i}}} E_j) - \overline{\nabla}_{E_{\overline{i}}}(L_{\widetilde{V}} E_j) - \overline{\nabla}_{(L_{\widetilde{V}} E_{\overline{i}})} E_j \\ &= \widetilde{\theta}(E_{\overline{i}}) E_j + \widetilde{\theta}(E_j) E_{\overline{i}}, \end{aligned} \tag{3.14}$$

$$\begin{aligned} \left(L_{\widetilde{V}} \overline{\nabla}\right)(E_i, E_j) &= L_{\widetilde{V}}(\overline{\nabla}_{E_i} E_j) - \overline{\nabla}_{E_i}(L_{\widetilde{V}} E_j) - \overline{\nabla}_{(L_{\widetilde{V}} E_i)} E_j \\ &= \widetilde{\theta}(E_i) E_j + \widetilde{\theta}(E_j) E_i. \end{aligned} \tag{3.15}$$

From (3.13), by virtue of (2.2) and Lemma 3.2 we get

$$\left\{ \partial_{\overline{i}} (\partial_{\overline{j}} v^{\overline{a}} \right\} E_{\overline{a}} = \widetilde{\theta}_{\overline{i}} E_{\overline{j}} + \widetilde{\theta}_{\overline{j}} E_{\overline{i}}. \tag{3.16}$$

Similarly, from (3.14) we obtain

$$\left\{-v^{c}R_{jci}{}^{a}+(E_{\bar{i}}v^{\bar{b}})\Gamma^{a}_{bj}+E_{\bar{i}}(E_{j}v^{\bar{a}})\right\}E_{\bar{a}}=\widetilde{\theta}_{\bar{i}}E_{j}+\widetilde{\theta}_{j}E_{\bar{i}}, \tag{3.17}$$

from which we have

$$\widetilde{\theta}_{\bar{i}}=0. \tag{3.18}$$

Due to $\widetilde{\theta}_{\bar{i}}=0$, (3.16) reduces to

$$\partial_{\bar{i}}(\partial_{\bar{j}}v^{\bar{a}})=0$$

and we obtain

$$v^{\bar{a}}=y^{s}A^{a}_{s}+B^{a}, \tag{3.19}$$

where A^{a}_{s} and B^{a} are certain functions which depend only (x^{h}). Hence, the fiber-preserving projective vector field $\widetilde{V}$ on TM can be expressed as

$$\begin{aligned}\widetilde{V}&=v^{h}E_{h}+v^{\bar{h}}E_{\bar{h}}=v^{h}E_{h}+\{y^{s}A^{a}_{s}+B^{a}\}E_{\bar{h}}\\&={}^{H}V+{}^{V}B+\gamma A.\end{aligned} \tag{3.20}$$

Substituting (3.19) into (3.17), we obtain

$$R_{aji}{}^{h}v^{a}+\nabla_{j}A^{h}_{i}=\delta^{h}_{i}\theta_{j}. \tag{3.21}$$

Substituting (3.19) and (3.21) into (3.15), we have

$$\begin{aligned}&\{\nabla_{i}\nabla_{j}v^{h}+R_{aij}{}^{h}v^{a}\}E_{h}+\{\nabla_{i}\nabla_{j}B^{h}+\\&R_{aij}{}^{h}B^{a}+B_{j}R^{h}_{i}-B^{h}R_{ij}+y^{s}[\nabla_{i}\nabla_{j}A^{h}_{s}+A^{a}_{s}R_{aij}{}^{h}-R_{sij}{}^{a}A^{h}_{a}+\\&v^{a}\nabla_{a}R_{sij}{}^{h}-v^{a}\nabla_{i}R_{jas}{}^{h}+\nabla_{j}v^{a}R_{sia}{}^{h}+\nabla_{i}v^{a}R_{sja}{}^{h}\\&-\delta^{h}_{s}[v^{a}\nabla_{a}R_{ij}+\nabla_{i}v^{a}R_{aj}+\nabla_{j}v^{a}R_{ia}]+v^{a}g_{sj}\nabla_{a}R^{h}_{i}\\&+\nabla_{j}v^{a}g_{sa}R^{h}_{i}+\nabla_{i}v^{a}g_{sj}R^{h}_{a}+A^{a}_{s}g_{aj}R^{h}_{i}-g_{sj}R^{a}_{i}A^{h}_{a}]\}E_{\bar{h}}\\=\;&\widetilde{\theta}_{i}E_{j}+\widetilde{\theta}_{j}E_{i}.\end{aligned} \tag{3.22}$$

From (3.22), we have

$$\nabla_{i}\nabla_{j}v^{h}+R_{aij}{}^{h}v^{a}=\widetilde{\theta}_{i}\delta^{h}_{j}+\widetilde{\theta}_{j}\delta^{h}_{i}, \tag{3.23}$$

$$\nabla_{i}\nabla_{j}B^{h}+R_{aij}{}^{h}B^{a}+B_{j}R^{h}_{i}-B^{h}R_{ij}=0, \tag{3.24}$$

$$\begin{aligned}&\nabla_{i}\nabla_{j}A^{h}_{s}+A^{a}_{s}R_{aij}{}^{h}-R_{sij}{}^{a}A^{h}_{a}+v^{a}\nabla_{a}R_{sij}{}^{h}\\&-v^{a}\nabla_{i}R_{jas}{}^{h}+\nabla_{j}v^{a}R_{sia}{}^{h}+\nabla_{i}v^{a}R_{sja}{}^{h}\\&-\delta^{h}_{s}L_{V}R_{ij}+v^{a}g_{sj}\nabla_{a}R^{h}_{i}+\nabla_{j}v^{a}g_{sa}R^{h}_{i}\\&+\nabla_{i}v^{a}g_{sj}R^{h}_{a}+A^{a}_{s}g_{aj}R^{h}_{i}-g_{sj}R^{a}_{i}A^{h}_{a}=0.\end{aligned} \tag{3.25}$$

From the equation (3.23), we can say that the induced vector field $V=v^{h}\frac{\partial}{\partial x^{h}}$ denotes a projective vector field in accordance with the Riemannian connection ∇. By using the relation on the Lie derivative of the curvature tensor, we get

$$L_{V}R_{ijk}{}^{h}=\nabla_{i}(L_{v}\Gamma_{jk})-\nabla_{j}(L_{v}\Gamma_{ik})$$

$$L_{V}R_{ij}=-(n-1)\nabla_{i}\theta_{j}. \tag{3.26}$$

Contracting h and s in (3.25) and using (3.21) and (3.26), we find for $n\neq 2$,

$$\nabla_{i}\theta_{j}=0. \tag{3.27}$$

This showes $\theta_{a}\theta^{a}=c$ (constant). By using (3.27) in (3.24), we get

$$R_{aij}{}^{h}B^{a}=B^{h}R_{ij}-B_{j}R^{h}_{i}.$$

Conversely, without prejudice to the above steps, if B^{h},v^{h},θ_{h} and A^{h}_{i} are taken to satisfy *(i)-(vi)*, we obtain $\widetilde{X}={}^{H}V+{}^{V}B+\gamma A$ is a fiber-preserving projective vector field on TM in accordance with the RQSMC $\overline{\nabla}$. Hence, the proof is completed. □

It is a well-established fact that every fiber-preserving vector field $\widetilde{V}$ induces a vector field V on M with components (v^h), where $V = (v^h, v^{\overline{h}})$ represents the fiber-preserving vector field on TM. The following result directly emerges from Theorem 3.1 and its proof.

Corollary 3.1. *Consider TM as the tangent bundle of a (pseudo-)Riemannian manifold M equipped with the RQSMC $\overline{\nabla}$. Every fiber-preserving projective vector field $\widetilde{V}$ is of the form (3.20) and it naturally induces a projective vector field V on M.*

If we have $\widetilde{V} = (v^h, v^{\overline{h}})$ as a vector field on TM according to $\{E_\beta\}$ and $v^h = 0$, $\widetilde{V}$ is referred to as a vertical vector field on TM. In the present case, the vector field $\widetilde{V}$ in Theorem 3.1 simplifies to $\widetilde{V} =^V B + \gamma A$. Consequently, based on Theorem 3.1, we can draw the following conclusion.

Corollary 3.2. *Consider TM as the tangent bundle of a (pseudo-)Riemannian manifold M equipped with the RQSMC $\overline{\nabla}$. If TM possesses a vertical projective vector field $\widetilde{V}$, then the vector field $\widetilde{V}$ can be expressed as:*

$$\widetilde{V} =^V B + \gamma A,$$

where the vector field $B = \left(B^h\right)$, the $(1,1)$-tensor field $A = \left(A_i^h\right)$ and 1-form $\widetilde{\theta}$ satisfy the following conditions

$$(i)\ \overline{\theta} = \theta_i dx^i,$$
$$(ii)\ \nabla_i \theta_j = 0,$$
$$(iii) \nabla_j A_i^h = \theta_j \delta_i^h,$$
$$(iv)\ R_{aij}^{\ \ h} B^a = B^h R_{ij} - B_j R_i^h.$$

3.5. $\widetilde{\varphi}(Ric)$ vector fields

The vector field φ, locally represented as $\varphi = \varphi^m \frac{\partial}{\partial x^m}$ on a Riemannian manifold M, is identified as a $\varphi(Ric)$ vector field under the condition:

$$\nabla(\overline{\varphi}) = \lambda Ric,$$

where λ is a non-zero scalar function,∇ signifies the Riemannian connection associated with the metric g, Ric represents the Ricci tensor of (M, g) and $g(\varphi, \xi) = \overline{\varphi}\xi$. This equation can be expressed locally as:

$$\nabla_j \overline{\varphi}_i = \lambda R_{ij}, \tag{3.28}$$

where $\overline{\varphi}_i = g_{im}\varphi^m$ and R_{ij} is the Ricci tensor [9].

Definition 3.6. A vector field $\widetilde{\varphi} = \widetilde{\varphi}^h E_h + \widetilde{\varphi}^{\overline{h}} E_{\overline{h}}$ on TM is said to be a $\widetilde{\varphi}(Ric)$-vector field in accordance with the RQSMC $\overline{\nabla}$ if it satisfies

$$\overline{\nabla}_K \overline{\varphi}_M = \widetilde{\lambda} \widetilde{R}_{KM},$$

where $\widetilde{\lambda}$ is a non-zero scalar function and $\widetilde{R}_{KM}$ is the Ricci tensor on TM. Here $\overline{\varphi}_M = \widetilde{g}_{MK}\widetilde{\varphi}^K$.

Proposition 3.8. *Consider TM as the tangent bundle of a (pseudo-)Riemannian manifold M equipped with the RQSMC $\overline{\nabla}$. The vector field $\widetilde{\varphi}$ on TM is a fiber-preserving $\widetilde{\varphi}(Ric)$-vector field in accordance with the RQSMC $\overline{\nabla}$ if and only if $\widetilde{\varphi}$ has the form*

$$\widetilde{\varphi} = \begin{pmatrix} \widetilde{\varphi}^h(x^h) \\ \widetilde{\varphi}^{\overline{h}}(x^h) \end{pmatrix}$$

and the following conditions are satisfied

$$(i) \nabla_i \widetilde{\varphi}^h = 0,$$
$$(ii) \nabla_i \widetilde{\varphi}_{\overline{j}} = \lambda R_{ij},$$
$$(iii)(R_{sia}^{\ \ \ h} + g_{sa} R_i^h - \delta_s^h R_{ia})\widetilde{\varphi}^a = 0.$$

Proof. By using (2.4) if we take $K = \overline{i}, M = \overline{j}$ in (3.28), it follows that

$$\begin{aligned}
\overline{\nabla}_{\overline{i}} \widetilde{\varphi}_{\overline{j}} &= \widetilde{\lambda} \widetilde{R}_{\overline{i}\overline{j}} \Rightarrow \overline{\nabla}_{\overline{i}}(\widetilde{\varphi}^\epsilon \widetilde{g}_{\epsilon\overline{j}}) = 0 \\
&\Rightarrow \overline{\nabla}_{\overline{i}}(\widetilde{\varphi}^h \widetilde{g}_{h\overline{j}}) = 0 \\
&\Rightarrow (\overline{\nabla}_{\overline{i}} \widetilde{\varphi}^h) \widetilde{g}_{h\overline{j}} = 0 \\
&\Rightarrow (E_{\overline{i}} \widetilde{\varphi}^h + \Gamma_{\overline{i}a}^h \widetilde{\varphi}^a + \Gamma_{\overline{i}\overline{a}}^h \widetilde{\varphi}^{\overline{a}}) g_{hj} = 0 \\
&\Rightarrow (E_{\overline{i}} \widetilde{\varphi}^h) g_{hj} = 0 \\
&\Rightarrow \widetilde{\varphi}^h = \widetilde{\varphi}^h(x^h).
\end{aligned}$$

Similarly putting $K=\bar{i}, M=j$ and $K=i, M=\bar{j}$, we respectively get

$$\begin{aligned}
\overline{\nabla}_{\bar{i}}\widetilde{\varphi}_j &= \widetilde{\lambda}\widetilde{R}_{\bar{i}j} \Rightarrow \overline{\nabla}_{\bar{i}}(\widetilde{\varphi}^{\epsilon}\widetilde{g}_{\epsilon j}) = 0 \\
&\Rightarrow \overline{\nabla}_{\bar{i}}(\widetilde{\varphi}^{\bar{h}}\widetilde{g}_{\bar{h}j}) = 0 \\
&\Rightarrow (\overline{\nabla}_{\bar{i}}\widetilde{\varphi}^{\bar{h}})\widetilde{g}_{\bar{h}j} = 0 \\
&\Rightarrow (E_{\bar{i}}\widetilde{\varphi}^{\bar{h}} + \Gamma^{\bar{h}}_{\bar{i}a}\widetilde{\varphi}^{a} + \Gamma^{\bar{h}}_{\bar{i}\bar{a}}\widetilde{\varphi}^{\bar{a}})g_{hj} = 0 \\
&\Rightarrow (E_{\bar{i}}\widetilde{\varphi}^{\bar{h}})g_{hj} = 0 \\
&\Rightarrow \widetilde{\varphi}^{\bar{h}} = \widetilde{\varphi}^{\bar{h}}(x^h)
\end{aligned}$$

and

$$\begin{aligned}
\overline{\nabla}_{i}\widetilde{\varphi}_{\bar{j}} &= \widetilde{\lambda}\widetilde{R}_{i\bar{j}} \Rightarrow \overline{\nabla}_{i}(\widetilde{\varphi}^{\epsilon}\widetilde{g}_{\epsilon\bar{j}}) = 0 \\
&\Rightarrow \overline{\nabla}_{i}(\widetilde{\varphi}^{h}\widetilde{g}_{h\bar{j}}) = 0 \\
&\Rightarrow (\overline{\nabla}_{i}\widetilde{\varphi}^{h})\widetilde{g}_{h\bar{j}} = 0 \\
&\Rightarrow (E_{i}\widetilde{\varphi}^{h} + \Gamma^{h}_{ia}\widetilde{\varphi}^{a} + \Gamma^{h}_{i\bar{a}}\widetilde{\varphi}^{\bar{a}})g_{hj} = 0 \\
&\Rightarrow (\partial_{i}\widetilde{\varphi}^{h} + \Gamma^{h}_{ia}\widetilde{\varphi}^{a})g_{hj} = 0 \\
&\Rightarrow (\nabla_{i}\widetilde{\varphi}^{h})g_{hj} = 0.
\end{aligned}$$

Finally putting $K=i, M=j$, we find

$$\begin{aligned}
\overline{\nabla}_{i}\widetilde{\varphi}_j &= \widetilde{\lambda}\widetilde{R}_{ij} \Rightarrow \overline{\nabla}_{i}(\widetilde{\varphi}^{\epsilon}\widetilde{g}_{\epsilon j}) = \widetilde{\lambda}\widetilde{R}_{ij} \\
&\Rightarrow \overline{\nabla}_{i}(\widetilde{\varphi}^{\bar{h}}\widetilde{g}_{\bar{h}j}) = \widetilde{\lambda}(3-n)R_{ij} \\
&\Rightarrow (\overline{\nabla}_{i}\widetilde{\varphi}^{\bar{h}})\widetilde{g}_{\bar{h}j} = \widetilde{\lambda}(3-n)R_{ij} \\
&\Rightarrow (E_{i}\widetilde{\varphi}^{\bar{h}} + \overline{\Gamma}^{\bar{h}}_{ia}\widetilde{\varphi}^{a} + \overline{\Gamma}^{\bar{h}}_{i\bar{a}}\widetilde{\varphi}^{\bar{a}})g_{hj} = \widetilde{\lambda}(3-n)R_{ij} \\
&\Rightarrow \left[(\partial_i - y^s\Gamma^h_{si}\partial_{\bar{h}})\widetilde{\varphi}^{\bar{h}} + (y^sR^h_{sia} + y_aR^h_i - y^hR_{ia})\widetilde{\varphi}^{a} + \Gamma^h_{ia}\widetilde{\varphi}^{\bar{a}}\right]g_{hj} \\
&= \widetilde{\lambda}(3-n)R_{ij} \\
&\Rightarrow \left[\partial_i\widetilde{\varphi}^{\bar{h}} + \Gamma^h_{ia}\widetilde{\varphi}^{\bar{a}} + (y^sR^h_{sia} + y_aR^h_i - y^hR_{ia})\widetilde{\varphi}^{a}\right]g_{hj} \\
&= \widetilde{\lambda}(3-n)R_{ij} \\
&\Rightarrow (\nabla_i\widetilde{\varphi}^{\bar{h}})g_{hj} + \left[y^s(R_{sia}{}^{h} + g_{sa}R^h_i - \delta^h_sR_{ia})\widetilde{\varphi}^{a}\right]g_{hj} \\
&= \widetilde{\lambda}(3-n)R_{ij} \\
&\Rightarrow \nabla_i\widetilde{\varphi}_{\bar{j}} = \widetilde{\lambda}(3-n)R_{ij} \text{ and } y^s(R_{sia}{}^{h} + g_{sa}R^h_i - \delta^h_sR_{ia})\widetilde{\varphi}^{a} = 0 \\
&\Rightarrow \nabla_i\widetilde{\varphi}_{\bar{j}} = \lambda R_{ij}.
\end{aligned}$$

□

4. Soliton structures on TM in the context of the RQSMC $\overline{\nabla}$

4.1. Generalized Ricci-Yamabe soliton structure

The concept of Ricci solitons emerged after Hamilton introduced the Ricci flow in 1982. The Ricci flow is represented as:

$$\frac{\partial}{\partial t}g(t) = -2Ric(g(t)),$$

where g is the Riemannian metric, t denotes time, and Ric signifies the Ricci tensor of M. Ricci solitons correspond to self-similar solutions of the Ricci flow and provide a model for understanding the development of singularities in the flow. A smooth vector field V on a Riemannian manifold (M, g) is considered to define a Ricci soliton if it fulfills:

$$\frac{1}{2}L_V g + Ric = \lambda g,$$

where $L_V g$ represents the Lie derivative of the metric tensor g with respect to V, Ric is the Ricci tensor of (M, g) and λ is a constant. The vector field V is termed the potential vector field of the Ricci soliton.

Hamilton introduced the notion of Yamabe flow inspired by Yamabe's conjecture, which states that a metric of a complete Riemannian manifold can be conformally related to a metric with constant scalar curvature. The Yamabe flow takes the form:

$$\frac{\partial}{\partial t}g(t) = -r(t)g(t),$$

where $r(t)$ denotes the scalar curvature of the metric $g(t)$. This flow is employed to find solutions for the Yamabe problem. Yamabe solitons are special solutions of the Yamabe flow. A smooth vector field V on a Riemannian manifold (M, g) defines a Yamabe soliton if it satisfies:

$$\frac{1}{2}L_V g = (r - \lambda)g,$$

where r is the scalar curvature of M. A Yamabe soliton (or Ricci soliton) is categorized as shrinking, steady or expanding based on whether $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively.

In recent decades, geometric flows and their associated solitons have captured the attention of numerous geometers. In 2018, Chen and Deshmukh [1] introduced the concept of quasi-Yamabe solitons, defined on a Riemannian manifold as follows:

$$(L_V g)(X, Y) = 2(\lambda - r)g(X, Y) + 2\rho V^{\#}(X)V^{\#}(Y),$$

where $V^{\#}$ is the dual $1-$form of V, λ is a constant and ρ is a smooth function.

In 2019, Güler and Crâşmăreanu [6] introduced a novel geometric flow named the Ricci-Yamabe map on a Riemannian manifold (M, g). This flow is a combination of the Ricci flow and the Yamabe flow, formulated as:

$$\frac{\partial g}{\partial t}(t) + 2\alpha Ric(t) + \beta r(t)g(t) = 0, \tag{4.1}$$

where g, Ric, r are the (pseudo-)Riemannian metric, Ricci tensor and scalar curvature, respectively. Also, α and β are two constants whose signs can be chosen arbitrarily. Depending on the signs of the associated scalars, the Ricci-Yamabe flow can be Riemannian, semi-Riemannian or singular Riemannian. This variety of options is very useful in differential geometry and theory of relativity.

Now, we aim to extend these concepts to a more generalized framework as follows:

Definition 4.1. A (pseudo-)Riemannian manifold (M, g) of dimension $n > 2$ is deemed to admit a generalized Ricci-Yamabe soliton $(g, V, \lambda, \alpha, \beta, \rho)$ if it satisfies the following equation:

$$L_V g + 2\alpha Ric = (2\lambda - \beta r)g + 2\rho V^{\#} \otimes V^{\#}, \tag{4.2}$$

where $\lambda, \alpha, \beta, \rho \in \mathbb{R}$ and $V^{\#}$ is the 1-form dual to V. This notion provides a generalization encompassing a broad class of soliton-like equations. Specifically, a generalized Ricci-Yamabe soliton can be categorized as:

* proper Ricci-Yamabe soliton if $\rho = 0$ and $\alpha \neq 0, 1$;
* Ricci soliton if $\alpha = 1,\ \beta = \rho = 0$;
* Yamabe soliton if $\alpha = \rho = 0,\ \beta = 2$;
* Quasi-Yamabe soliton if $\alpha = 0$ and $\beta = 2$;
* Einstein soliton if $\alpha = 1,\ \beta = -1$ and $\rho = 0$;
* $\epsilon-$Einstein soliton if $\alpha = 1,\ \beta = -2\epsilon$ and $\rho = 0$.

Theorem 4.1. *Consider TM as the tangent bundle of a (pseudo-)Riemannian manifold M equipped with the RQSMC $\overline{\nabla}$. The $({}^C g, \widetilde{V}, \lambda, \alpha, \beta, \rho)$ is a generalized Ricci-Yamabe soliton if and only if following conditions are satisfying:*

$$i)\, {}^C V = (v^a, v^{\bar{a}}) = (v^a, y^s \nabla_s v^a),$$
$$ii)\ \lambda = \frac{1}{n}[\nabla_h v^h - \rho(y^s \nabla_s v_j)v^j],$$
$$iii) R_{ij} = 0,$$
$$iv)(\nabla_i \nabla_s v^h + R_{sia}{}^h v^a)g_{hj} + (\nabla_j \nabla_s v^h + R_{sja}{}^h v^a)g_{hi} = 0,$$
$$v) \nabla_s v_i \nabla_t v_j = 0,$$

where the potential vector field is the complete lift ${}^C V$ of a vector field V on M to the tangent bundle TM.

Proof. To establish the existence of the scalar λ, we start from the definition (4.2) and proceed as follows:

$$L_{\widetilde{V}}\widetilde{g}_{\varepsilon\delta}+2\alpha\widetilde{R}_{\varepsilon\delta}=(2\lambda-\beta r)\widetilde{g}_{\varepsilon\delta}+2\rho(V^{\#}\otimes V^{\#})_{\varepsilon\delta} \tag{4.3}$$

If we consider $(\varepsilon,\delta)=(\bar{i},j)$, the previous equation gives

$$L_{\widetilde{V}}\widetilde{g}_{\bar{i}j}+2\alpha\widetilde{R}_{\bar{i}j}=(2\lambda-\beta r)\widetilde{g}_{\bar{i}j}+2\rho(V^{\#}\otimes V^{\#})_{\bar{i}j}$$

and from the expression of $L_{\widetilde{V}}\widetilde{g}$ in (3.6) we have

$$\left(E_{\bar{i}}v^{\bar{h}}\right)g_{hj}+\left(\nabla_j v^h\right)g_{hi}=2\lambda g_{ij}+2\rho(V^{\#}\otimes V^{\#})_{\bar{i}j}, \tag{4.4}$$

where the potential vector field ${}^{C}V$ and $1-$form dual to ${}^{C}V$ are written as

$${}^{C}V=\begin{pmatrix} v^m \\ v^{\bar{m}} \end{pmatrix}=\begin{pmatrix} v^m \\ y^s\nabla_s v^m \end{pmatrix} \tag{4.5}$$

and

$$V^{\#}_{j}=V^I\tilde{g}_{Ij}=v^i\tilde{g}_{ij}+v^{\bar{i}}g_{\bar{i}j}=v^i.0+\ y^s\nabla_s v^i g_{ij}=y^s\nabla_s v_j,$$

$$V^{\#}_{\bar{j}}=V^I\tilde{g}_{I\bar{j}}=v^i\tilde{g}_{i\bar{j}}+v^{\bar{i}}g_{\bar{i}\bar{j}}=v^i.g_{ij}+0=v_j,$$

from which

$$\left(V^{\#}\right)=\begin{pmatrix} y^s\nabla_s v_j \\ v_j \end{pmatrix}. \tag{4.6}$$

If equations (4.5) and (4.6) are used in (4.4), we get

$$[E_{\bar{i}}(y^s\nabla_s v^h)g_{hj}+\left(\nabla_j v^h\right)g_{hi}=2\lambda g_{ij}+2\rho(y^s\nabla_s v_j)v_i,$$

which gives

$$\delta_i^s(\nabla_s v^h)g_{hj}+\left(\nabla_j v^h\right)g_{hi}=2\lambda g_{ij}+2\rho(y^s\nabla_s v_j)v_i.$$

Contracting with g^{ij} both sides of the last equation we have

$$\lambda=\frac{1}{n}[\nabla_h v^h-\rho(y^s\nabla_s v_j)v^j].$$

If we consider $(\varepsilon,\delta)=(i,j)$ in (4.3), we write

$$L_{\widetilde{V}}\widetilde{g}_{ij}+2\alpha\widetilde{R}_{ij}=(2\lambda-\beta r)\widetilde{g}_{ij}+2\rho(V^{\#}\otimes V^{\#})_{ij}.$$

From (3.6), we have

$$\begin{gathered}\left[E_i v^{\bar{h}}+\left(y^s R_{sia}^{\ \ \ h}+y_a R_i^h-y^h R_{ia}\right)v^a+\Gamma_{ia}^{\ h}v^{\bar{a}}\right]g_{hj}\\+\left[E_j v^{\bar{h}}+\left(y^s R_{sja}^{\ \ \ h}+y_a R_j^h-y^h R_{ja}\right)v^a+\Gamma_{ja}^{\ h}v^{\bar{a}}\right]g_{hi}\\+2\alpha(3-n)R_{ij}=2\rho(y^s\nabla_s v_i)(y^t\nabla_t v_j)\end{gathered}$$

and it follows

$$\begin{gathered}\left[y^s\nabla_i\nabla_s v^h+\left(y^s R_{sia}^{\ \ \ h}+y_a R_i^h-y^h R_{ia}\right)v^a\right]g_{hj}\\+\left[y^s\nabla_j\nabla_s v^h+\left(y^s R_{sja}^{\ \ \ h}+y_a R_j^h-y^h R_{ja}\right)v^a\right]g_{hi}\\+2\alpha(3-n)R_{ij}=2\rho y^s y^t(\nabla_s v_i)(\nabla_t v_j),\end{gathered}$$

from which we get

$$n\neq 3,\ R_{\alpha\delta}=0,$$

$$(\nabla_s v_i)(\nabla_t v_j)=0$$

and

$$(\nabla_i\nabla_s v^h+R_{sia}^{\ \ \ h}v^a)g_{hj}+(\nabla_j\nabla_s v^h+R_{sja}^{\ \ \ h}v^a)g_{hi}=0.$$

If we reverse the above calculations, it becomes evident that the sufficiency of the theorem can be readily demonstrated. □

Theorem 4.2. *Consider TM as the tangent bundle of a (pseudo-)Riemannian manifold M equipped with the RQSMC $\overline{\nabla}$. $({}^{C}g, \widetilde{V}, \lambda, \alpha, \beta, \rho)$ is a generalized Ricci-Yamabe soliton if and only if following conditions are satisfying:*

$$i)\ {}^{V}V = (v^{a}, v^{\bar{a}}) = (0, v^{a}),$$
$$ii)\ \lambda = 0,$$
$$iii)\ (L_{V}g)_{ij} = 2[\rho v_i v_j - \alpha(3-n)R_{ij}],$$

where the potential vector field is the vertical lift ${}^{V}V$ of a vector field V on M to the tangent bundle TM.

Proof. We will show the existence of the scalar λ. From the definition (4.2) we write

$$L_{\widetilde{V}}\widetilde{g}_{\varepsilon\delta} + 2\alpha\widetilde{R}_{\varepsilon\delta} = (2\lambda - \beta r)\widetilde{g}_{\varepsilon\delta} + 2\rho(V^{\#}\otimes V^{\#})_{\varepsilon\delta}. \tag{4.7}$$

If we consider $(\varepsilon,\delta) = (\bar{i}, j)$, from the previous equation we find

$$L_{\widetilde{V}}\widetilde{g}_{\bar{i}j} + 2\alpha\widetilde{R}_{\bar{i}j} = (2\lambda - \beta r)\widetilde{g}_{\bar{i}j} + 2\rho(V^{\#}\otimes V^{\#})_{\bar{i}j}$$

and using the expression of $L_{\widetilde{V}}\widetilde{g}$ in (3.6) we have

$$\left(E_{\bar{i}}v^{\overline{h}}\right)g_{hj} + \left(\nabla_j v^{h}\right)g_{hi} = 2\lambda g_{ij} + 2\rho(V^{\#}\otimes V^{\#})_{\bar{i}j}, \tag{4.8}$$

where the potential vector field ${}^{V}V$ and $1-$form dual to ${}^{V}V$ are written as

$${}^{V}V = \begin{pmatrix} v^{m} \\ v^{\overline{m}} \end{pmatrix} = \begin{pmatrix} 0 \\ v^{m} \end{pmatrix} \tag{4.9}$$

and

$$V^{\#}_{j} = V^{I}\widetilde{g}_{Ij} = v^{i}\widetilde{g}_{ij} + v^{\bar{i}}g_{\bar{i}j} = 0 + \ v^{i}g_{ij} = v_j,$$
$$V^{\#}_{\bar{j}} = V^{I}\widetilde{g}_{I\bar{j}} = v^{i}\widetilde{g}_{i\bar{j}} + v^{\bar{i}}g_{\bar{i}\bar{j}} = 0 + 0 = 0,$$

from which

$$\left(V^{\#}\right) = \begin{pmatrix} v_j \\ 0 \end{pmatrix}. \tag{4.10}$$

If equations (4.9) and (4.10) are used in (4.8), we get

$$2\lambda g_{ij} + 2\rho.0 = 0$$

$$\lambda = 0.$$

When we consider $(\varepsilon,\delta) = (i,j)$ in (4.7), we get

$$L_{\widetilde{V}}\widetilde{g}_{ij} + 2\alpha\widetilde{R}_{ij} = (2\lambda - \beta r)\widetilde{g}_{ij} + 2\rho(V^{\#}\otimes V^{\#})_{ij}.$$

From (3.6) we have

$$\left[E_i v^{\overline{h}} + \left(y^{s}R_{sia}{}^{h} + y_a R^{h}_{i} - y^{h}R_{ia}\right)v^{a} + \Gamma_{ia}^{h}v^{\overline{a}}\right]g_{hj}$$
$$+\left[E_j v^{\overline{h}} + \left(y^{s}R_{sja}{}^{h} + y_a R^{h}_{j} - y^{h}R_{ja}\right)v^{a} + \Gamma_{ja}^{h}v^{\overline{a}}\right]g_{hi}$$
$$+2\alpha(3-n)R_{ij} = 2\rho v_i v_j$$

and it follows

$$[E_i v^{h} + \Gamma_{ia}^{h}v^{a}]g_{hj} + [E_j v^{h} + \Gamma_{ja}^{h}v^{a}]g_{hi} + 2\alpha(3-n)R_{ij} = 2\rho v_i v_j.$$

From above equation we obtain

$$\nabla_i v_j + \nabla_j v_i + 2\alpha(3-n)R_{ij} = 2\rho v_i v_j$$

and

$$(L_V g)_{ij} = 2[\rho v_i v_j - \alpha(3-n)R_{ij}].$$

If the above calculations are followed in reverse, the sufficiency of the theorem can be easily proved. □

Theorem 4.3. *Consider TM as the tangent bundle of a (pseudo-)Riemannian manifold M equipped with the RQSMC $\overline{\nabla}$. $({}^{C}g, \widetilde{V}, \lambda, \alpha, \beta, \rho)$ is a generalized Ricci-Yamabe soliton if and only if following conditions are satisfying:*

$$\begin{gathered} i)\ {}^{H}V = (v^{a}, v^{\bar{a}}) = (v^{a}, 0), \\ ii)\ \lambda = \tfrac{1}{2n}(\nabla_{h} v^{h}), \\ iii)\ \text{For } n \neq 3,\ R_{ij} = 0, \\ iv)\ v^{a}(R_{siaj} + R_{sjai}) = 0, \end{gathered}$$

where the potential vector field is the horizontal lift ${}^{H}V$ of a vector field V on M to the tangent bundle TM.

Proof. Let us demonstrate the existence of the scalar λ. Starting from the definition (4.2) we have:

$$L_{\widetilde{V}}\widetilde{g}_{\varepsilon\delta} + 2\alpha\widetilde{R}_{\varepsilon\delta} = (2\lambda - \beta r)\widetilde{g}_{\varepsilon\delta} + 2\rho(V^{\#} \otimes V^{\#})_{\varepsilon\delta}.$$

Considering $(\varepsilon, \delta) = (\bar{i}, j)$, we obtain:

$$L_{\widetilde{V}}\widetilde{g}_{\bar{i}j} + 2\alpha\widetilde{R}_{\bar{i}j} = (2\lambda - \beta r)\widetilde{g}_{\bar{i}j} + 2\rho(V^{\#} \otimes V^{\#})_{\bar{i}j}.$$

Using the expression of $L_{\widetilde{V}}\widetilde{g}$ in (3.6) we have

$$\left(E_{\bar{i}} v^{\bar{h}}\right) g_{hj} + \left(\nabla_{j} v^{h}\right) g_{hi} = 2\lambda g_{ij} + 2\rho(V^{\#} \otimes V^{\#})_{\bar{i}j}, \tag{4.11}$$

where the potential vector field ${}^{H}V$ and $1-$form dual to ${}^{H}V$ are written as

$${}^{H}V = \begin{pmatrix} v^{m} \\ v^{\overline{m}} \end{pmatrix} = \begin{pmatrix} v^{m} \\ 0 \end{pmatrix} \tag{4.12}$$

and

$$V^{\#}_{j} = V^{I}\tilde{g}_{Ij} = v^{i}\tilde{g}_{ij} + v^{\bar{i}}g_{\bar{i}j} = 0,$$

$$V^{\#}_{\bar{j}} = V^{I}\tilde{g}_{I\bar{j}} = v^{i}\tilde{g}_{i\bar{j}} + v^{\bar{i}}g_{\bar{i}\bar{j}} = v^{i}g_{ij} + 0 = v_{j},$$

which yields

$$(V^{\#}) = \begin{pmatrix} 0 \\ v_{j} \end{pmatrix}. \tag{4.13}$$

By substituting equations (4.13) and (4.12) into equation (4.11), we get

$$\left(E_{\bar{i}} v^{h}\right) g_{hj} + \left(\nabla_{j} v^{h}\right) g_{hi} = 2\lambda g_{ij} + 2\rho.0$$

$$\left(\nabla_{j} v^{h}\right) g_{hi} = 2\lambda g_{ij}.$$

Contracting both sides of the last equation with g^{ij}, we obtain

$$\lambda = \frac{1}{2n}(\nabla_{h} v^{h}).$$

Now, considering $(\varepsilon, \delta) = (i, j)$ in equation (4.7), we find

$$L_{\widetilde{V}}\widetilde{g}_{ij} + 2\alpha\widetilde{R}_{ij} = (2\lambda - \beta r)\widetilde{g}_{ij} + 2\rho(V^{\#} \otimes V^{\#})_{ij}.$$

From (3.6), we have

$$\begin{gathered} \left[E_{i} v^{\overline{h}} + \left(y^{s} R_{sia}{}^{h} + y_{a} R^{h}_{i} - y^{h} R_{ia}\right) v^{a} + \Gamma_{ia}^{h} v^{\bar{a}}\right] g_{hj} \\ + \left[E_{j} v^{\overline{h}} + \left(y^{s} R_{sja}{}^{h} + y_{a} R^{h}_{j} - y^{h} R_{ja}\right) v^{a} + \Gamma_{ja}^{h} v^{\bar{a}}\right] g_{hi} \\ + 2\alpha(3 - n) R_{ij} = 2\rho v_{i} v_{j}, \end{gathered}$$

which gives

$$\begin{aligned} & \left[\left(y^{s} R_{sia}{}^{h} + y_{a} R^{h}_{i} - y^{h} R_{ia}\right) v^{a}\right] g_{hj} \\ & + \left[\left(y^{s} R_{sja}{}^{h} + y_{a} R^{h}_{j} - y^{h} R_{ja}\right) v^{a}\right] g_{hi} + 2\alpha(3 - n) R_{ij} \\ = \quad & 0. \end{aligned}$$

From the above equation, for $n \neq 3$, we deduce:

$$R_{ij} = 0$$

and

$$v^a(R_{siaj} + R_{sjai}) = 0.$$

If we follow the above calculations in reverse, we can easily establish the sufficiency of the theorem. □

4.2. Riemannian soliton

A natural generalization of the Hamilton-Ricci flow is the concept of a Riemannian flow, defined by: $\frac{\partial}{\partial t}G(t) = -2Rg(t),\ G = \frac{1}{2}g \wedge g$, where R is the Riemannian curvature tensor and $\wedge$ represents the Kulkarni-Nomizu product. For $C, D \in \Im_2^0(M)$, the Kulkarni-Nomizu product is defined as:

$$\begin{aligned}(C \wedge D)(W, X, Y, Z) &= C(W, Z)\, D(X, Y) + C(X, Y)\, D(W, Z)\\ &\quad -C(W, Y)\, D(X, Z) - C(X, Z)\, D(W, Y).\end{aligned}$$

In local coordinates, this can be expressed as:

$$C \wedge D = C_{il}D_{jk} + C_{jk}D_{il} - C_{ik}D_{jl} - C_{jl}D_{ik}.$$

The notion of a Riemannian soliton, similar to the Ricci soliton, was introduced by Hirica and Udriste [20]. A Riemannian metric g on a smooth manifold M is considered a Riemannian soliton if there exists a differentiable vector field X and a real constant λ such that

$$R + \frac{1}{2}g \wedge L_X g = \lambda G,$$

where L_X is the Lie derivative along $X,\ \lambda$ is a constant and R is the Riemannian tensor of g. Such a vector field X is known as the potential of the soliton. A Riemannian soliton is referred to as the potential of the soliton. A Riemannian soliton can further be classified as shrinking, steady, or expanding depending on whether $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively. Moreover, $R = R_{ijkl}$ is the Riemannian curvature tensor and $G = G_{ijkl} = g \wedge g = g_{il}g_{jk} - g_{ik}g_{jl}$.

Theorem 4.4. *Consider TM as the tangent bundle of a (pseudo-)Riemannian manifold M equipped with the RQSMC $\overline{\nabla}$.The $(TM, {}^C g, \widetilde{X}, \lambda)$ is a Riemannian soliton if and only if the following conditions are satisfying:*

$$\begin{gathered} i)\widetilde{X} = (v^l, v^{\bar{l}}) = (v^l, y^a A_a^l + B^l),\\ ii)\lambda = \frac{1}{n}\left(\nabla_h v^h + E_{\overline{h}} v^{\overline{h}}\right),\\ iii)\ \nabla_s R_{ijkl} = 0, \end{gathered}$$

where $\tilde{X} = v^a E_a + v^{\bar{a}} E_{\bar{a}}$ be a fiber-preserving vector field on $TM,\ \ \lambda \in \mathbb{R},\ B = \left(B^l\right)$ and $A = (A_s^h)$ are the $(1,0)$ and $(1,1)$ tensor fields on M, respectively.

Proof. We will begin by stating that we will only calculate the expressions:

$$\begin{aligned} i)\quad & \tilde{R}_{ijkl} + \frac{1}{2}({}^C g)_{ij} \wedge \left(L_{\tilde{X}}\, {}^C g\right)_{kl} = \lambda \widetilde{G}_{ijkl},\\ ii)\quad & \tilde{R}_{i\bar{j}k\bar{l}} + \frac{1}{2}({}^C g)_{i\bar{j}} \wedge \left(L_{\tilde{X}}\, {}^C g\right)_{k\bar{l}} = \lambda \widetilde{G}_{i\bar{j}k\bar{l}}. \end{aligned}$$

These calculations will suffice for our purposes, as the other components beyond these will yield the same results as these equations. Therefore, it is adequate for us to work with these expressions.

Given that $\widetilde{G} = \widetilde{G}_{IJKL} = ({}^C g)_{IL}({}^C g)_{JK} - ({}^C g)_{IK}({}^C g)_{JL}$, the Kulkarni-Nomizu products we will employ in our proof are as follows:

$$\begin{aligned} &\widetilde{G}_{ij\overline{kl}} = g_{il}g_{jk} - g_{ik}g_{jl}, && \widetilde{G}_{\bar{i}jk\bar{l}} = -g_{ik}g_{jl}, && \widetilde{G}_{i\bar{j}k\bar{l}} = g_{il}g_{jk},\\ &\widetilde{G}_{\overline{ij}kl} = g_{il}g_{jk} - g_{ik}g_{jl}, && \widetilde{G}_{\bar{i}j\bar{k}l} = g_{il}g_{jk}, && \widetilde{G}_{i\bar{j}\bar{k}l} = -g_{ik}g_{jl} \end{aligned}$$

and other components are zero.

Starting from:

$$\tilde{R}_{i\bar{j}k\bar{l}} + \frac{1}{2}({}^{C}g)_{i\bar{j}} \wedge \left(L_{\tilde{X}}\, {}^{C}g\right)_{k\bar{l}} = \lambda\widetilde{G}_{i\bar{j}k\bar{l}},$$

we derive:

$$\begin{aligned} &\frac{1}{2}[({}^{C}g)_{i\bar{l}}(L_{\tilde{X}}\, {}^{C}g)_{\bar{j}k} + ({}^{C}g)_{\bar{j}k}(L_{\tilde{X}}\, {}^{C}g)_{i\bar{l}} \\ &-({}^{C}g)_{ik}(L_{\tilde{X}}\, {}^{C}g)_{\bar{j}\bar{l}} - ({}^{C}g)_{\bar{j}\bar{l}}(L_{\tilde{X}}\, {}^{C}g)_{ik}] \\ = \; &\lambda g_{il}g_{jk} \end{aligned}$$

$$[g_{il}(L_{\tilde{X}}\, {}^{C}g)_{\bar{j}k} + g_{jk}(L_{\tilde{X}}\, {}^{C}g)_{i\bar{l}}] = 2\lambda g_{il}g_{jk}.$$

Contracting both sides with g^{ij}, we obtain:

$$\delta^l_j(L_{\tilde{X}}\, {}^{C}g)_{\bar{j}k} + \delta^i_k(L_{\tilde{X}}\, {}^{C}g)_{i\bar{l}} = 2\lambda\delta^j_l g_{jk}$$

and using the expression of $(L_{\tilde{X}}\, {}^{C}g)$ in (3.6):

$$\delta^l_j[(E_{\bar{j}}v^{\bar{h}})g_{hk} + (\nabla_k v^h)g_{hj}] + \delta^i_k[(E_{\bar{l}}v^{\bar{h}})g_{hi} + (\nabla_i v^h)g_{hl}] = 2\lambda g_{lk}.$$

Contracting both sides with g^{lk}, we get:

$$\delta^l_j[(E_{\bar{j}}v^{\bar{h}})\delta^l_h + (\nabla_k v^h)g_{hj}g^{lk}] + \delta^i_k[(E_{\bar{l}}v^{\bar{h}})g_{hi}g^{lk} + (\nabla_i v^h)\delta^k_h] = 2\lambda n,$$

which leads to:

$$2(E_{\bar{h}}v^{\bar{h}} + \nabla_h v^h) = 2\lambda n$$

and

$$\lambda = \frac{1}{n}(E_{\bar{h}}v^{\bar{h}} + \nabla_h v^h). \tag{4.14}$$

Applying $E_{\bar{h}}$ to both sides in the equation (4.14), we obtain

$$v^{\bar{l}} = y^a A^l_a + B^l.$$

Furthermore, starting from:

$$\tilde{R}_{ijkl} + \frac{1}{2}({}^{C}g)_{ij} \wedge \left(L_{\tilde{X}}\, {}^{C}g\right)_{kl} = \lambda\widetilde{G}_{ijkl},$$

we can derive:

$$\begin{aligned} &y^s\nabla_s R_{ijkl} + \frac{1}{2}[({}^{C}g)_{il}\left(L_{\tilde{X}}\, {}^{C}g\right)_{jk} + ({}^{C}g)_{jk}\left(L_{\tilde{X}}\, {}^{C}g\right)_{il} \\ &-({}^{C}g)_{ik}\left(L_{\tilde{X}}\, {}^{C}g\right)_{jl} - ({}^{C}g)_{jl}\left(L_{\tilde{X}}\, {}^{C}g\right)_{ik}] \\ = \; &0 \end{aligned}$$

and it follows that:

$$\nabla_s R_{ijkl} = 0.$$

This is easily derived from the obtained results, completing the proof of the necessity of the theorem. It demonstrates that the base manifold M is a local symmetric manifold ($\nabla_s R_{ijkl} = 0$). The sufficiency of the theorem can be easily proven by reversing the above calculations. □

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author’s contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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