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Research Article

Optimal variable acceptance sampling plan for exponential distribution using Bayesian estimate under Type-I hybrid censoring

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Abstract

Acceptance sampling plans with censoring schemes are crucial for improving quality control by efficiently managing incomplete information. This approach improves cost and time effectiveness compared to traditional methods, providing a more accurate assessment of product quality. In this study, a variable acceptance sampling plan under Type-I hybrid censoring is designed for a lot of independent and identical units with exponential lifetimes using Bayesian estimation of the mean life. This novel approach diverges from conventional methods in acceptance sampling plans, which rely on maximum likelihood estimation and the minimization of Bayes risk. Bayesian estimation is obtained using both squared error loss and Linex loss functions. Under each method, a nonlinear optimization problem is solved to minimize the testing cost, and the optimal values of the plan parameters are determined. The proposed plans are illustrated using various numerical examples, with each plan presented in tables. The acceptance sampling plan using the squared error loss function proves to be more cost-effective than the plan using the Linex loss function. A comparative analysis of the proposed plans with existing work in the literature demonstrates that our cost is much lower than the cost of existing plans using maximum likelihood estimation. Additionally, a real-life case study is conducted to validate the approach.

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1. Introduction

In the world of manufacturing and quality control, ensuring the consistent quality of products is of paramount importance. However, inspecting every item produced can be impractical, time-consuming, and costly. This is where acceptance sampling plans (ASPs) come into play. An acceptance sampling plan (ASP) is a statistical technique used to make informed decisions about accepting or rejecting a batch or lot of items based on the inspection of a sample.

Understanding and implementing a well-designed ASP is crucial for manufacturers seeking to maintain consistent product quality, optimize resources, and meet customer expectations. By employing statistically sound sampling techniques, organizations can make reliable decisions regarding batch acceptance, leading to improved efficiency, reduced costs, and enhanced customer satisfaction.

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Censoring schemes play a crucial role in ASP and are designed to optimize the efficiency and effectiveness of the sampling process. Type-I and Type-II censoring schemes are widely used in statistical analysis, particularly in reliability testing and acceptance sampling. In a Type-I censoring scheme, the time of the experiment is predetermined, but the total number of failures that occurred becomes a random variable. Conversely, in a Type-II censoring scheme, the testing time is a random variable since the number of observed failures is fixed. When a combination of both Type-I and Type-II censoring schemes is employed, it is referred to as a hybrid censoring scheme (HCS).

Here, we consider an experiment where a sample of n units is subjected to testing. Also, assume that the sample units' lifespans are independent random variables and identically follow an exponential distribution. The sample units ordered lifetimes are represented as $X_{1,n}, X_{2,n}, ..., X_{n,n}$. The test is concluded either when a predetermined number, γ (where $\gamma < n$), of the n items, has failed or when a predetermined time, $\mathcal{T}$, has been reached. In other words, the life test ends at a random time $\mathcal{T}^*$, which is the minimum of either the ordered lifetime $(X_{\gamma,n})$ or the predetermined time $\mathcal{T}$. Moreover, a widely accepted assumption is that the failed units in the experiment are not substituted or replaced.

The Type-I HCS, proposed by [14], has found extensive application in reliability acceptance testing, such as in MIL-STD-781-C[†] [25]. Since its introduction, extensive research has been conducted on hybrid censoring and its various variations. Epstein initially introduced the Type-I HCS and discussed estimation methods for the exponential distribution. Additionally, a two-sided confidence interval was put forth without a mathematical proof of how it was obtained. Subsequent researchers, such as Fairbanks et al. [15], Chen and Bhattacharyya [9], Barlow et al. [5] and Bartholomew [6] made slight modifications to Epstein's proposition and worked on the derivation of the conditional moment generating function approach to derive the exact distribution of the conditional maximum likelihood estimator (MLE) of the mean life parameter. Childs et al. [12] obtained a simplified but equivalent form of the exact distribution of the MLE as derived by [9]. Draper and Guttman [13] investigated the Bayesian inference within the Type-I HCS and obtained Bayesian estimates and two-sided credible intervals using an inverted gamma prior.

Various studies have investigated the application of ASPs with hybrid censoring. With the Type-I generalized HCS, Chakrabarty et al. [7] developed an optimal reliability ASP for non-repairable products sold under the general rebate warranty for products having Weibull distributed lifetimes. Later, in their work [8], Chakrabarty et al. employed an accelerated life test setting and introduced a decision model to obtain an optimal sampling plan for products covered under warranty using the Type-I HCS.

In [19], Kumar and Ramyamol designed a cost-efficient ASP for exponentially distributed lifetime data under Type-I censoring based on MLE of the mean life. Most of the work in ASPs is centered on the MLE of the parameter for deciding the acceptance and rejection of a lot, even though the Bayesian estimator is more reliable than MLE.

Bayesian ASPs have gained considerable attention due to their ability to incorporate prior information and make informed decisions based on observed data. However, existing Bayesian sampling plans focus on minimizing Bayes risk, while consideration of testing cost optimization remains limited. Lam [34], Lam and Choy [35], Lin et al. [23], and Huang and Lin [16, 17] focused on conventional Type-I, Type-II, and random censored samples.

Recently, Chen et al. [11] constructed a curtailed Bayesian sampling plan using Type-I hybrid censored samples, demonstrating reduced risk compared to traditional Bayesian sampling plans. Prajapati et al. [29] proposed a decision-theoretic method for analyzing the exponential distribution under a generalized Type-I HCS. Furthermore, in [30], they expanded this method to include the generalized Type-II HCS, employing an effective loss function. Prajapat et al. [28] derived an optimal Bayesian ASP for the two-parameter exponential distribution under a Type-I HCS based on a four-parameter conjugate prior. In [32], Sharma introduced multiple

[†]MIL-STD-781-C is a military standard that provides guidelines for reliability testing, and quality control of electronic and electrical equipment to ensure they meet required performance and reliability standards.

deferred state repetitive group sampling plans for generalized Gamma distribution under hybrid censoring. These studies collectively highlight the potential benefit of hybrid censoring in Bayesian ASPs.

Researchers like Chen et al. [10], Lin et al. [22], and Prajapati et al. [31] have specifically addressed Bayesian sampling plans for exponential distributions by reducing Bayes risk utilizing Type-I hybrid censored samples. However, these plans are not based on the Bayesian estimator of the mean life parameter, which is better than MLE [21].

In this work, we develop an ASP for a batch of units where we assume that the failure time follows an exponential distribution with mean ϑ. Further, it is assumed that the lifetime data are independent and identically distributed. Our aim is to design an optimal Bayesian ASP under Type-I HCS by using a Bayesian estimator of the mean life parameter, ϑ, as the decision function. Our plan stands out from all other existing Bayesian sampling plans by centering the decision of acceptance or rejection of a lot based upon the value of the Bayesian estimator of the mean lifetime parameter. As a novel approach, we have considered the minimization of the testing cost subject to the requirements of Type I and Type II error constraints. This departure from traditional Bayes risk-oriented approaches allows us to focus explicitly on the optimization of testing costs, a critical consideration in practical industrial problems.

Our plan has significant applications in fields like reliability engineering and survival analysis. In reliability engineering, the plan helps to estimate failure rates of components and systems (electronics or machinery), incorporating prior knowledge and updating predictions with new data, even when tests are terminated early due to time constraints or limited failures. This improves the precision of product lifetime predictions and maintenance schedules. In survival analysis, in medical studies, Bayesian estimation with Type-I hybrid censoring estimates the time until events such as death or relapse occur. This methodology allows for continuous updating of survival probabilities as new patient data becomes available, resulting in better-informed treatment plans and more accurate prognostic models, even in cases where data is limited due to early termination of follow-up.

The rest of the paper is organized as follows: Section 2 provides detailed information about Bayesian inference of exponentially distributed lifetime data under Type-I HCS. The Bayesian estimator of the mean life parameter, ϑ, is considered using both the squared error loss (SEL) function and the Linex loss function. Section 3 presents the formulation and optimization framework of the proposed ASP. Here, the plan parameters (n, t_1, t_2) are obtained by minimizing the expected testing cost (ETC) subject to required probability conditions using both the loss functions, namely, the SEL and Linex loss functions. The distribution of the Bayesian estimator of ϑ under Type-I HCS for both loss functions is derived using the delta method. In Section 4, we present the results of numerical computations of the proposed ASP and a comparison with existing work in [19]. A real-life case study is done to illustrate the performance of the sampling plan presented here. Finally, Section 5 concludes the paper with a summary of the contributions and future research directions.

2. Bayesian estimation for Type-I hybrid censoring scheme

Consider a lot of units, all of which experience failures over time following an exponential distribution. The probability density function (PDF) that characterizes the failure time of each unit is given by:

$$f(x, \vartheta) = \frac{1}{\vartheta} exp\left\{-\frac{-x}{\vartheta}\right\} \qquad x \geq 0, \vartheta > 0 \tag{2.1}$$

Here we conduct Type-I hybrid censored life testing, and we obtain one of the following two forms of observations as our observed data:

$$\begin{aligned} &\text{Data type I} : \{X_{1,n} < X_{2,n} < \cdots < X_{\gamma,n}\} \quad \text{if} \quad \mathcal{T}^* = X_\gamma, \\ &\text{Data type II} : \{X_{1,n} < X_{2,n} < \cdots < X_{\mathcal{D},n}\} \quad \text{if} \quad \mathcal{T}^* = \mathcal{T}, \end{aligned} \tag{2.2}$$

where $\mathcal{D}$ denotes the number of failures observed before time $\mathcal{T}^*$. Thus, $\mathcal{D} = \gamma$ when $\mathcal{T}^* = X_\gamma$.

This section discusses the method of obtaining a Bayesian estimator of the unknown parameter ϑ under the SEL and Linex loss functions and their corresponding distributions. The

Bayesian estimator of the parameter ϑ is used to design the ASP in the next sections, and this idea is new in ASPs.

Under Type-I HCS, the likelihood function is obtained as (see [9])

$$L(x,\vartheta) = \frac{n!}{(n-\mathcal{D})!}\frac{1}{\vartheta^{\mathcal{D}}} exp\left\{-\frac{1}{\vartheta}\sum_{i=1}^{\mathcal{D}} x_{i,n} - \frac{(n-\mathcal{D})}{\vartheta}\mathcal{T}^*\right\}, \tag{2.3}$$

and log-likelihood function as

$$L(\vartheta,x) = \log\left[\frac{n!}{(n-\mathcal{D})!}\right] - \mathcal{D}\log\vartheta - \frac{1}{\vartheta}\sum_{i=1}^{\mathcal{D}} x_{i,n} - \frac{(n-\mathcal{D})}{\vartheta}\mathcal{T}^*,$$

where $\mathcal{T}^* = \mathrm{Min}\{\mathcal{T}, X_{\gamma,n}\}$ and $\mathcal{D}$ = Number of units failed before $\mathcal{T}^*$.

Hence, by referring to the observed sample, the MLE of ϑ can be determined as follows (see [9]):

$$\hat{\vartheta}_{MLE} = \begin{cases} \frac{1}{\mathcal{D}}\left[\sum\limits_{i=1}^{\mathcal{D}} x_{i,n} + (n-\mathcal{D})\mathcal{T}\right] & \text{if} \quad 1 \le \mathcal{D} \le \gamma - 1 \\ \frac{1}{\gamma}\left[\sum\limits_{i=1}^{\gamma} x_{i,n} + (n-\gamma)X_{\gamma,n}\right] & \text{if} \quad \mathcal{D} = \gamma \\ n\mathcal{T} & \text{if} \quad \mathcal{D} = 0. \end{cases} \tag{2.4}$$

It is evident that $\widehat{\vartheta}_{MLE}$ is a conditioned MLE of ϑ, with the condition that at least one observed failure is present.

Here Bayesian inference is done by using two kinds of loss functions, namely, SEL and Linex loss function. SEL is a symmetric function and its symmetric nature is demonstrated as (see [27])

$$l_1(\eta,\widehat{\eta}) = (\widehat{\eta} - \eta)^2 \tag{2.5}$$

where $\widehat{\eta}$ is the estimate of the unknown parameter η.

The Bayesian estimation of any function $g = g(\vartheta)$ under the SEL in (2.5), is obtained by

$$\widehat{g}_s = E(g \mid Data) = \frac{\int_{\vartheta} g(\vartheta)L(x,\vartheta)\rho(\vartheta)d\vartheta}{\int_{\vartheta} L(x,\vartheta)\rho(\vartheta)d\vartheta}. \tag{2.6}$$

where ϑ follows the prior distribution with PDF $\rho(\vartheta)$.

Using Type-I HCS, Draper and Guttman [13] investigated the application of Bayesian inference to estimate the unknown parameter ϑ under SEL function. They assumed in the study that ϑ follows an inverted gamma prior distribution, with the PDF given below

$$\rho(\vartheta) = \frac{a^b}{\Gamma(b)}\vartheta^{-(b+1)}e^{-a/\vartheta}, \quad \vartheta > 0,\ a > 0,\ \text{and } b > 0. \tag{2.7}$$

The prior in (2.7) becomes non-informative prior when the hyper-parameters $a = b = 0$.

The posterior density function of ϑ, based on the knowledge from the observed data and the inverted gamma prior given by (2.7), becomes (see [4])

$$h(\vartheta \mid Data) = \frac{\left(\mathcal{D}\ \hat{\vartheta}_{MLE} + a\right)^{(\mathcal{D}+b)}}{\Gamma(\mathcal{D}+b)}\vartheta^{-(\mathcal{D}+b+1)} \exp\left\{-\left(\mathcal{D}\ \hat{\vartheta}_{MLE} + a\right)/\vartheta\right\} \tag{2.8}$$

Considering that $\frac{(\mathcal{D}\ \hat{\vartheta}_{MLE}+a)}{\vartheta}$ is distributed as $\chi^2_{2(b+a)}/2$ a posteriori, the posterior mean serves as the Bayesian estimate of ϑ under the SEL function. If $\mathcal{D} + b > 1$, this estimator is simply produced as (see [4])

$$\widehat{\vartheta}_s = \frac{\mathcal{D}\ \hat{\vartheta}_{MLE} + a}{\mathcal{D} + b - 1}. \tag{2.9}$$

When $a = b = 0$, that is, when the prior becomes non-informative, it is evident that the MLE of ϑ provided by (2.4) agrees with the Bayesian estimator in (2.9).

The second is the Linex loss function, which is referred to as an asymmetric function and is represented by the following formula (see [27]):

$$l_2(\eta, \widehat{\eta}) \propto e^{c(\widehat{\eta}-\eta)} - c(\widehat{\eta}-\eta) - 1, \quad c \neq 0. \tag{2.10}$$

The direction and magnitude of asymmetry are both determined by the parameter c's value. Bayesian estimation tends to overestimate when $c < 0$, but it favors underestimating when $c > 0$. The Linex loss value resembles the SEL when c approaches 0, leading to an almost symmetric behavior. Applying the Linex loss function, we can derive the Bayesian estimation of $g(\vartheta)$ as ([27])

$$\widehat{g}_l = -\frac{1}{c} \ln E\left(e^{-cg} \mid Data\right), \tag{2.11}$$

where

$$E\left(e^{-cg} \mid Data\right) = \frac{\int_\vartheta e^{-cg(\vartheta)} L(x,\vartheta)\rho(\vartheta)d\vartheta}{\int_\vartheta L(x,\vartheta)\rho(\vartheta)d\vartheta}. \tag{2.12}$$

But in most cases, the ratio of integrals in (2.6) and (2.12) is not possible to derive an analytical expression for it. As a result, Lindley [24] developed approximate procedures for determining the ratio of two integrals, such as (2.6) and (2.12). Several researchers have used this to derive approximate Bayesian estimates [20].

Thus, based on Linux loss functions, we get the Bayesian estimator of $g(\vartheta) = \vartheta$ as

$$\widehat{\vartheta}_l = -\frac{1}{c} \ln E\left(e^{-c\vartheta} \mid Data\right). \tag{2.13}$$

By applying Lindley's approximate method in (2.13) for solving the ratio of integrals without a closed form, we get the Bayesian estimate of ϑ as follows:

$$\widehat{\vartheta}_l = \widehat{\vartheta}_{MLE} - \frac{1}{c} \ln\left[1 + \frac{c}{2\mathcal{D}}\left(c\widehat{\vartheta}^2_{MLE} - 2a + 2\widehat{\vartheta}_{MLE}(b-1)\right)\right]. \tag{2.14}$$

3. Acceptance sampling plans under Type-I hybrid censoring scheme

In this section, we design ASPs for a lot of units having exponential failure time with PDF given by (2.1). For that, a sample comprising n elements is taken from the lot and is tested under Type-I HCS.

Let ϑ_A be the acceptable quality level (AQL) and ϑ_U be the unacceptable quality level (UQL) of a unit item in the lot. The acceptance and rejection of the lot depend on the probability conditions given by:

$$\begin{aligned} &P\left(\text{Lot is rejected} \mid \vartheta = \vartheta_A\right) \leq \alpha, \\ &P\left(\text{Lot is accepted} \mid \vartheta = \vartheta_U\right) \leq \beta, \end{aligned} \tag{3.1}$$

where β is the consumer's risk, α is the producer's risk, and $0 < \alpha, \beta < 1$.

3.1. Normal approximation of the Bayesian estimators of ϑ using delta method

Here, the Bayesian estimator of the unknown parameter ϑ is used to design the ASP. In order to proceed, it is necessary to derive the distribution of the Bayesian estimator. Recall that both the Bayesian estimators are derived in terms of $\widehat{\vartheta}_{MLE}$, the MLE of ϑ. In [12], Childs et al. derived the exact distribution for the conditional MLE of ϑ, $\widehat{\vartheta}_{MLE}$ for $\mathcal{D} \geq 1$ with PDF as

$$\begin{aligned} f_{\widehat{\vartheta}_{MLE}}(x) = (1-v^n)^{-1} \Bigg[&\sum_{\mathcal{D}=1}^{\gamma-1} \sum_{k=0}^{\mathcal{D}} A_{k,\mathcal{D}}\, q\left(x - \mathcal{T}_{k,\mathcal{D}};\ \frac{\mathcal{D}}{\vartheta}, \mathcal{D}\right) + q\left(x;\ \frac{\gamma}{\vartheta}, \gamma\right) \\ &+ \gamma \binom{n}{\gamma} \sum_{k=1}^{\gamma} \frac{(-1)^k v^{n-\gamma+k}}{n-\gamma+k} \binom{\gamma-1}{k-1} q\left(x - T_{k,\gamma};\ \frac{\gamma}{\vartheta}, \gamma\right) \Bigg], \quad 0 < x < n\mathcal{T} \end{aligned} \tag{3.2}$$

where

$$v = \exp\left\{-\frac{\mathcal{T}}{\vartheta}\right\}, \quad T_{k,\mathcal{D}} = \frac{(n-\mathcal{D}+k)\mathcal{T}}{\mathcal{D}},$$

$$A_{k,\mathcal{D}} = (-1)^k \left(\begin{array}{c} n \\ \mathcal{D} \end{array} \right) \left(\begin{array}{c} \mathcal{D} \\ k \end{array} \right) u^{n-\mathcal{D}+k},$$

and

$$q(x;\ p, t) = \begin{cases} \frac{p^t}{\Gamma(t)} x^{t-1} e^{-px}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

From (3.2), the expectation of $\widehat{\vartheta}_{MLE}$ and $E(\widehat{\vartheta}^2_{MLE})$ are obtained respectively as

$$E(\widehat{\vartheta}_{MLE}) = (1 - v^n)^{-1} \left[\sum_{\mathcal{D}=1}^{\gamma-1} \sum_{k=0}^{\mathcal{D}} A_{k,\mathcal{D}} \left(\vartheta + T_{k,\mathcal{D}} \right) + \vartheta \right.$$
$$\left. + \gamma \left(\begin{array}{c} n \\ \gamma \end{array} \right) \sum_{k=1}^{\gamma} \frac{(-1)^k v^{n-\gamma+k}}{n - \gamma + k} \left(\begin{array}{c} \gamma - 1 \\ k - 1 \end{array} \right) (\vartheta + T_{k,\gamma}) \right], \quad (3.3)$$

and

$$E\left(\widehat{\vartheta}^2_{MLE}\right) = (1 - v^n)^{-1} \left[\sum_{\mathcal{D}=1}^{\gamma-1} \sum_{k=0}^{\mathcal{D}} A_{k,\mathcal{D}} \left\{ \frac{\vartheta^2}{\mathcal{D}} (1 + \mathcal{D}) + 2T_{k,\mathcal{D}} \vartheta + \left(T_{k,\mathcal{D}} \right)^2 \right\} \right.$$
$$\left. + \frac{\vartheta^2}{\gamma} (1 + \gamma) + \gamma \left(\begin{array}{c} n \\ \gamma \end{array} \right) \cdot \sum_{k=1}^{\gamma} \frac{(-1)^k v^{n-\gamma+k}}{n - \gamma + k} \left(\begin{array}{c} \gamma - 1 \\ k - 1 \end{array} \right) \left\{ \frac{\vartheta^2}{\gamma} (1 + \gamma) + 2T_{k,\gamma} \vartheta + (T_{k,\gamma})^2 \right\} \right]. \quad (3.4)$$

It is important to note that the variance of $\widehat{\vartheta}_{MLE}$, denoted as $\sigma^2 \left(\widehat{\vartheta}_{MLE} \right)$, can be obtained by applying equations (3.3) and (3.4).

Define the Bayesian estimator of ϑ using SEL, $\widehat{\vartheta}_s = U_1 \left(\widehat{\vartheta}_{MLE} \right)$ then by delta method (as in [3]), $\widehat{\vartheta}_s$ follows Normal distribution with mean,

$$E\left(\widehat{\vartheta}_s\right) = U_1 \left(E\left(\widehat{\vartheta}_{MLE}\right) \right) = \frac{\mathcal{D}\ E\left(\widehat{\vartheta}_{MLE}\right) + a}{\mathcal{D} + b - 1}, \quad (3.5)$$

and variance,

$$\begin{aligned} V\left(\widehat{\vartheta}_s\right) &= \left(U_1' \left(E\left(\widehat{\vartheta}_{MLE}\right) \right) \right)^2 \sigma^2 \left(\widehat{\vartheta}_{MLE} \right) \\ &= \left(\frac{\mathcal{D}}{\mathcal{D} + b - 1} \right)^2 \sigma^2 \left(\widehat{\vartheta}_{MLE} \right). \end{aligned} \quad (3.6)$$

Similarly, from (2.14), we can define the Bayesian estimator of ϑ using Linex loss function, $\widehat{\vartheta}_l$ as $U_2 \left(\widehat{\vartheta}_{MLE} \right)$. Then by delta method, $\widehat{\vartheta}_l$ follows Normal distribution with mean,

$$\begin{aligned} E\left(\widehat{\vartheta}_l\right) &= U_2 \left(E\left(\widehat{\vartheta}_{MLE}\right) \right) \\ &= E\left(\widehat{\vartheta}_{MLE}\right) - \frac{1}{c} \ln \left[1 + \frac{c}{2\mathcal{D}} \left(cE\left(\widehat{\vartheta}_{MLE}\right)^2 - 2a + 2E\left(\widehat{\vartheta}_{MLE}\right)(b - 1) \right) \right], \end{aligned} \quad (3.7)$$

and variance,

$$\begin{aligned} V\left(\widehat{\vartheta}_l\right) &= \left(U_2' \left(E\left(\widehat{\vartheta}_{MLE}\right) \right) \right)^2 \sigma^2 \left(\widehat{\vartheta}_{MLE} \right) \\ &= \left(1 - \frac{2cE\left(\widehat{\vartheta}_{MLE}\right) + 2b - 2}{2\mathcal{D} + c\left(cE\left(\widehat{\vartheta}_{MLE}\right)^2 - 2a + 2E\left(\widehat{\vartheta}_{MLE}\right)(b - 1) \right)} \right)^2 \sigma^2 \left(\widehat{\vartheta}_{MLE} \right). \end{aligned} \quad (3.8)$$

3.2. ASP with Bayesian estimator of ϑ using SEL

For this ASP, a sample consisting of n items is taken from the lot and examined using Type-I hybrid censoring. The probability requirements stated in (3.1) are used to determine whether to accept or reject a lot. In this case, our ASP is as follows:

Step 1: A random sample of size n is taken from the lot and is tested up to time $\mathcal{T}^* = \mathrm{Min}\{\mathcal{T}, X_\gamma\}$ where $\mathcal{T}$ is the pre-fixed time and $\gamma < n$ is the pre-fixed number of failures.The number of failures that happened before the time $\mathcal{T}^*$ with associated lifetimes provided by (2.2), i.e, either data type I or type II, is denoted by $\mathcal{D}$. Also, fix the values of AQL, UQL, producer's risk (α), and consumer's risk (β).

Step 2: Calculate the Bayesian estimate of ϑ using SEL ($\widehat{\vartheta}_s$), which is given by

$$\widehat{\vartheta}_s = \frac{\mathcal{D}\ \hat{\vartheta}_{MLE} + a}{\mathcal{D} + b - 1}.$$

Step 3: Continue the testing process, by repeating Steps 1, and 2 if $t_1 \le \widehat{\vartheta}_s < t_2$. If not, go to Step 4.

Step 4: Accept the lot if $\widehat{\vartheta}_s \ge t_2$ and reject the lot if $\widehat{\vartheta}_s < t_1$.

The proposed plan is also illustrated using the flowchart in Figure 1. The key objective is to minimize the total testing cost subject to the constraints given in (3.1) in order to obtain the optimal values of n, t_1 and t_2. The total testing cost is the product of the testing cost of an item for unit time and the total testing time. Based on the sampling plan described above, the total testing time is the product of the time taken to reach a decision for each sample, $\widehat{\vartheta}_s$, by the number of samples tested. But, $\widehat{\vartheta}_s$ and number of samples are random variables. So, here, the expected testing cost is obtained.

Consider p_c, p_a, and p_r as the probabilities of continuation of testing, accepting the lot, and rejecting the lot, respectively. Then, the corresponding long-run probabilities of acceptance and rejection are given by $P_a = \frac{p_a}{1-p_c}$ and $P_r = \frac{p_r}{1-p_c}$, and the expected number of items that failed is given by $\frac{1}{1-p_c}$ (see [33]). Thus, the expected testing cost for the ASP is obtained as

$$\mathrm{ETC} = \frac{C\ E\left(\widehat{\vartheta}_s\right)}{1 - p_c},$$

where C is the testing cost of an item for unit time.

From equations (3.5) and (3.6), we get that, $\widehat{\vartheta}_s$ follows normal distribution with mean,

$$E\left(\widehat{\vartheta}_s\right) = \frac{\mathcal{D}\ E\left(\hat{\vartheta}_{MLE}\right) + a}{\mathcal{D} + b - 1},$$

and variance,

$$V\left(\widehat{\vartheta}_s\right) = \left(\frac{\mathcal{D}}{\mathcal{D} + b - 1}\right)^2 \sigma^2\left(\widehat{\vartheta}_{MLE}\right).$$

Then, the required probabilities are given by:

$$p_a = P\left(\widehat{\vartheta}_s \ge t_2\right) = P\left[Z \ge \frac{t_2 - E\left(\widehat{\vartheta}_s\right)}{\sqrt{V\left(\widehat{\vartheta}_s\right)}}\right], \tag{3.9}$$

$$p_r = P\left(\widehat{\vartheta}_s < t_1\right) = P\left[Z < \frac{t_1 - E\left(\widehat{\vartheta}_s\right)}{\sqrt{V\left(\widehat{\vartheta}_s\right)}}\right], \tag{3.10}$$

$$p_c = P\left(t_1 \le \widehat{\vartheta}_s < t_2\right) = P\left[\frac{t_1 - E\left(\widehat{\vartheta}_s\right)}{\sqrt{V\left(\widehat{\vartheta}_s\right)}} \le Z < \frac{t_2 - E\left(\widehat{\vartheta}_s\right)}{\sqrt{V\left(\widehat{\vartheta}_s\right)}}\right]. \tag{3.11}$$

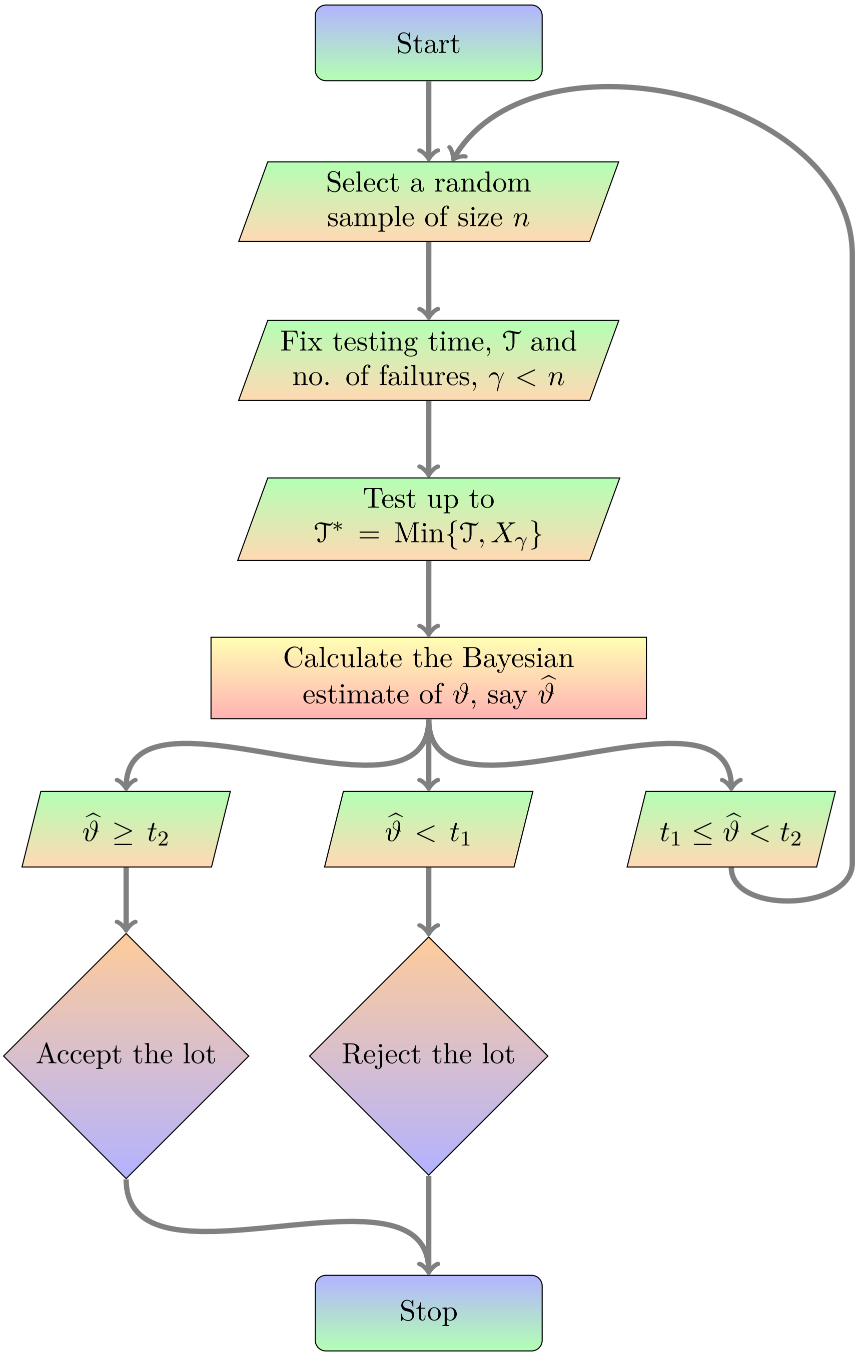

Figure 1. Flowchart of the ASP.

As the expected testing cost is a function of the unknown parameter ϑ, here we consider the expected testing cost at ϑ_A as the objective of the optimization problem. Thus, the required optimization problem, $P_1\,(n, t_1, t_2)$, is obtained by minimising the expected testing cost at ϑ_A, subject to the probability conditions in (3.1).

$$
\begin{aligned}
P_1(n, t_1, t_2): \quad & \underset{(n,t_1,t_2)}{\text{Min}} \left(\frac{C\ E\left(\widehat{\vartheta}_s\right)}{1 - p_c} \right) \text{ at } \vartheta_A \\
\text{subject to} \quad & P\left(\text{Reject the lot} \mid \vartheta = \vartheta_A\right) \leq \alpha, \\
& P\left(\text{Accept the lot} \mid \vartheta = \vartheta_U\right) \leq \beta.
\end{aligned}
$$

where $t_1, t_2 > 0$ and $t_2 > t_1$. That is,

$$\begin{aligned}
&\underset{(n,t_1,t_2)}{\text{Min}} \left(\frac{C\ E\left(\widehat{\vartheta}_s\right)}{1-p_c} \right) \text{ at } \vartheta_A \\
\text{subject to}\quad & (P_r \mid \vartheta = \vartheta_A) \le \alpha, \\
& (P_a \mid \vartheta = \vartheta_U) \le \beta,
\end{aligned} \tag{3.12}$$

where $t_1, t_2 > 0$ and $t_2 > t_1$. Thus, we get,

$$\begin{aligned}
&\underset{(n,t_1,t_2)}{\text{Min}} \left(\frac{C\ E\left(\widehat{\vartheta}_s\right)}{1-p_c} \right) \text{ at } \vartheta_A \\
\text{subject to}\quad & \left(\frac{p_r}{1-p_c} \mid \vartheta = \vartheta_A \right) \le \alpha, \\
& \left(\frac{p_a}{1-p_c} \mid \vartheta = \vartheta_U \right) \le \beta,
\end{aligned} \tag{3.13}$$

where $t_1, t_2 > 0$ and $t_2 > t_1$. Thus, by substituting the expressions of p_a, p_r and p_c from equations (3.9), (3.10) and (3.11), the non-linear optimization problem, $P_1(n, t_1, t_2)$, becomes:

$$\begin{aligned}
P_1(n,t_1,t_2):\quad & \underset{(n,t_1,t_2)}{\text{Min}} \left(\frac{C\ E\left(\widehat{\vartheta}_s\right)}{1-P\left[\frac{t_1-E\left(\widehat{\vartheta}_s\right)}{\sqrt{V\left(\widehat{\vartheta}_s\right)}} \le Z < \frac{t_2-E\left(\widehat{\vartheta}_s\right)}{\sqrt{V\left(\widehat{\vartheta}_s\right)}}\right]} \right) \text{ at } \vartheta_A \\
\text{subject to}\quad & \left(\frac{P\left[Z < \frac{t_1-E\left(\widehat{\vartheta}_s\right)}{\sqrt{V\left(\widehat{\vartheta}_s\right)}}\right]}{1-P\left[\frac{t_1-E\left(\widehat{\vartheta}_s\right)}{\sqrt{V\left(\widehat{\vartheta}_s\right)}} \le Z < \frac{t_2-E\left(\widehat{\vartheta}_s\right)}{\sqrt{V\left(\widehat{\vartheta}_s\right)}}\right]} \mid \vartheta = \vartheta_A \right) \le \alpha, \\
& \left(\frac{P\left[Z \ge \frac{t_2-E\left(\widehat{\vartheta}_s\right)}{\sqrt{V\left(\widehat{\vartheta}_s\right)}}\right]}{1-P\left[\frac{t_1-E\left(\widehat{\vartheta}_s\right)}{\sqrt{V\left(\widehat{\vartheta}_s\right)}} \le Z < \frac{t_2-E\left(\widehat{\vartheta}_s\right)}{\sqrt{V\left(\widehat{\vartheta}_s\right)}}\right]} \mid \vartheta = \vartheta_U \right) \le \beta,
\end{aligned} \tag{3.14}$$

where $t_1, t_2 > 0$ and $t_2 > t_1$.

For n, t_1, and t_2, this non-linear optimization problem P_1 can be solved by the following steps:

(1) Compute the minimum value of γ which satisfies the constraints of the problem P_1.
(2) Utilising the obtained value of γ, from Step 1, solve P_1 for the optimal values of n, t_1, and t_2.

The optimal values of n, t_1, and t_2 are obtained by solving the above non-linear optimization problem using the Genetic Algorithm (GA) solver in MATLAB. The GA solver in MATLAB employs an iterative process that emulates natural selection to obtain an optimal solution to the stated optimization problem. It evolves a population of potential solutions through selection, crossover, and mutation operations to identify the most optimal solution satisfying the constraints. It is particularly effective for nonlinear, non-convex, and high-dimensional optimization problems. Some examples are presented in Table 1.

3.3. ASP with Bayesian estimator of ϑ using Linex loss function

In this section, the Bayesian estimate of ϑ using the Linex loss function is applied in the ASP. For that, a sample of size n is taken from the lot. The testing is terminated at time $\mathfrak{T}^*$ and the observed lifetime data will be as given in (2.2). Here, we define an ASP as follows:

Step 1: Take a random sample of size n from the lot. Fix the time $\mathfrak{T}$, failure number, $\gamma < n$, AQL, UQL, producer's risk (α), and consumer's risk (β). Testing is stopped at time, $\mathfrak{T}^* = \text{Min}\{\mathfrak{T}, X_\gamma\}$.

Step 2: Calculate the Bayesian estimate of ϑ using the Linex loss function ($\widehat{\vartheta}_l$), given by

$$\widehat{\vartheta}_l = \widehat{\vartheta}_{MLE} - \frac{1}{c}\ln\left[1 + \frac{c}{2\mathfrak{D}}\left(c\widehat{\vartheta}^2_{MLE} - 2a + 2\widehat{\vartheta}_{MLE}(b-1)\right)\right].$$

Step 3: Continue the test, if $t_1 \leq \widehat{\vartheta}_l < t_2$ and repeat Steps 1 and 2 .
Else, go to Step 4.

Step 4: Accept the lot, if $\widehat{\vartheta}_l \geq t_2$ and reject the lot, if $\widehat{\vartheta}_l < t_1$.

The flow chart shown in Figure 1 demonstrates the proposed plan. The main problem here is determining the optimal values of n, t_1, and t_2 while minimizing the testing cost under probability constraints as mentioned in (3.1). Total testing cost is a product of the testing cost of a unit item in unit time and total testing time. Total testing time is obtained by multiplying the overall number of samples tested with decision-making time while testing a sample, given by $\widehat{\vartheta}_l$. But they are both random variables. Thus, the total testing time becomes a random quantity with expectation, $\frac{1}{1-p_c}\, E\left(\widehat{\vartheta}_l\right)$. As a result, here we use the expected testing cost as the objective function and it is given by

$$\text{ETC} = \frac{C\; E\left(\widehat{\vartheta}_l\right)}{1-p_c},$$

where C is the testing cost of an item for unit time.

From equations (3.7) and (3.8), we get that, $\widehat{\vartheta}_l$ follows Normal distribution with mean,

$$E\left(\widehat{\vartheta}_l\right) = E\left(\widehat{\vartheta}_{MLE}\right) - \frac{1}{c}\ln\left[1 + \frac{c}{2\mathfrak{D}}\left(cE\left(\widehat{\vartheta}_{MLE}\right)^2 - 2a + 2E\left(\widehat{\vartheta}_{MLE}\right)(b-1)\right)\right],$$

and variance,

$$V\left(\widehat{\vartheta}_l\right) = \left(1 - \frac{2cE\left(\widehat{\vartheta}_{MLE}\right) + 2b - 2}{2\mathfrak{D} + c\left(cE\left(\widehat{\vartheta}_{MLE}\right)^2 - 2a + 2E\left(\widehat{\vartheta}_{MLE}\right)(b-1)\right)}\right)^2 \sigma^2\left(\widehat{\vartheta}_{MLE}\right).$$

Using this information, we can evaluate the required probabilities as

$$p_a = P\left(\widehat{\vartheta}_l \geq t_2\right) = P\left[Z \geq \frac{t_2 - E\left(\widehat{\vartheta}_l\right)}{\sqrt{V\left(\widehat{\vartheta}_l\right)}}\right], \tag{3.15}$$

$$p_r = P\left(\widehat{\vartheta}_l < t_1\right) = P\left[Z < \frac{t_1 - E\left(\widehat{\vartheta}_l\right)}{\sqrt{V\left(\widehat{\vartheta}_l\right)}}\right], \tag{3.16}$$

$$p_c = P\left(t_1 \leq \widehat{\vartheta}_l < t_2\right) = P\left[\frac{t_1 - E\left(\widehat{\vartheta}_l\right)}{\sqrt{V\left(\widehat{\vartheta}_l\right)}} \leq Z < \frac{t_2 - E\left(\widehat{\vartheta}_l\right)}{\sqrt{V\left(\widehat{\vartheta}_l\right)}}\right]. \tag{3.17}$$

Here, we evaluate the objective function of the optimization problem at ϑ_A since the overall testing cost includes the unknown parameter ϑ. Thus, the expected testing cost at ϑ_A is minimized by satisfying the probability conditions given in (3.1). Hence, the required non-linear optimization problem is formulated as

$$\begin{aligned} P_2(n,t_1,t_2): \quad & \underset{(n,t_1,t_2)}{\mathrm{Min}} \left(\frac{C\ E\left(\widehat{\vartheta}_l\right)}{1-p_c}\right) \text{ at } \vartheta_A \\ \text{subject to} \quad & P\left(\text{Reject the lot} \mid \vartheta=\vartheta_A\right) \le \alpha, \\ & P\left(\text{Accept the lot} \mid \vartheta=\vartheta_U\right) \le \beta, \end{aligned}$$

where $t_1, t_2 > 0$ and $t_2 > t_1$. This implies,

$$\begin{aligned} P_2(n,t_1,t_2): \quad & \underset{(n,t_1,t_2)}{\mathrm{Min}} \left(\frac{C\ E\left(\widehat{\vartheta}_l\right)}{1-p_c}\right) \text{ at } \vartheta_A \\ \text{subject to} \quad & \left(P_r \mid \vartheta=\vartheta_A\right) \le \alpha, \\ & \left(P_a \mid \vartheta=\vartheta_U\right) \le \beta, \end{aligned} \tag{3.18}$$

where $t_1, t_2 > 0$ and $t_2 > t_1$. That is,

$$\begin{aligned} & \underset{(n,t_1,t_2)}{\mathrm{Min}} \left(\frac{C\ E\left(\widehat{\vartheta}_l\right)}{1-p_c}\right) \text{ at } \vartheta_A \\ \text{subject to} \quad & \left(\frac{p_r}{1-p_c} \mid \vartheta=\vartheta_A\right) \le \alpha, \\ & \left(\frac{p_a}{1-p_c} \mid \vartheta=\vartheta_U\right) \le \beta, \end{aligned} \tag{3.19}$$

where $t_1, t_2 > 0$ and $t_2 > t_1$. The non-linear optimization problem, $P_2(n,t_1,t_2)$, is modified by substituting the expressions for p_a, p_r, and p_c from equations (3.15), (3.16), and (3.17) as

$$\begin{aligned} P_2(n,t_1,t_2): \quad & \underset{(n,t_1,t_2)}{\mathrm{Min}} \left(\frac{C\ E\left(\widehat{\vartheta}_l\right)}{1-P\left[\frac{t_1-E\left(\widehat{\vartheta}_l\right)}{\sqrt{V\left(\widehat{\vartheta}_l\right)}} \le Z < \frac{t_2-E\left(\widehat{\vartheta}_l\right)}{\sqrt{V\left(\widehat{\vartheta}_l\right)}}\right]}\right) \text{ at } \vartheta_A \\ \text{subject to} \quad & \left(\frac{P\left[Z < \frac{t_1-E\left(\widehat{\vartheta}_l\right)}{\sqrt{V\left(\widehat{\vartheta}_l\right)}}\right]}{1-P\left[\frac{t_1-E\left(\widehat{\vartheta}_l\right)}{\sqrt{V\left(\widehat{\vartheta}_l\right)}} \le Z < \frac{t_2-E\left(\widehat{\vartheta}_l\right)}{\sqrt{V\left(\widehat{\vartheta}_l\right)}}\right]} \mid \vartheta=\vartheta_A\right) \le \alpha, \\ & \left(\frac{P\left[Z \ge \frac{t_2-E\left(\widehat{\vartheta}_l\right)}{\sqrt{V\left(\widehat{\vartheta}_l\right)}}\right]}{1-P\left[\frac{t_1-E\left(\widehat{\vartheta}_l\right)}{\sqrt{V\left(\widehat{\vartheta}_l\right)}} \le Z < \frac{t_2-E\left(\widehat{\vartheta}_l\right)}{\sqrt{V\left(\widehat{\vartheta}_l\right)}}\right]} \mid \vartheta=\vartheta_U\right) \le \beta, \end{aligned} \tag{3.20}$$

where $t_1, t_2 > 0$ and $t_2 > t_1$.

The following steps are used to resolve the above-mentioned nonlinear optimization problem P_2 for (n, t_1, t_2):

(1) First, determine the least value of γ subjected to the constrains of the problem P_2.
(2) Then solve P_2, using the value of γ obtained in Step 1.

Similarly, as in Subsection 3.2, the non-linear optimization problem is solved using the GA solver in MATLAB and the obtained optimal values of n, t_1, and t_2 are tabulated in Table 2 and Table 3 for $c = 0.5$ and $c = -0.5$, respectively.

4. Computational results, comparisons, and real data case study

Numerical computations of all the above sections are discussed in this section and are displayed in Table 1 to 3. The hyper-parameters, $a = 1.25$ and $b = 2.5$ are used to compute the Bayesian estimator throughout this work since they give more approximate Bayesian estimate for ϑ (see [22] and [34]). ETC of the ASP under Type I hybrid censoring using Bayesian estimate of ϑ subject to Linex loss function for both $c = 0.5$ and $c = -0.5$ is computed and is shown in Table 2 and Table 3, respectively. From that, we can observe that the ETC is less for $c = 0.5$ than that of $c = -0.5$. For example, with $C = 1$, $\vartheta_A = 200$, $\vartheta_U = 100$, $\mathfrak{T} = 100$, $\alpha = 0.05$ and $\beta = 0.05$, the ETC is 127.2029 for $c = 0.5$ and ETC is increased to 228.1767 for $c = -0.5$. A comparison of testing costs obtained from ASPs under Type-I hybrid censoring using a Bayesian estimate of the parameter ϑ using the SEL and Linex loss functions is also done using the results in Tables 1, 2 and 3. Consider the following example for the set of parameters: $C = 1$, $\vartheta_A = 500$, $\vartheta_U = 200$, $\mathfrak{T} = 50$, $\alpha = 0.05$ and $\beta = 0.05$, the ETC obtained for ASP with SEL and Linex loss functions are 475.5810, 557.9282 ($c = 0.5$) and 627.7466 ($c = -0.5$), respectively. These comparisons show that the ASP has the lowest test cost when the Bayesian estimate of ϑ is computed using the SEL function.

Table 4 presents the comparison of ETC computed for the ASP under Type-I hybrid censoring with Bayesian estimate of ϑ using SEL function and Linex loss function with $c = 0.5$ against the ASP under Type-I censoring scheme designed using MLE of ϑ in [19]. Thus, one can observe that the cost of our plan is less than the one described in [19].

Furthermore, a sensitivity analysis is conducted to analyze how changes in producer's risk (α) and consumer's risk (β) affect the ETCs of proposed ASPs. Figure 2 depicts the impact of varying α while maintaining β constant, revealing a decrease in the ETC for both ASPs as α increases. Similarly, Figure 3 shows that as β increases, with α held constant, the ETCs of both ASPs also decrease. Figure 4 illustrates a consistent trend of ETC when both α and β are varied simultaneously. Additionally, the proposed ASPs are applied to real-life data.

Table 1. ASP under Type-I HCS using SEL function for C=1.

ϑ_A	ϑ_U	$\mathfrak{T}$	α	β	γ	t_1	t_2	n	ETC
200	100	100	0.05	0.05	26	162.3926	162.3957	31	123.4077
			0.01	0.05	20	74.7316	74.7322	31	124.7417
			0.01	0.01	25	146.3421	146.3394	36	125.6045
500	200	50	0.05	0.05	21	313.5638	313.5638	26	475.5810
			0.01	0.05	22	310.5940	310.5948	27	480.9310
			0.01	0.01	30	225.7734	225.7735	32	563.9248
500	200	100	0.05	0.05	26	313.3180	313.3188	37	509.9392
			0.01	0.05	14	311.1179	311.1179	37	513.4323
			0.01	0.01	24	390.4518	390.4526	38	566.6305
3000	1500	1000	0.05	0.05	16	2005.3416	2005.3425	62	2675.7067
			0.01	0.05	10	2000.6681	2000.6682	92	2680.1320
			0.01	0.01	15	2022.5679	2022.568	98	2684.0441

Table 2. ASP under Type-I HCS using Linex loss for $c = 0.5$ and C=1.

ϑ_A	ϑ_U	$\mathcal{T}$	α	β	γ	t_1	t_2	n	ETC
200	100	100	0.05	0.05	21	63.1133	63.1134	35	127.2029
			0.01	0.05	19	56.3674	56.3675	38	136.7482
			0.01	0.01	28	140.9731	140.9733	40	143.1099
500	200	50	0.05	0.05	21	367.4209	367.421	34	557.9282
			0.01	0.05	26	361.9554	361.9555	34	558.9578
			0.01	0.01	28	371.9294	371.9295	36	595.2252
500	200	100	0.05	0.05	22	367.4918	367.4918	39	521.9267
			0.01	0.05	27	354.8226	354.8229	39	523.5307
			0.01	0.01	25	364.1113	364.1115	42	524.21
3000	1500	1000	0.05	0.05	15	2473.6173	2473.6174	24	3379.9555
			0.01	0.05	11	2437.8132	2437.8133	33	3461.2578
			0.01	0.01	6	2428.6943	2428.6944	34	3705.1855

Table 3. ASP under Type-I HCS using Linex loss for $c = -0.5$ and C=1.

ϑ_A	ϑ_U	$\mathcal{T}$	α	β	γ	t_1	t_2	n	ETC
200	100	100	0.05	0.05	28	186.2786	186.2793	34	228.1767
			0.01	0.05	26	178.3335	178.3342	35	232.5816
			0.01	0.01	23	177.5453	177.5455	45	250.3530
500	200	50	0.05	0.05	12	400.025	400.0248	25	627.7466
			0.01	0.05	15	396.9158	396.9158	26	635.9504
			0.01	0.01	14	405.41	405.4198	26	636.7619
500	200	100	0.05	0.05	18	423.5114	423.5116	37	600.3209
			0.01	0.05	22	405.5332	405.5335	38	620.4932
			0.01	0.01	24	411.5643	411.5644	43	623.4914
3000	1500	1000	0.05	0.05	19	3783.5928	3783.5929	29	3449.2903
			0.01	0.05	11	2507.4709	2507.471	30	3449.8618
			0.01	0.01	9	2557.26	2557.2699	30	3450.2116

Table 4. Comparison of ETCs for ASP using two different loss functions and testing costs under Type-I censoring.

ϑ_A	ϑ_U	$\mathcal{T}$	α	β	A	B	C
200	100	100	0.05	0.25	123.4077	151.0479	156.53
3000	1500	1500	0.05	0.1	1423.5748	2256.9985	2290.35

ETCs are calculated using the SEL function (column A), Linex loss with c = 0.5 (column B), and ASP under Type-I censoring (column C).

4.1. Case study

In this section, the ASPs presented in this work are demonstrated using a real-world example from [26]. The failure rates for 36 appliances that underwent an automated life test are listed below; the lifetime shown here refers to the number of cycles of use that the appliances can withstand before failing.

$$11, 35, 49, 170, 329, 381, 708, 958, 1062, 1167, 1594, 1925, 1990, 2223, 2327, 2400, 2451,$$
$$2471, 2551, 2565, 2568, 2694, 2702, 2761, 2831, 3034, 3059, 3112, 3214, 3478, 3504, 4329,$$
$$6367, 6976, 7846, 13403.$$

The above data follows an exponential distribution. For the application of the proposed sampling plans in Sections 3 and 4, we take our plan parameters as $\vartheta_A = 3000$, $\vartheta_U = 600$, $\mathcal{T} = 2000$,

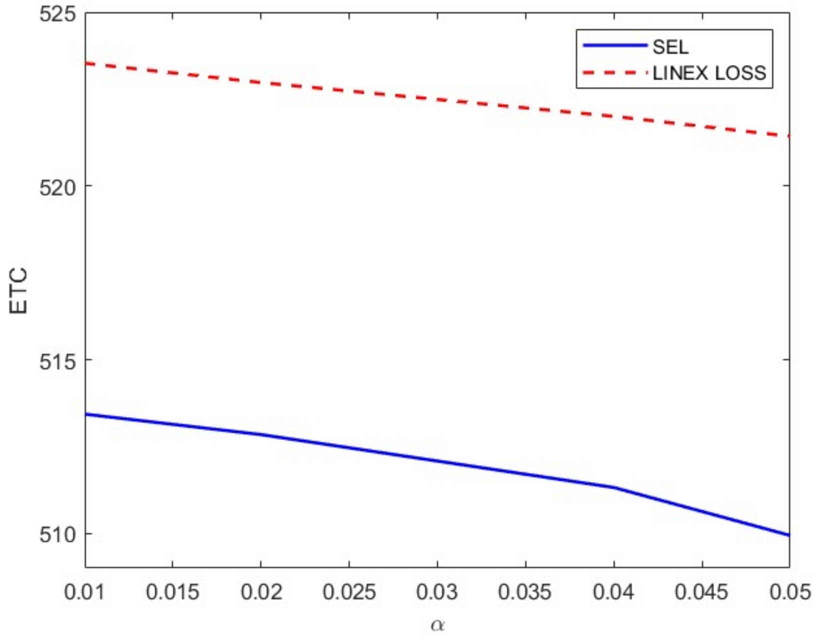

Figure 2. Changes in the ETC of ASPs based on SEL and Linex loss function for fixed $\beta = 0.05$ and varying values of α with $\vartheta_A = 500$, $\vartheta_U = 200$, $\mathfrak{T} = 100$ and $C = 1$.

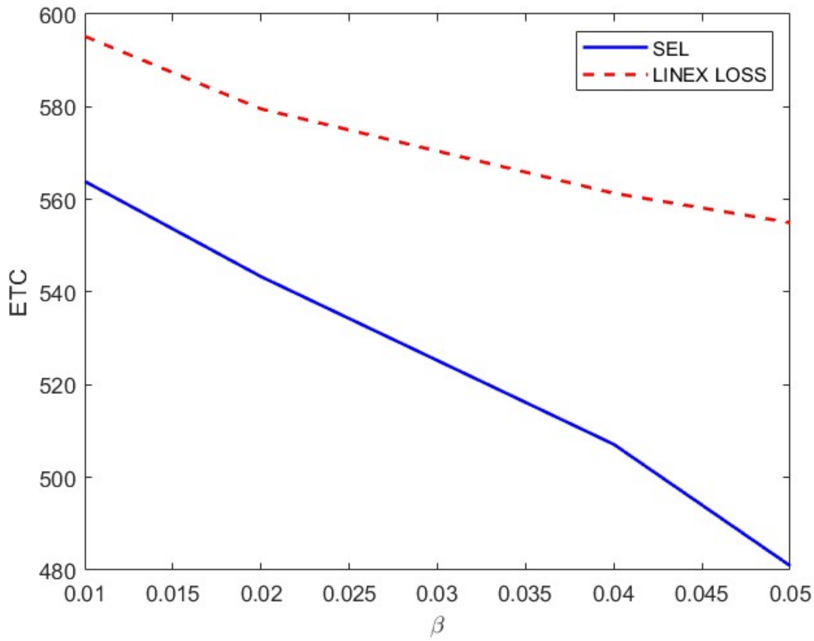

Figure 3. Changes in the ETC of ASPs based on SEL and Linex loss function for fixed $\alpha = 0.01$ and varying values of β with $\vartheta_A = 500$, $\vartheta_U = 200$, $\mathfrak{T} = 50$ and $C = 1$.

$\alpha = 0.1$ and $\beta = 0.2$ as in [19]. Now, we can solve the nonlinear optimization problems P_1 and P_2 to obtain the optimal values of plan parameters t_1 and t_2. Note that, t_1 and t_2 are real numbers but for the application of our ASPs we need them to be integers. So we first solve P_1 and P_2, if we get integer values for t_1 and t_2, then we can look for the acceptance and rejection of the lot with appropriate conditions. If t_1 and t_2 are non-integers then we restrict them to be integers. Hence our optimization problems P_1 and P_2 become non-linear integer programming problems with integer solutions.

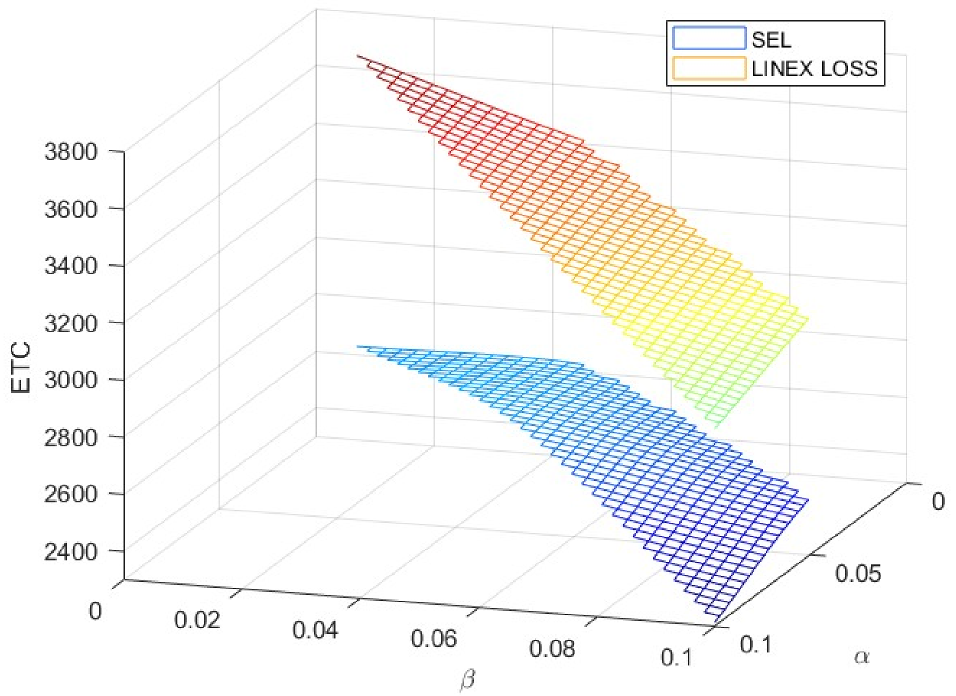

Figure 4. Sensitivity of the ETC of ASPs, based on SEL and Linex loss function to changes in α and β for fixed $\vartheta_A = 3000$, $\vartheta_U = 1500$, $\mathfrak{T} = 1000$ and $C = 1$.

For the above-mentioned data, the different sampling plans proposed in our work yield the following outcomes:

(1) ASP under Type-I HCS using SEL function gives the parameters as $\gamma = 9$, $t_1 = 2064$, $t_2 = 2065$, and $n = 31$. Next, we consider the first 31 samples from the above data. The samples are tested upto time $\mathfrak{T}^* = \text{Min}\{\mathfrak{T}, X_9\} = 1062$ since $X_9 = 1062$. Now we compute the Bayesian estimate of ϑ with SEL function as $\widehat{\vartheta}_s = 2577.9286$ and it is greater than $t_2 = 2065$. So, we can accept the lot with ETC $= 2405$.

(2) Based on ASP under Type-I HCS using Linex loss function for $c = 0.5$, we get the optimal values for test parameters as $\gamma = 11$, $t_1 = 2156$, $t_2 = 2157$, and $n = 27$. Then, the first 27 samples are taken from the data and the testing is stopped at $\mathfrak{T}^* = \text{Min}\{\mathfrak{T}, X_{11}\} = 1594$ as $X_{11} = 1594$. Next, we calculate the Bayesian estimate of ϑ using Linex loss function and get $\widehat{\vartheta}_l = 2883.2339$. Hence, we accept the lot as $\widehat{\vartheta}_l > t_2$ and the ETC $= 2909$.

In [19], Kumar et al. utilized the same real-life data to illustrate their ASP under Type-I censoring for the exponential distribution. With the same parameter values, they accepted the lot with an ETC of 2918, based on 11 samples. This comparison demonstrates that the proposed ASPs in our work result in lower costs, indicating an improvement over the existing ASP presented in [19].

5. Conclusion

In conclusion, using Bayesian inference, a novel approach for Type-I hybrid censoring in ASPs is proposed here. Unlike conventional methods that rely on MLE of ϑ and minimizing Bayes risk, this study considered the Bayesian estimators of ϑ, for designing the ASP and incorporating the ETC. The SEL and linex loss were employed to compute the Bayesian estimators of ϑ.

The aim of all ASPs was to compute the optimal values of n, t_1, and t_2 by solving nonlinear optimization problems of minimizing ETC. A comparative study of ETC involved in ASPs using SEL and Linex loss functions is illustrated here using Table 1 to 3 for different values of AQL (ϑ_A), UQL (ϑ_U), producer's risk (α), consumer's risk (β), $\mathfrak{T}$ and γ. The lower ETC for SEL compared to Linex loss functions signifies that SEL provides a more cost-effective approach for our plan. This insight can have substantial implications for resource allocation, budget optimization, and overall project efficiency. ETC obtained for ASP using the Linex loss function is computed for both $c = 0.5$ and $c = -0.5$ and is tabulated in Table 2 and Table 3, respectively. From that, we get the ETC for $c = -0.5$ is more than the ETC obtained for $c = 0.5$. This is mainly because the Bayesian estimator of ϑ is overestimated for $c = -0.5$ using the Linex loss function (see [27]).

Table 4 shows the comparison of expected costs for our plan using Bayesian inference and the ASP under Type-I censoring in [19] which uses MLE of ϑ. The comparison results have revealed the importance of utilizing Bayesian inference to minimize costs, as the ETC involved in ASP under Type-I censoring in [19] is more than the ASPs discussed in this work. Thus, by adopting Bayesian inference and incorporating SEL as the loss function, we can better manage costs and improve the overall performance of our plan. In addition, the sensitivity analysis performed for the ASPs based on SEL and Linex loss functions, as shown in Figures 2, 3, and 4, reveals that the ETC decreases as α and β increase. This behavior is similar for both the proposed ASPs. Moreover, a real-life case study is conducted to explain the practical application of the various ASPs discussed here. However, our proposed plans have some limitations, that they will not be effective if the lifetime data are coming from an uncertain environment, such as the one considered by authors in papers [1, 2, 18]. Hence, our work can be extended using neutrosophic statistics, which can be a future scope for research.

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