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The Solutions of the Space-Time Fractional Cubic Nonlinear Schrödinger Equation by Using the Unified Method

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Abstract. Representing physical processes by introducing fractional derivatives in partial differential equations provides more realistic and flexible mathematical models. The solutions of nonlinear partial differential equations (NPDEs) can be derived from the solutions of the fractional nonlinear partial differential equations (FNPDEs) when the fractional derivatives go to 1 because FNPDEs are a generalization of NPDEs. Most of the exact solution methods for NPDEs based on the ansatz method can be extended easily to solve FNPDEs. In this study, we employ the unified method to obtain exact solutions in a more general form for the space-time fractional cubic nonlinear Schrödinger equation (stFCSE). Compared to other methods, this method not only gives more general solution forms with free parameters for the stFCSE, but also provides many novel solutions including hyperbolic, trigonometric, and rational function solutions. The solutions of the stFCSE approach the solutions of the cubic nonlinear Schrödinger equation when the fractional orders go to 1 for time and space. Moreover, three-dimensional graphs of some selected solutions with specific values of the parameters are presented to visualize the behavior and physical structures of the stFCSE.

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1. Introduction

Nonlinear partial differential equations (NPDEs) and fractional nonlinear partial differential equations (FNPDEs) containing fractional order derivatives appear in modeling problems in many scientific fields. Therefore, using a straightforward and robust method [9, 13, 15, 24, 26, 28, 29, 32] to find numerical or exact solutions for NPDEs and FNPDEs is a crucial step in shifting the theory to real-life applications. The nonlinear Schrödinger equation (NLSE) and its different forms, one of the distinguished equations particularly in science and engineering, have attracted great attention in the last decades because of their leading role in the most important areas of mathematical physics. Therefore, numerous researchers have sought new methods to obtain exact and numerical solutions for the class of NLSE due to its significant applications in the dynamics of particles, water waves, nonlinear acoustics, hydrodynamics, optics, and telecommunications.

Sulaiman et al. [30] studied (2 + 1)-dimensional Heisenberg ferromagnetic spin chain equation describing nonlinear dynamics of magnets to find exact solutions using the sine-Gordon expansion method and modified $\phi(\xi)$-expansion function method. The sine-Gordon expansion method was used by Bulut et al. [12] to obtain exact solutions of the (1+1)

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and (2 + 1)-dimensional Chiral nonlinear Schrödinger equations which describe the edge states of the fractional quantum hall effect. Biswas et al. [10] applied a semi-inverse variational principle to obtain an analytical soliton solution to the perturbed nonlinear Schrödinger equation with anti-cubic nonlinearity. The exact solutions of the Kundu-Eckhaus equation modeling soliton propagation in a dispersive media were obtained using the modified simple equation method by Biswas et al. [11]. Eslami and Neirameh [16] employed the $\phi(\xi)$-expansion function method to find new exact solutions of higher order nonlinear Schrödinger equation. Lan and Guo [22] used the Hirota method to find the 1, 2, 3, and N-soliton solutions for a coupled generalized nonlinear Schrödinger-Boussinesq system in a homogeneous magnetized plasma. Kudryashov [21] utilized the traveling wave reductions to find solutions to the generalization of the resonant nonlinear Schrödinger equation which describes the propagation of nonlinear waves in a resonant medium. Using the Kudryashov method and sub-equation method, Akinyemi et al. [2] found exact solutions for the generalized nonlinear Schrödinger-Korteweg-de Vries equations. The optical soliton solutions of the nonlinear Schrödinger equation with parabolic law nonlinearity are presented by Akinyemi et al. [3] applying the generalized auxiliary equation method. The perturbed Biswas-Milovic equation with Kudryashov's law of refractive index a form of the generalized nonlinear Schrödinger equation was studied by Mirzazadeh et al. [23] using the improved F-expansion method. A perturbed nonlinear Schrödinger equation with $\Gamma = 0$ was investigated by Akinyemi et al. [4] both analytically and numerically employing the modified sub-equation method and the split-step Fourier method, respectively.

The governing model of the space-time fractional cubic nonlinear Schrödinger equation (stFCSE) is the generalized (1 + 1)-dimensional unstable space-time fractional nonlinear Schrödinger [8, 14] given by

$$iD_t^{\alpha}u + D_x^{2\beta}u + \sigma|u|^2u + \tau u = 0.$$

When taking $\sigma = \gamma$ and $\tau = 0$, then it reduces the space-time fractional cubic NLS^+ equation

$$iD_t^{\alpha}u + D_x^{2\beta}u + \gamma|u|^2u = 0$$

and when taking $\sigma = -\gamma$ and $\tau = 0$, then it reduces the space-time fractional cubic NLS^- equation

$$iD_t^{\alpha}u + D_x^{2\beta}u - \gamma|u|^2u = 0.$$

The notion of fractional comes from the fractional order derivatives of the form $\frac{d^{\alpha}}{dx^{\alpha}}$ where $\alpha > 0$ is not necessarily an integer. Fractional derivatives contribute to generalizing classical results, capturing memory effects in physical systems, and improving mathematical modeling. The FNPDEs involving fractional order derivatives as a subfield of the partial differential equations have gained growing interest from researchers in recent times due to their wide utilization in control theory, signal and image processing, environmental science, medicine, mechanics, and other various disciplines including mathematics, physics, chemistry, and biology. The He's variational iteration method was used to solve numerically by Wazwaz [31] for (1 + 1)-dimensional nonlinear Schrödinger with $\gamma = \mp 2$. Bilal et al. [8] worked on the generalized (1 + 1)-dimensional unstable space-time fractional nonlinear Schrödinger to obtain exact solutions by extended sinh-Gordon equation expansion method. Bright, dark, and singular optical soliton solutions of space-time were found by Darvishi et al. [14]. Pandir and Agir [25] applied the extended trial equation method to the cubic nonlinear Schrödinger equation to find exact solutions. The Jacobi elliptic functions expansion method is used to solve the space-time fractional cubic nonlinear Schrödinger equation by Gundogdu and Gozukizil [17]. Seadawy and Tariq [27] applied the extended modified auxiliary equation mapping method to the generalized (1+1)-dimensional space-time fractional unstable nonlinear Schrödinger equation to obtain bright, dark, singular, combo, optical, singular optical, and bright-singular combo soliton solutions.

In this study, we apply the unified method to the space-time fractional cubic nonlinear Schrödinger equation (stFCSE). The stFCSE is given by

$$iD_t^{\alpha}u + D_x^{2\beta}u + \gamma|u|^2u = 0, \tag{1.1}$$

where $\gamma \neq 0$ and $\alpha, \beta \in (0, 1]$. Eq. (1.1) is defined as attractive if $\gamma > 0$, otherwise repulsive. The unified method used in many articles to solve NPDEs [1, 5–7, 18, 19] provides many exact hyperbolic, trigonometric, and rational-type solutions with free parameters without needing extra hardware support.

This paper is organized as follows. Some important definitions and properties of the conformable fractional derivatives and a brief description of the unified method for FNPDEs are presented in Section 2. Application of the method to the space-time fractional cubic nonlinear Schrödinger equation (stFCSE) is given in Section 3. Finally, result and discussion and conclusions are given in Section 4 and Section 5, respectively.

2. Preliminaries

2.1. **Conformable Fractional Derivative.** Assume that $\phi : [0, \infty) \to R$ is α-differentiable function, then α-order conformable fractional derivative is

$$D_x^{\alpha}\phi(x) = \lim_{\epsilon \to 0} \frac{\phi(x + \epsilon x^{1-\alpha}) - \phi(x)}{\epsilon}$$

for all $t > 0$, $\alpha \in (0, 1)$. Some important properties of the conformable fractional derivative can be reached from [20]. Let $\phi(x)$, $\psi(x)$ are α-order conformable fractional differentiable functions and a, b are real constants, then

- $D_x^{\alpha}(a) = 0$,
- $D_x^{\alpha}(x^b) = bx^{b-\alpha}$,
- $D_x^{\alpha}\phi(x) = x^{1-\alpha}\dfrac{d\phi(x)}{dx}$,
- $D_x^{\alpha}[a\phi(x) + b\psi(x)] = aD_x^{\alpha}\phi(x) + bD_x^{\alpha}\psi(x)$,
- $D_x^{\alpha}[\phi(x)\psi(x)] = \phi(x)D_x^{\alpha}\psi(x) + \psi(x)D_x^{\alpha}\phi(x)$,
- $D_x^{\alpha}\left[\dfrac{\phi(x)}{\psi(x)}\right] = \dfrac{\psi(x)D_x^{\alpha}(\phi(x)) - \phi(x)D_x^{\alpha}(\psi(x))}{\psi^2(x)}$.

2.2. **Description of the Unified Method.** In this section, we briefly describe the unified method and explain in the following steps how to apply it to the fractional nonlinear partial differential equations (FNPDEs) with complex-valued terms. Let a general form for FNPDEs in two independent variables x and t be defined as follows:

$$P(u, u_t, u_x, u_{xt}, u_{tt}, u_{xx}, ...) = 0, \tag{2.1}$$

where $u(x, t)$ is a complex-valued unknown function and P is an equation of $u = u(x, t)$ and its various partial derivatives, where highest order derivative and nonlinear terms are involved.
Step 1: Firstly, we perform the transformation $u(x, t) = U(\eta)e^{i\zeta}$ where $\eta = \frac{x^{\alpha}}{\alpha} - c\frac{t^{\alpha}}{\alpha}$ and $\zeta = w\frac{x^{\beta}}{\beta} + \lambda\frac{t^{\alpha}}{\alpha}$ are. Substituting this transformation into Eq. (2.1) reduces it to the following nonlinear ordinary differential equation (NODE)

$$P(U, U', U'', U''', ...) = 0. \tag{2.2}$$

Step 2: Supposed the solution of FNPDE can be expressed by an ansatz as follows:

$$V(\eta) = a_0 + \sum_{m=1}^{M} \left[a_m\phi^m + b_m\phi^{-m}\right], \tag{2.3}$$

where $\phi = \phi(\eta)$ satisfies the Riccati differential equation

$$\phi'(\eta) = \mu + \phi^2(\eta) \tag{2.4}$$

and $\phi' = \frac{d\phi}{d\eta}$, and a_m, b_m are coefficients of ϕ and μ is parameter. The general solutions of Eq. (2.4) as follows:
Family 1. When $\mu < 0$, the solutions of Eq. (2.4)

$$\phi(\eta) = \begin{cases} \dfrac{\mp\sqrt{-(A^2 + B^2)\mu} - A\sqrt{-\mu}\cosh\left(2\sqrt{-\mu}(\eta + \eta_0)\right)}{A\sinh\left(2\sqrt{-\mu}(\eta + \eta_0)\right) + B} \\ \mp\sqrt{-\mu} \pm \dfrac{2A\sqrt{-\mu}}{A + \cosh\left(2\sqrt{-\mu}(\eta + \eta_0)\right) \pm \sinh\left(2\sqrt{-\mu}(\eta + \eta_0)\right)} \end{cases}, \tag{2.5}$$

where $A \neq 0$ and B are two real arbitrary parameters, and η_0 arbitrary parameter.
Family 2. When $\mu > 0$, the solutions of Eq. (2.4)

$$\phi(\eta) = \begin{cases} \dfrac{\mp\sqrt{(A^2 - B^2)\mu} - A\sqrt{\mu}\cos\left(2\sqrt{\mu}(\eta + \eta_0)\right)}{A\sin\left(2\sqrt{\mu}(\eta + \eta_0)\right) + B} \\ \mp i\sqrt{\mu} \pm \dfrac{2Ai\sqrt{\mu}}{A + \cos\left(2\sqrt{\mu}(\eta + \eta_0)\right) \pm i\sin\left(2\sqrt{\mu}(\eta + \eta_0)\right)} \end{cases}, \tag{2.6}$$

where $A \neq 0$ and B are two real arbitrary parameters, and η_0 arbitrary parameter.
Family 3. When $\mu = 0$, the solution of Eq. (2.4)

$$\phi(\eta) = -\frac{1}{\eta + \eta_0}, \tag{2.7}$$

where η_0 arbitrary parameter.

It should be pointed out that the unified method yields the same solutions in family 1 and family 2. In other words, using the identities $\sinh(ix) = i\sin(x)$ and $\cosh(ix) = \cos(x)$, the solutions in (2.5) and (2.6) are exactly the same. Therefore, the solutions in (2.5) and (2.6) can be easily converted to each other by applying these identities.
Step 3: After determining positive integer M by considering the homogeneous balance between the linear term of the highest order with the nonlinear term of the highest degree, Eq. (2.3) and (2.4) substitute into Eq. (2.2) and collect all terms with the same powers of ϕ in the final equation. Then, equating each power of ϕ to zero gives a set of algebraic equations of a_m, b_m, c, w, λ and μ.
Step 4: Substituting a_m, b_m, c, w, λ and μ obtained in step 3 into Eq. (2.3) and using the general solutions of Eq. (2.4) given (2.5), (2.6) and (2.7), we obtain the exact solutions of Eq. (2.1) in closed form.

3. The Space-Time Fractional Cubic Nonlinear Schrödinger Equation

The space-time fractional cubic nonlinear Schrödinger equation (stFCSE) is given by

$$iD_t^{\alpha}u + D_x^{2\beta}u + \gamma|u|^2u = 0, \tag{3.1}$$

where $\gamma \neq 0$ and $\alpha, \beta \in (0, 1]$. We substitute $u(x,t) = U(\eta)e^{i(w\frac{x^{\beta}}{\beta} + \lambda\frac{t^{\alpha}}{\alpha})}$ and its derivatives

$$\begin{aligned}
iD_t^{\alpha}(x,t) &= \left(-\lambda U(\eta) - icU'(\eta)\right)e^{i(w\frac{x^{\beta}}{\beta} + \lambda\frac{t^{\alpha}}{\alpha})},\\
D_x^{\beta}(x,t) &= \left(iwU(\eta) + U'(\eta)\right)e^{i(w\frac{x^{\beta}}{\beta} + \lambda\frac{t^{\alpha}}{\alpha})},\\
D_x^{2\beta}(x,t) &= \left(U''(\eta) + 2iwU'(\eta) - w^2U(\eta)\right)e^{i(w\frac{x^{\beta}}{\beta} + \lambda\frac{t^{\alpha}}{\alpha})},\\
|u(x,t)|^2u(x,t) &= U^3(\eta)e^{i(w\frac{x^{\beta}}{\beta} + \lambda\frac{t^{\alpha}}{\alpha})},
\end{aligned}$$

into Eq. (3.1) considering the wave transformation $\eta = \frac{x^{\beta}}{\beta} - c\frac{t^{\alpha}}{\alpha}$, then the stFCSE is reduced to the nonlinear ordinary differential equations as follows:

$$U'' - (w^2 + \lambda)U + \gamma U^3 + i(2w - c)U' = 0. \tag{3.2}$$

$c = 2w$ obtained from the imaginary part of Eq. (3.2); therefore, the final reduced nonlinear ordinary differential equation for the stFCSE is

$$U'' - (w^2 + \lambda)U + \gamma U^3 = 0. \tag{3.3}$$

Balancing between the highest order U'' with the nonlinear term U^3 gives the simple equation $M + 2 = 3M$. From here, it is $M = 1$. So the solutions of the stFCSE (3.1) can be written in the form

$$U(\eta) = a_0 + a_1\phi + \frac{b_1}{\phi}, \tag{3.4}$$

where a_0, a_1 and b_1 are coefficients of ϕ which are determined later. Substituting Eq. (3.4) and its derivatives into Eq. (3.3), then equating each coefficients of ϕ to zero gives a set of nonlinear algebraic equations for a_0, a_1, b_1 and c, w, λ. Solving this algebraic equations system by using Maple, the following sets of parameters are obtained.

Set 1. $a_0 = 0, \quad a_1 = \mp i\sqrt{\frac{2}{\gamma}}, \quad b_1 = 0, \quad \lambda = 2\mu - w^2.$

Set 2. $a_0 = 0, \quad a_1 = 0, \quad b_1 = \mp i\sqrt{\frac{2}{\gamma}}\mu, \quad \lambda = 2\mu - w^2.$

Set 3. $a_0 = 0, \quad a_1 = \mp i\sqrt{\frac{2}{\gamma}}, \quad b_1 = \mp i\sqrt{\frac{2}{\gamma}}\mu, \quad \lambda = -4\mu - w^2.$

Set 4. $a_0 = 0, \quad a_1 = \pm i\sqrt{\frac{2}{\gamma}}, \quad b_1 = \mp i\sqrt{\frac{2}{\gamma}}\mu, \quad \lambda = 8\mu - w^2.$

The exact solutions of the stFCSE are listed under three categories hyperbolic, trigonometric, and rational with respect to these solution sets as follows:

Hyperbolic function solutions

$$u_1(x,t) = \pm i\sqrt{\frac{2}{\gamma}}\left[\frac{\mp\sqrt{-(A^2+B^2)\mu} - A\sqrt{-\mu}\cosh\left(2\sqrt{-\mu}(\eta+\eta_0)\right)}{A\sinh\left(2\sqrt{-\mu}(\eta+\eta_0)\right)+B}\right]e^{i(w\frac{x^\beta}{\beta}+(2\mu-w^2)\frac{t^\alpha}{\alpha})}, \tag{3.5}$$

$$u_2(x,t) = \pm i\sqrt{\frac{2}{\gamma}}\left[\mp\sqrt{-\mu} \pm \frac{2A\sqrt{-\mu}}{A+\cosh\left(2\sqrt{-\mu}(\eta+\eta_0)\right)\pm\sinh\left(2\sqrt{-\mu}(\eta+\eta_0)\right)}\right]e^{i(w\frac{x^\beta}{\beta}+(2\mu-w^2)\frac{t^\alpha}{\alpha})}, \tag{3.6}$$

$$u_3(x,t) = \pm i\sqrt{\frac{2}{\gamma}}\mu\left[\frac{A\sinh\left(2\sqrt{-\mu}(\eta+\eta_0)\right)+B}{\mp\sqrt{-(A^2+B^2)\mu}-A\sqrt{-\mu}\cosh\left(2\sqrt{-\mu}(\eta+\eta_0)\right)}\right]e^{i(w\frac{x^\beta}{\beta}+(2\mu-w^2)\frac{t^\alpha}{\alpha})}, \tag{3.7}$$

$$u_4(x,t) = \pm\frac{i\sqrt{2}\mu}{\sqrt{\gamma}\left(\mp\sqrt{-\mu}\pm\frac{2A\sqrt{-\mu}}{A+\cosh\left(2\sqrt{-\mu}(\eta+\eta_0)\right)\pm\sinh\left(2\sqrt{-\mu}(\eta+\eta_0)\right)}\right)}e^{i(w\frac{x^\beta}{\beta}+(2\mu-w^2)\frac{t^\alpha}{\alpha})}, \tag{3.8}$$

$$u_5(x,t) = \mp i\sqrt{\frac{2}{\gamma}}\left(\begin{array}{l}\mu\left(\frac{A\sinh\left(2\sqrt{-\mu}(\eta+\eta_0)\right)+B}{\mp\sqrt{-(A^2+B^2)\mu}-A\sqrt{-\mu}\cosh\left(2\sqrt{-\mu}(\eta+\eta_0)\right)}\right)\\+\left(\frac{\mp\sqrt{-(A^2+B^2)\mu}-A\sqrt{-\mu}\cosh\left(2\sqrt{-\mu}(\eta+\eta_0)\right)}{A\sinh\left(2\sqrt{-\mu}(\eta+\eta_0)\right)+B}\right)\end{array}\right)e^{i(w\frac{x^\beta}{\beta}-(4\mu+w^2)\frac{t^\alpha}{\alpha})}, \tag{3.9}$$

$$u_6(x,t) = \mp\left(\begin{array}{l}\frac{i\sqrt{2}\mu}{\sqrt{\gamma}\left(\mp\sqrt{-\mu}\pm\frac{2A\sqrt{-\mu}}{A+\cosh\left(2\sqrt{-\mu}(\eta+\eta_0)\right)\pm\sinh\left(2\sqrt{-\mu}(\eta+\eta_0)\right)}\right)}\\+\frac{i\sqrt{2}\left(\mp\sqrt{-\mu}\pm\frac{2A\sqrt{-\mu}}{A+\cosh\left(2\sqrt{-\mu}(\eta+\eta_0)\right)\pm\sinh\left(2\sqrt{-\mu}(\eta+\eta_0)\right)}\right)}{\sqrt{\gamma}}\end{array}\right)e^{i(w\frac{x^\beta}{\beta}-(4\mu+w^2)\frac{t^\alpha}{\alpha})}, \tag{3.10}$$

$$u_7(x,t) = \mp i\sqrt{\frac{2}{\gamma}}\left(\begin{array}{l}\mu\left(\frac{A\sinh\left(2\sqrt{-\mu}(\eta+\eta_0)\right)+B}{\mp\sqrt{-(A^2+B^2)\mu}-A\sqrt{-\mu}\cosh\left(2\sqrt{-\mu}(\eta+\eta_0)\right)}\right)\\-\left(\frac{\mp\sqrt{-(A^2+B^2)\mu}-A\sqrt{-\mu}\cosh\left(2\sqrt{-\mu}(\eta+\eta_0)\right)}{A\sinh\left(2\sqrt{-\mu}(\eta+\eta_0)\right)+B}\right)\end{array}\right)e^{i(w\frac{x^\beta}{\beta}+(8\mu+w^2)\frac{t^\alpha}{\alpha})}, \tag{3.11}$$

$$u_8(x,t) = \mp\left(\begin{array}{l}\frac{i\sqrt{2}\mu}{\sqrt{\gamma}\left(\mp\sqrt{-\mu}\pm\frac{2A\sqrt{-\mu}}{A+\cosh\left(2\sqrt{-\mu}(\eta+\eta_0)\right)\pm\sinh\left(2\sqrt{-\mu}(\eta+\eta_0)\right)}\right)}\\-\frac{i\sqrt{2}\left(\mp\sqrt{-\mu}\pm\frac{2A\sqrt{-\mu}}{A+\cosh\left(2\sqrt{-\mu}(\eta+\eta_0)\right)\pm\sinh\left(2\sqrt{-\mu}(\eta+\eta_0)\right)}\right)}{\sqrt{\gamma}}\end{array}\right)e^{i(w\frac{x^\beta}{\beta}+(8\mu+w^2)\frac{t^\alpha}{\alpha})}, \tag{3.12}$$

where $\mu < 0$ and $\eta = \frac{x^\beta}{\beta} - 2w\frac{t^\alpha}{\alpha}$.

Trigonometric function solutions

$$u_9(x,t) = \pm i\sqrt{\frac{2}{\gamma}}\left[\frac{\mp\sqrt{(A^2-B^2)\mu} - A\sqrt{\mu}\cos\left(2\sqrt{\mu}(\eta+\eta_0)\right)}{A\sin\left(2\sqrt{\mu}(\eta+\eta_0)\right)+B}\right]e^{i(w\frac{x^\beta}{\beta}+(2\mu-w^2)\frac{t^\alpha}{\alpha})}, \tag{3.13}$$

$$u_{10}(x,t) = \mp\sqrt{\frac{2}{\gamma}}\left[\pm\sqrt{\mu} \pm \frac{2A\sqrt{\mu}}{A+\cos\left(2\sqrt{\mu}(\eta+\eta_0)\right)\pm i\sin\left(2\sqrt{\mu}(\eta+\eta_0)\right)}\right]e^{i(w\frac{x^\beta}{\beta}+(2\mu-w^2)\frac{t^\alpha}{\alpha})}, \tag{3.14}$$

$$u_{11}(x,t) = \pm i\sqrt{\frac{2}{\gamma}}\mu\left[\frac{A\sin\left(2\sqrt{\mu}(\eta+\eta_0)\right)+B}{\mp\sqrt{(A^2-B^2)\mu}-A\sqrt{\mu}\cos\left(2\sqrt{\mu}(\eta+\eta_0)\right)}\right]e^{i(w\frac{x^\beta}{\beta}+(2\mu-w^2)\frac{t^\alpha}{\alpha})}, \tag{3.15}$$

$$u_{12}(x,t) = \frac{\pm\sqrt{2}\mu}{\sqrt{\gamma}\left(\mp\sqrt{\mu}\pm\frac{2A\sqrt{\mu}}{A+\cos\left(2\sqrt{\mu}(\eta+\eta_0)\right)\pm i\sin\left(2\sqrt{\mu}(\eta+\eta_0)\right)}\right)}e^{i(w\frac{x^\beta}{\beta}+(2\mu-w^2)\frac{t^\alpha}{\alpha})}, \tag{3.16}$$

$$u_{13}(x,t) = \mp i\sqrt{\frac{2}{\gamma}}\begin{pmatrix}\mu\left(\frac{A\sin\left(2\sqrt{\mu}(\eta+\eta_0)\right)+B}{\mp\sqrt{(A^2-B^2)\mu}-A\sqrt{\mu}\cos\left(2\sqrt{\mu}(\eta+\eta_0)\right)}\right)\\ +\left(\frac{\mp\sqrt{(A^2-B^2)\mu}-A\sqrt{\mu}\cos\left(2\sqrt{\mu}(\eta+\eta_0)\right)}{A\sin\left(2\sqrt{\mu}(\eta+\eta_0)\right)+B}\right)\end{pmatrix}e^{i(w\frac{x^\beta}{\beta}-(4\mu+w^2)\frac{t^\alpha}{\alpha})}, \tag{3.17}$$

$$u_{14}(x,t) = \mp\begin{pmatrix}\frac{\sqrt{2}\mu}{\sqrt{\gamma}\left(\mp\sqrt{\mu}\pm\frac{2A\sqrt{\mu}}{A+\cos\left(2\sqrt{\mu}(\eta+\eta_0)\right)\pm i\sin\left(2\sqrt{\mu}(\eta+\eta_0)\right)}\right)}\\ +\frac{\sqrt{2}\left(\mp\sqrt{\mu}\pm\frac{2A\sqrt{\mu}}{A+\cos\left(2\sqrt{\mu}(\eta+\eta_0)\right)\pm i\sin\left(2\sqrt{\mu}(\eta+\eta_0)\right)}\right)}{\sqrt{\gamma}}\end{pmatrix}e^{i(w\frac{x^\beta}{\beta}-(4\mu+w^2)\frac{t^\alpha}{\alpha})}, \tag{3.18}$$

$$u_{15}(x,t) = \mp i\sqrt{\frac{2}{\gamma}}\begin{pmatrix}\mu\left(\frac{A\sin\left(2\sqrt{\mu}(\eta+\eta_0)\right)+B}{\mp\sqrt{(A^2-B^2)\mu}-A\sqrt{\mu}\cos\left(2\sqrt{\mu}(\eta+\eta_0)\right)}\right)\\ -\left(\frac{\mp\sqrt{(A^2-B^2)\mu}-A\sqrt{\mu}\cos\left(2\sqrt{\mu}(\eta+\eta_0)\right)}{A\sin\left(2\sqrt{\mu}(\eta+\eta_0)\right)+B}\right)\end{pmatrix}e^{i(w\frac{x^\beta}{\beta}+(8\mu+w^2)\frac{t^\alpha}{\alpha})}, \tag{3.19}$$

$$u_{16}(x,t) = \mp\begin{pmatrix}\frac{\sqrt{2}\mu}{\sqrt{\gamma}\left(\mp\sqrt{\mu}\pm\frac{2A\sqrt{\mu}}{A+\cos\left(2\sqrt{\mu}(\eta+\eta_0)\right)\pm i\sin\left(2\sqrt{\mu}(\eta+\eta_0)\right)}\right)}\\ -\frac{\sqrt{2}\left(\mp\sqrt{\mu}\pm\frac{2A\sqrt{\mu}}{A+\cos\left(2\sqrt{\mu}(\eta+\eta_0)\right)\pm i\sin\left(2\sqrt{\mu}(\eta+\eta_0)\right)}\right)}{\sqrt{\gamma}}\end{pmatrix}e^{i(w\frac{x^\beta}{\beta}+(8\mu+w^2)\frac{t^\alpha}{\alpha})}, \tag{3.20}$$

where $\mu > 0$ and $\eta = \frac{x^\beta}{\beta} - 2w\frac{t^\alpha}{\alpha}$.

Rational function solutions

$$u_{17}(x,t) = \pm i\sqrt{\frac{2}{\gamma}}\frac{e^{i(w\frac{x^\beta}{\beta}-w^2\frac{t^\alpha}{\alpha})}}{(\eta+\eta_0)}, \tag{3.21}$$

where $\mu = 0$ and $\eta = \frac{x^\beta}{\beta} - 2w\frac{t^\alpha}{\alpha}$.

4. Result and Discussion

In this section, the reduced form of the solutions found in the former section and the graphical representations of some solutions have been illustrated.

We have obtained 17 general solutions for the space-time fractional cubic nonlinear Schrödinger equation in the former section. Using trigonometric and hyperbolic identities and taking $B = 0$, hyperbolic and trigonometric solutions obtained by the modified extended tanh method can be attained as follows:

Reduced hyperbolic function solutions

$$u_{h1}(x,t) = \pm i\sqrt{\frac{-2\mu}{\gamma}}\tanh\left(\sqrt{-\mu}(\eta+\eta_0)\right)e^{i(w\frac{x^\beta}{\beta}+(2\mu-w^2)\frac{t^\alpha}{\alpha})},$$

$$u_{h2}(x,t) = \pm i\sqrt{\frac{-2\mu}{\gamma}}\coth\left(\sqrt{-\mu}(\eta+\eta_0)\right)e^{i(w\frac{x^\beta}{\beta}+(2\mu-w^2)\frac{t^\alpha}{\alpha})},$$

$$u_{h3}(x,t) = \pm i\sqrt{\frac{-2\mu}{\gamma}}\left(\frac{-A + e^{\mp 2\sqrt{-\mu}(\eta+\eta_0)}}{A + e^{\mp 2\sqrt{-\mu}(\eta+\eta_0)}}\right)e^{i(w\frac{x^\beta}{\beta}+(2\mu-w^2)\frac{t^\alpha}{\alpha})},$$

$$u_{h4}(x,t) = \pm i\sqrt{\frac{-8\mu}{\gamma}}\coth\left(2\sqrt{-\mu}(\eta+\eta_0)\right)e^{i(w\frac{x^\beta}{\beta}-(4\mu+w^2)\frac{t^\alpha}{\alpha})},$$

$$u_{h5}(x,t) = \pm i\sqrt{\frac{-8\mu}{\gamma}}\operatorname{cosech}\left(2\sqrt{-\mu}(\eta+\eta_0)\right)e^{i(w\frac{x^\beta}{\beta}-(4\mu+w^2)\frac{t^\alpha}{\alpha})},$$

$$u_{h6}(x,t) = \pm i\sqrt{\frac{-8\mu}{\gamma}}\left(\frac{A^2 + e^{\mp 4\sqrt{-\mu}(\eta+\eta_0)}}{-A^2 + e^{\mp 4\sqrt{-\mu}(\eta+\eta_0)}}\right)e^{i(w\frac{x^\beta}{\beta}-(4\mu+w^2)\frac{t^\alpha}{\alpha})},$$

$$u_{h7}(x,t) = \pm i\sqrt{\frac{-8\mu}{\gamma}}\coth\left(2\sqrt{-\mu}(\eta+\eta_0)\right)e^{i(w\frac{x^\beta}{\beta}+(8\mu+w^2)\frac{t^\alpha}{\alpha})},$$

$$u_{h8}(x,t) = \pm i\sqrt{\frac{-8\mu}{\gamma}}\operatorname{cosech}\left(2\sqrt{-\mu}(\eta+\eta_0)\right)e^{i(w\frac{x^\beta}{\beta}+(8\mu+w^2)\frac{t^\alpha}{\alpha})},$$

$$u_{h9}(x,t) = \pm i\sqrt{\frac{-8\mu}{\gamma}}\left(\frac{4Ae^{\mp 2\sqrt{-\mu}(\eta+\eta_0)}}{-A^2 + e^{\mp 4\sqrt{-\mu}(\eta+\eta_0)}}\right)e^{i(w\frac{x^\beta}{\beta}+(8\mu+w^2)\frac{t^\alpha}{\alpha})},$$

where $\mu < 0$ and $\eta = \frac{x^\beta}{\beta} - 2w\frac{t^\alpha}{\alpha}$.

Reduced trigonometric function solutions

$$u_{t1}(x,t) = \pm i\sqrt{\frac{-2\mu}{\gamma}}\tan\left(\sqrt{\mu}(\eta+\eta_0)\right)e^{i(w\frac{x^\beta}{\beta}+(2\mu-w^2)\frac{t^\alpha}{\alpha})},$$

$$u_{t2}(x,t) = \pm i\sqrt{\frac{-2\mu}{\gamma}}\cot\left(\sqrt{\mu}(\eta+\eta_0)\right)e^{i(w\frac{x^\beta}{\beta}+(2\mu-w^2)\frac{t^\alpha}{\alpha})},$$

$$u_{t3}(x,t) = \pm i\sqrt{\frac{-2\mu}{\gamma}}\left(\frac{-A + e^{\mp 2i\sqrt{\mu}(\eta+\eta_0)}}{A + e^{\mp 2i\sqrt{\mu}(\eta+\eta_0)}}\right)e^{i(w\frac{x^\beta}{\beta}+(2\mu-w^2)\frac{t^\alpha}{\alpha})},$$

$$u_{t4}(x,t) = \pm i\sqrt{\frac{-8\mu}{\gamma}}\cot\left(2\sqrt{\mu}(\eta+\eta_0)\right)e^{i(w\frac{x^\beta}{\beta}-(4\mu+w^2)\frac{t^\alpha}{\alpha})},$$

$$u_{t5}(x,t) = \pm i\sqrt{\frac{-8\mu}{\gamma}}\operatorname{cosec}\left(2\sqrt{\mu}(\eta+\eta_0)\right)e^{i(w\frac{x^\beta}{\beta}-(4\mu+w^2)\frac{t^\alpha}{\alpha})},$$

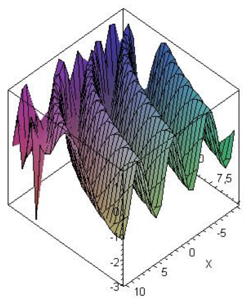
$$u_{t6}(x,t) = \pm i\sqrt{\frac{-8\mu}{\gamma}}\left(\frac{A^2 + e^{\mp 4i\sqrt{\mu}(\eta+\eta_0)}}{-A^2 + e^{\mp 4i\sqrt{\mu}(\eta+\eta_0)}}\right)e^{i(w\frac{x^\beta}{\beta}-(4\mu+w^2)\frac{t^\alpha}{\alpha})},$$

$$u_{t7}(x,t) = \pm i\sqrt{\frac{-8\mu}{\gamma}}\cot\left(2\sqrt{\mu}(\eta+\eta_0)\right)e^{i(w\frac{x^\beta}{\beta}+(8\mu+w^2)\frac{t^\alpha}{\alpha})},$$

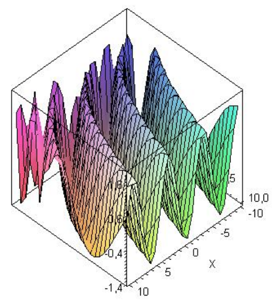
$$u_{t8}(x,t) = \pm i\sqrt{\frac{-8\mu}{\gamma}}\operatorname{cosec}\left(2\sqrt{\mu}(\eta+\eta_0)\right)e^{i(w\frac{x^\beta}{\beta}+(8\mu+w^2)\frac{t^\alpha}{\alpha})},$$

$$u_{t9}(x,t) = \pm i\sqrt{\frac{-8\mu}{\gamma}}\left(\frac{4Ae^{\mp 2i\sqrt{\mu}(\eta+\eta_0)}}{-A^2 + e^{\mp 4i\sqrt{\mu}(\eta+\eta_0)}}\right)e^{i(w\frac{x^\beta}{\beta}+(8\mu+w^2)\frac{t^\alpha}{\alpha})},$$

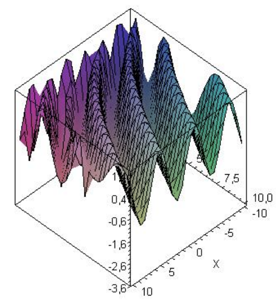
where $\mu > 0$ and $\eta = \frac{x^\beta}{\beta} - 2w\frac{t^\alpha}{\alpha}$.



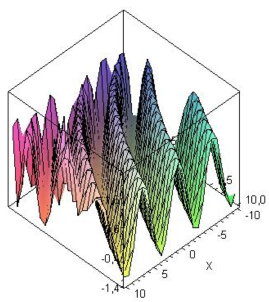
(A) 3D graph of of real part with $\alpha = 0.25$



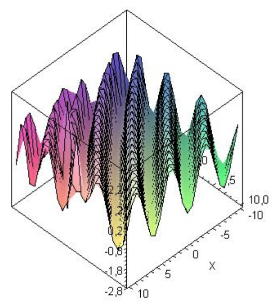
(B) 3D graph of of imaginary part with $\alpha = 0.25$



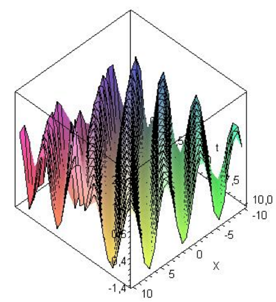
(C) 3D graph of of real part with $\alpha = 0.5$



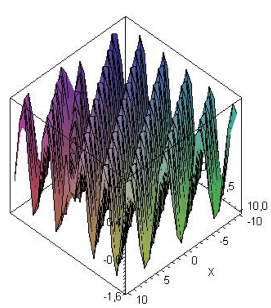
(D) 3D graph of of imaginary part with $\alpha = 0.5$



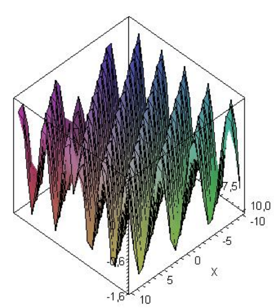
(E) 3D graph of of real part with $\alpha = 0.75$



(F) 3D graph of of imaginary part with $\alpha = 0.75$

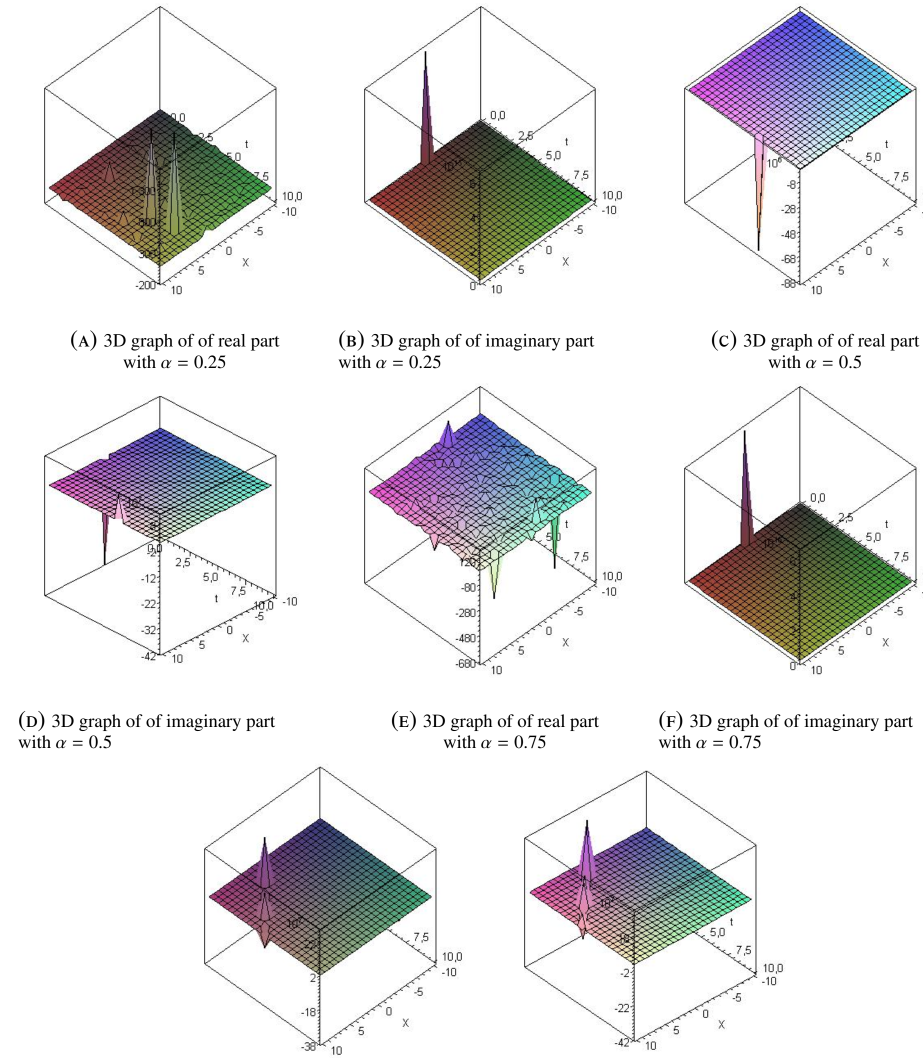


(G) 3D graph of of real part with $\alpha = 1$



(H) 3D graph of of imaginary part with $\alpha = 1$

Figure 1. Graphical representation of real and imaginary part for solution of u_1 is plotted above for $-10 < x < 10$, $0 < t < 10$, with parameters $\mu = -1$, $w = 1$, $\gamma = 1$, $\beta = 1$ $A = 1$, $B = 0$ and various alpha parameters.



(A) 3D graph of of real part with $\alpha = 0.25$

(B) 3D graph of of imaginary part with $\alpha = 0.25$

(C) 3D graph of of real part with $\alpha = 0.5$

(D) 3D graph of of imaginary part with $\alpha = 0.5$

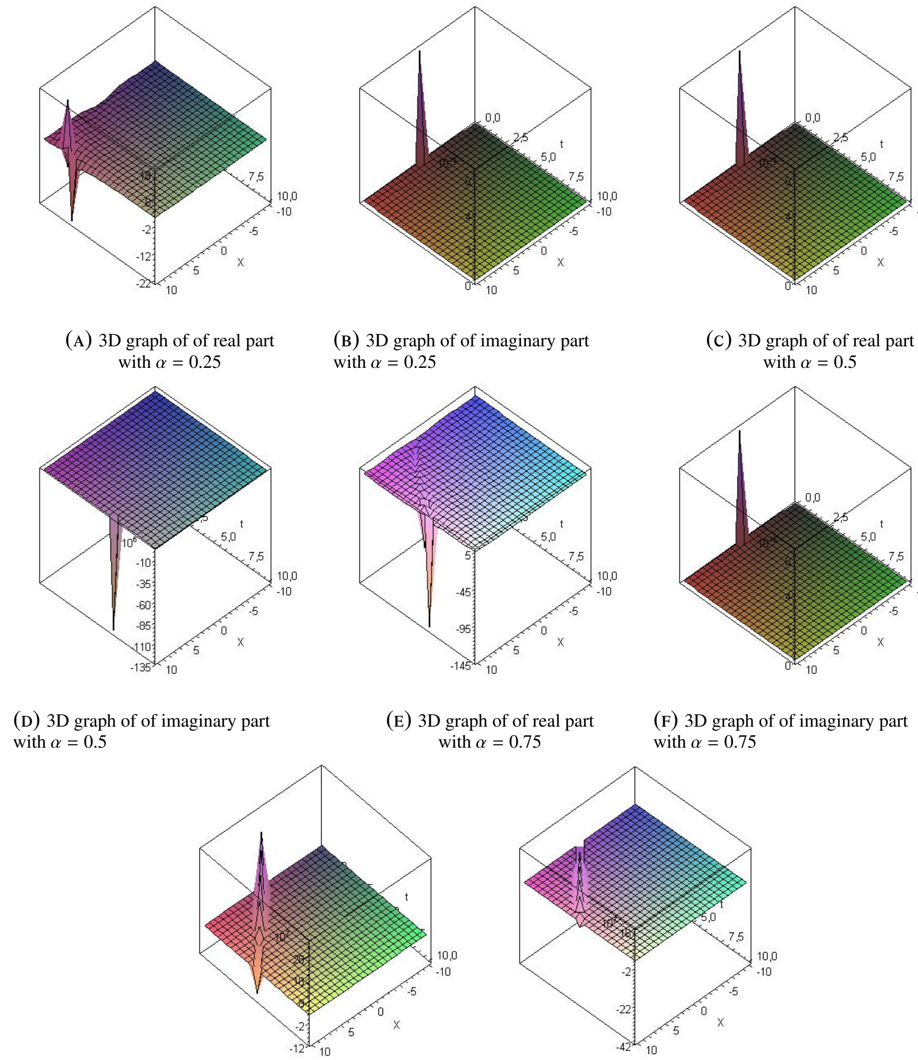
(E) 3D graph of of real part with $\alpha = 0.75$

(F) 3D graph of of imaginary part with $\alpha = 0.75$

(G) 3D graph of of real part with $\alpha = 1$

(H) 3D graph of of imaginary part with $\alpha = 1$

FIGURE 2. Graphical representation of real and imaginary part for the solution of u_{14} is plotted above for $-10 < x < 10$, $0 < t < 10$, with parameters $\mu = 1$, $w = 1$, $\gamma = 1$, $\beta = 1$ $A = 1$, $B = 0$ and various alpha parameters.



(A) 3D graph of of real part with $\alpha = 0.25$

(B) 3D graph of of imaginary part with $\alpha = 0.25$

(C) 3D graph of of real part with $\alpha = 0.5$

(D) 3D graph of of imaginary part with $\alpha = 0.5$

(E) 3D graph of of real part with $\alpha = 0.75$

(F) 3D graph of of imaginary part with $\alpha = 0.75$

(G) 3D graph of of real part with $\alpha = 1$

(H) 3D graph of of imaginary part with $\alpha = 1$

Figure 3. Graphical representation of real and imaginary part for the solution of u_{17} is plotted above for $-10 < x < 10$, $0 < t < 10$, with parameters $\mu = 0$, $w = 1$, $\gamma = 1$, $\beta = 1$ $A = 1$, $B = 0$ and various alpha parameters.

The studies in [8,14,17,25,27] used the extended sinh-Gordon equation expansion method, the extended trial equation method, the Jacobi elliptic functions expansion method, and the extended modified auxiliary equation mapping method respectively. Compared to the solutions in the aforementioned studies, the solutions obtained by the unified method represent a more general solution form with free parameters in closed forms. In other words, these methods provide hyperbolic and trigonometric function solution forms with the tanh, tan, coth, cot, sech, sec, cosech, cosec function and combinations of them. These all solutions can be derived from the solutions between (3.5)-(3.21) easily using some hyperbolic and trigonometric identities with certain values for free parameters.

The trigonometric type, hyperbolic type and rational type solutions are plotted in Fig.1, Fig.2, Fig.3, respectively. The selected solutions u_1 in (3.5), u_{14} in (3.18), and u_{17} in (3.21) are drawn using Maple with the real part on the left and the imaginary part on the right as above. The free parameters $A = 1$, and $B = 0$ are chosen because some of the obtained solutions are reduced those obtained by using modified extended tanh method solutions, known as the strongest tanh technique.

3D graphs of u_1, u_{14} and u_{17} solutions are drawn here for different α values from 0.25 to 1 with certain μ, w, γ, β values to show the physical structure of the wave solution. Therefore, the temporal changes of the wave solution can be followed as α goes to 1. This shows the structural change in solutions from fractional to non-fractional derivatives.

5. Conclusions

The fractional nonlinear partial differential equations (FNPDEs) are a generalization of nonlinear partial differential equations (NPDEs) that allows for more powerful and realistic modeling of systems with complex behaviors, such as those with memory effects or long-range interactions. For this reason, the FNPDEs are reduced to the NPDEs when the fractional orders go to integer values for time and space. The effect of the fractional order derivatives captures the memory or non-local effects in the system.

In this study, the unified method is applied successfully to the space-time fractional-order cubic nonlinear Schrödinger equation(stFCSE) with time and space fractional derivatives. This method based on the ansatz method is a straightforward and powerful method that gives many solutions. Different solutions can be produced by plugging various arbitrary parameters for A, B w, and μ from the obtained 17 solutions in Section 3. Therefore, it distinguishes this method from other methods that give many exact solutions such as bright, dark, singular, combo, optical, singular optical, and bright-singular combo soliton solutions because it is more general. For this reason, the obtained solutions of the stFCSE not only go to the solutions of the cubic nonlinear Schrödinger equation when the fractional orders go to 1 for time and space but also produce the solutions obtained in the previous studies.

The selected solutions are preferred to plot here with the free parameters $A = 1$, $B = 0$ because some of the solutions are the same as those obtained using modified extended tanh method solutions. The obtained solutions here can be plotted for different values.

The computations in this work have been performed by Maple.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed the published version of the manuscript.

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