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Lifts of Hypersurfaces on a Sasakian Manifold with a Quartersymmetric Semimetric Connection (QSSC) to Its Tangent Bundle

Mohammad Nazrul Islam Khan[1*] and Lovejoy Swapan Kumar Das[2]

[1]*Department of Computer Engineering, College of Computer, Qassim University, Buraydah-51452, Saudi Arabia*
[2]*Department of Mathematics, Kent State University,, Tuscarawas, New Philadelphia, OH, 44663, U.S.A.*
[*]*Corresponding author*

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Abstract

The aim of the present paper is to introduce a Sasakian manifold immersed with a quarter-symmetric semimetric connection to a tangent bundle. Some basic results are given on a Riemannian connection and a QSSC to the tangent bundle on a Sasakian manifold. The geometrical properties of a Sasakian manifold to its tangent bundle are also discussed.

1. Introduction

A quartersymmetric linear connection with an affine connection ∇ in differentiable manifolds was defined and studied by Golab [1]. Let $\tilde{T}_0$ be a torsion tensor defied as

$$\tilde{T}_0(X_0,Y_0) = u(Y_0)\phi X_0 - u(X_0)\phi Y_0, \tag{1.1}$$

where $u \in \Im_0^1(M), \phi \in \Im_1^1(M)$, then ∇ is known as a quarter symmetric connection.

Several authors made precious contributions to a QSSC including ([2], [3]). Dida et. al. ([4], [5]) studied the geometry of II order tangent bundle and Ricci soliton on the tangent bundle with semisymmetric metric connection. Golden Riemannian structure on tangent bundles were studied and some basic results was proved on it by Peyghan et. al. [6]. Recently, Altunbas et. al. [7] introduced and obtained fundamental results on Ricci soliton on tangent bundles by applying complete lifts. Some theorems on a Lorentzian para-Sasakian manifold with a quartersymmetric metric connection on tangent bundles are determined [8].

In this study, we apply the complete and vertical lifts on tensor fields and connections. The development of the theory of hypersurfaces prolonged to tangent bundle with respect to complete lifts of metric tensor of a Riemannian manifold is attributed to Tani [9]. In 2022, Khan [10] studied submanifolds of a Riemannian manifold endowed with a new type of semi-symmetric non-metric connection in the tangent bundle. Different geometers have studied and defined different types of connections and structures which can be seen in ([11]- [17]).

Lifts of hypersurface from a Sasakian manifold to its tangent bundle connected to a QSSC are examined in the proposed work. Key findings include the following:

Email addresses and ORCID numbers: **m.nazrul@qu.edu.sa, 0000-0002-9652-0355 (M.N.I. Khan), ldas@kent.edu, 0000-0002-2709-5113 (L.S.K. Das)**
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- We proved the induced connection on a Sasakian manifold with QSSC concerning the unit normal is also a QSSC.
- We determined the formula for $\bar{\nabla}^C$ and $\hat{\nabla}^C$ with a QSSC on TM.
- We developed a relation between a QSSC $\mathring{\nabla}^C$ with respect to Riemannian connection ∇^C in $(TS,\tilde{g})$.
- We proved some theorems on geometrical properties with respect to $\mathring{\nabla}^C$ and ∇^C.

2. Preliminaries

Let M (dim$=n$) be a Riemannian manifold. For $\phi \in \Im_1^1(M), \eta_0 \in \Im_1^0(M), \xi \in \Im_0^1(M)$ fulfilling

$$\phi^2 X = -X_0 + \eta_0(X_0)\xi, \tag{2.1}$$

M is called an almost contact manifold [18] and the structure (ϕ, ξ, η_0) is called an almost contact structure on M. In addition, there exists a metric tensor g satisfying

$$\begin{aligned} g(\phi X_0, \phi Y_0) =& g(X_0,Y_0) - \eta_0(X_0)\eta_0(Y_0), \\ g(X_0,\xi) =& \eta_0(X_0), \end{aligned}$$

then M is called an almost contact metric manifold [19].
The vector field ξ is said to be a Killing vector field if it generates a group of isometries or equivalently if $g(\nabla_{X_0}\xi, Y_0) + g(\nabla_{Y_0}\xi, X_0) = 0$.
If ξ is a Killing vector field then the contact metric manifold (ϕ, ξ, η_0) is called a K-contact structure, and such a manifold is called a K-contact manifold [19]. A K-contact Riemannian manifold (M,g) is called a Sasakian manifold ([2], [20]) if $\forall X_0, Y_0 \in \Im_0^1(M)$, we have

$$(\nabla_{X_0}\phi)(Y_0) = g(Y_0,X_0)\xi - \eta_0(Y_0)X_0. \tag{2.2}$$

Besides the relations (2.1) and (2.2), the following relations also hold in a Sasakian manifold

$$\phi\xi = 0, \ \ \eta_0(\xi) = 1, \ \ \nabla_{X_0}\xi = -\phi X_0, \ \ g(\phi X_0, Y_0) + g(X_0, \phi Y_0) = 0,$$

$\forall X_0, Y_0 \in \Im_0^1(M)$.
The torsion tensor $\hat{T}_0$ with the Levi-Civita connection ∇ and the linear connection $\hat{\nabla}$ is defined as

$$\hat{T}_0(\hat{X}_0, \hat{Y}_0) = \hat{\nabla}_{\hat{X}_0}\hat{Y}_0 - \hat{\nabla}_{\hat{Y}_0}\hat{X}_0 - [\hat{X}_0, \hat{Y}_0], \tag{2.3}$$

$\forall \hat{X}_0, \hat{Y}_0 \in \Im_0^1(M)$.
A QSSC $\overline{\nabla}$ in $(M,\hat{g})$ is defined as [21]

$$\overline{\nabla}_{\hat{X}_0}\hat{Y}_0 = \hat{\nabla}_{\hat{X}_0}\hat{Y}_0 - \hat{\eta}(\hat{X}_0)\hat{\phi}\hat{Y}_0 + \hat{g}(\hat{\phi}\hat{X}_0, \hat{Y}_0), \tag{2.4}$$

which satisfies

$$(\overline{\nabla}_{\hat{X}_0}\hat{g}(\hat{X}_0, \hat{Y}_0) = 2\hat{\eta}(\hat{X}_0)\hat{g}(\hat{Y}_0, \hat{Z}_0) - \hat{\eta}(\hat{Y}_0)\hat{g}(\hat{\phi}\hat{X}_0, \hat{Z}_0) + \hat{\eta}(\hat{Z}_0)\hat{g}(\hat{\phi}\hat{X}_0, \hat{Y}_0), \tag{2.5}$$

$\forall \hat{X}_0, \hat{Y}_0 \in \Im_0^1(M)$, where $\hat{\nabla}$ is a Riemannian connection in $(M,\hat{g})$ and $\hat{P} \in \Im_0^1(M)$ given by $\hat{g}(\hat{P}, \hat{X}_0) = \hat{\eta}(\hat{X}_0)$.

Let TM be the tangent bundle of M. Superscripts C and V denote the complete and vertical lifts of the tensor fields. The following characteristics of these lifts ([10, 22, 23]):

$$\begin{aligned} [\hat{X}_0{}^C, \hat{Y}_0{}^C] =& [\hat{X}_0, \hat{Y}_0]^C \ ; \ \hat{\phi}^C(\hat{X}_0{}^C) = (\hat{\phi}(\hat{X}_0))^C, \\ \hat{\phi}^V(\hat{X}_0{}^C) =& \hat{\phi}^C(\hat{X}_0{}^V) = (\hat{\phi}(\hat{X}_0))^V \ ; \ \hat{\phi}^V(\hat{X}_0{}^V) = 0, \\ \hat{\eta}_0{}^V(\hat{X}_0{}^C) =& (\hat{\eta}_0(\hat{X}_0))^V \ ; \ \hat{\eta}_0{}^C(\hat{X}_0{}^C) = (\hat{\eta}_0(\hat{X}_0))^C, \\ \hat{g}^C(\hat{X}_0{}^V, \hat{Y}_0{}^C) =& \hat{g}^C(\hat{X}_0{}^C, \hat{Y}_0{}^V) = (\hat{g}(\hat{X}_0, \hat{Y}_0))^V, \\ \hat{g}^C(\hat{X}_0{}^V, \hat{Y}_0{}^C) =& (\hat{g}(\hat{X}_0, \hat{Y}_0))^C, \\ \hat{\nabla}^C(\hat{X}_0{}^C, \hat{Y}_0{}^C) =& (\hat{\nabla}(\hat{X}_0, \hat{Y}_0))^C, \\ \hat{\nabla}^C(\hat{X}_0{}^C, \hat{Y}_0{}^V) =& (\hat{\nabla}(\hat{X}_0, \hat{Y}_0))^V, \end{aligned} \tag{2.6}$$

$\forall X_0, Y_0 \in \Im_0^1(M), \eta \in \Im_1^0(M), \phi \in \Im_1^1(M)$.

Let S (dim$=n-1$) be a manifold such that a mapping $B: S \to M$. The tangent map of B represented by $\tilde{B}: T(TS) \to T(TM)$, where $\tilde{B}: TS \to TM$. The hypersurface S is a Riemannian manifold and g is induce metric on S such that

$$g(X_0,Y_0)=\hat{g}(BX_0,BY_0),$$

and

$$\hat{\nabla}_{BX_0}BY_0 = B(\nabla_{X_0}Y_0)+h(X_0,Y_0)N, \tag{2.7}$$

$\forall X_0,Y_0 \in \Im_0^1(M)$, where $\hat{\nabla}$ is induced connection, N is the unit normal vector field and h is the second fundamental tensor field on (S,g) ([9, 24]). The relation

$$h(X_0,Y_0)=g(HX_0,Y_0), H \in \Im_1^1(S).$$

Definition 2.1 (i) If $h=0$ then S is said to be totally geodesic with respect to ∇.
(ii) If h is proportional to g then S is said to be totally umbilical with respect to ∇ [25].

3. Lifts of a QSSC to the Tangent Bundle on a Sasakian Manifold

Let $\mathring{\nabla}$ is a QSSC induced on the hypersurface S from $\bar{\nabla}$, fulfills

$$\bar{\nabla}_{BX_0}BY_0 = B(\mathring{\nabla}_{X_0}Y_0)+h(X_0,Y_0)N, \tag{3.1}$$

$\forall X_0,Y_0 \in \Im_0^1(S), m \in \Im_0^2(S)$.
Putting $M=H-\lambda I$, we get the relation

$$m(X_0,Y_0)=g(MX,Y_0),$$

$\forall I \in \Im_1^1(S)$.
Definition 3.1 (i) If $m=0$ then S is said to be totally geodesic with respect to $\mathring{\nabla}$.
(ii) If $m \propto g$ then S is said to be totally umbilical with respect to $\mathring{\nabla}$.
In view of (2.4), we infer

$$\bar{\nabla}_{BX}BY = \hat{\nabla}_{BX}BY - \hat{\eta}_0(BX)B\phi Y_0 + \hat{g}(B\phi X_0,BY)\hat{\xi}, \tag{3.2}$$

$\forall X_0,Y_0 \in \Im_0^1(S)$.
Using equations (2.7), (3.1) and (3.2), we obtain

$$B(\mathring{\nabla}_{X_0}Y_0)+m(X_0,Y_0)N \quad = \quad B(\nabla_{X_0}Y_0)+h(X_0,Y_0)N-\hat{\eta}_0(BX_0)B\phi Y_0+\hat{g}(B\phi X_0,BY_0)(B\xi+\lambda N),$$

Put $\hat{\xi}=B\xi+\lambda N$, where λ is a function, $\xi \in \Im_0^1(S)$ and $\eta_0 \in \Im_1^0(S)$ determined by $\eta_0(X_0)=\hat{\eta}_0(BX_0)$ ([19, 27, 28]).
Comparing the tangential and normal parts from both sides, we infer

$$\begin{aligned} \mathring{\nabla}_{X_0}Y_0 =& \nabla_X Y - \eta_0(X_0)\phi Y_0 + g(\phi X_0,Y_0)\xi, \\ m(X_0,Y_0) =& h(X_0,Y_0)+\lambda g(\phi X_0,Y_0). \end{aligned}$$

Hence, we state the following:

Theorem 3.1. *The connection induced on a Riemannian manifold's hypersurfaces with a QSSC on a Sasakian manifold with respect to the unit normal is also a QSSC.*

Let $\hat{g}$ be an element of M and the complete lift $\hat{g}^C$ be the element of TM. The induced metric on TS from $\hat{g}^C$ by $\hat{g}$. Then

$$\tilde{g}(X_0^C,Y_0^C)=\hat{g}^C(\tilde{B}X_0^C,\tilde{B}Y_0^C), \forall X_0,Y_0 \in \Im_1^0(S).$$

Let Riemannian connection $\hat{\nabla}$ be an element of $(M,\hat{g})$, then $\hat{\nabla}^C$ will be an element of $(TM,\hat{g}^C)$. Let ∇ be an induced connection in (S,g), then ∇^C is an element of $(TM,\tilde{g})$.
We shall first state known results ([6, 29])

Theorem 3.2. *If $\hat{T}_0$ is torsion tensor of $\hat{\nabla}$ in $(M,\hat{g})$, then $\hat{T}_0{}^C$ is torsion tensor of $\hat{\nabla}^C$ in $(TM,\hat{g}^C)$.*

Theorem 3.3. $\forall X_0,Y_0 \in \Im_0^1(S)$

$$\begin{aligned} \hat{\eta}_0{}^V(\tilde{B}X_0^C) =& \hat{\eta}_0{}^V(\tilde{B}X_0^{\bar{C}}) = \#(\hat{\eta}_0{}^V(\tilde{B}X_0^C) = \#(\hat{\eta}_0(\tilde{B}X_0)^V = (\hat{\eta}_0(\tilde{B}X_0)^{\bar{V}}, \\ \hat{\eta}_0{}^C(\tilde{B}X_0^C) =& \hat{\eta}_0{}^C(\tilde{B}X_0^{\bar{C}}) = \#(\hat{\eta}_0{}^V(\tilde{B}X_0^) = \#(\hat{\eta}_0(\tilde{B}X_0)^C = (\hat{\eta}_0(\tilde{B}X_0)^{\bar{C}}, \end{aligned}$$

where # represents an operation of restriction and $\bar{C}$ and $\bar{V}$ represent complete and vertical lifts operatons on $\pi_M^{-1}(\tau(S))$. Applying the complete lifts on (2.4) and with the help of (2.6), we infer

$$\begin{aligned}
(\bar{\nabla}_{BX}BY)^{\bar{C}} =&(\hat{\nabla}_{BX}BY)^{\bar{C}} - (\hat{\eta}_0(BX)B\phi Y_0)^{\bar{C}} + (\hat{g}(B\phi X_0, BY)\hat{\xi})^{\bar{C}}\\
(\bar{\nabla}_{BX}BY)^{\bar{C}} =&(\hat{\nabla}_{BX}BY)^{\bar{C}} - (\hat{\eta}_0(BX))^{\bar{C}}(B\phi Y_0)^{\bar{C}} + \hat{\eta}_0(BX))^{\bar{V}}(B\phi Y_0)^{\bar{V}})\\
&+(\hat{g}(B\phi X_0, BY))^{\bar{C}}\hat{\xi})^{\bar{V}} + (\hat{g}(B\phi X_0, BY))^{\bar{V}}\hat{\xi})^{\bar{C}}\\
\bar{\nabla}^C_{\tilde{B}X_0^C}\tilde{B}Y_0^C =&(\hat{\nabla}^C_{\tilde{B}X_0^C}\tilde{B}Y_0^C - \hat{\eta_0}^C(\tilde{B}X_0^C)(\tilde{B}(\phi Y_0)^V) - \hat{\eta_0}^V(\tilde{B}X_0^C)(\tilde{B}(\phi Y_0)^C)\\
&+(\hat{g}^C(\tilde{B}(\phi X_0)^C, \tilde{B}(\phi X_0)^C\hat{\xi})^{\bar{V}} + (\hat{g}^C(\tilde{B}(\phi X_0)^V, \tilde{B}(\phi X_0)^C\hat{\xi})^{\bar{C}}.
\end{aligned}$$

We have

$$\begin{aligned}
\bar{\nabla}^C_{\tilde{B}X_0^C}\tilde{B}Y_0^C - \bar{\nabla}^C_{\tilde{B}Y_0^C}\tilde{B}X_0^C - [X_0^C, Y_0^C] =& - \hat{\eta_0}^C(\tilde{B}X_0^C)(\tilde{B}(\phi Y_0)^V) - \hat{\eta_0}^V(\tilde{B}X_0^C)(\tilde{B}(\phi Y_0)^C) + \hat{\eta_0}^C(\tilde{B}Y_0^C)(\tilde{B}(\phi X_0)^V)\\
&+\hat{\eta_0}^V(\tilde{B}Y_0^C)(\tilde{B}(\phi X_0)^C).
\end{aligned}$$

From equation (2.3) and Theorem 3.2, we get

$$\bar{T}^C(BX^C, BY^C) =\hat{\eta_0}^C(\tilde{B}Y_0^C)(\tilde{B}(\phi X_0)^V) + \hat{\eta_0}^V(\tilde{B}Y_0^C)(\tilde{B}(\phi X_0)^C) - \hat{\eta_0}^C(\tilde{B}X_0^C)(\tilde{B}(\phi Y_0)^V) - \hat{\eta_0}^V(\tilde{B}X_0^C)(\tilde{B}(\phi Y_0)^C). \tag{3.3}$$

Now,

$$\begin{aligned}
\hat{g}^C(\bar{\nabla}^C_{\tilde{B}X_0^C}\tilde{B}Y_0^C, \tilde{B}Z_0^C) + \hat{g}^C(\tilde{B}Y_0^C, \bar{\nabla}^C_{\tilde{B}X_0^C}\tilde{B}Z_0^C) =&\hat{g}^C(\hat{\nabla}^C_{\tilde{B}X_0^C}\tilde{B}Y_0^C - \hat{\eta_0}^C(\tilde{B}X_0^C)(\tilde{B}(\phi Y_0)^V) - \hat{\eta_0}^C(\tilde{B}Y_0^C)(\tilde{B}(\phi X_0)^V)\\
&+\hat{g}^C(\tilde{B}(\phi X_0)^C, \tilde{B}Y_0^C)\hat{\xi}^{\bar{V}} + \hat{g}^C(\tilde{B}(\phi X_0)^V, \tilde{B}Y_0^C)\hat{\xi}^{\bar{C}}, \tilde{B}Z_0^C)\\
&+\hat{g}^C(\tilde{B}Y_0, \hat{\nabla}^C_{\tilde{B}X_0^C}\tilde{B}Z_0^C - \hat{\eta_0}^C(\tilde{B}X_0^C)(\tilde{B}(\phi Z_0)^V)\\
&-\hat{\eta_0}^V(\tilde{B}X_0^C)(\tilde{B}(\phi Z_0)^C) + \hat{g}^C(\tilde{B}(\phi X_0)^C, \tilde{B}Z_0^C)\hat{\xi}^{\bar{V}}\\
&+\hat{g}^C(\tilde{B}(\phi X_0)^V, \tilde{B}Z_0^C)\hat{\xi}^{\bar{C}})\\
=&\hat{g}^C(\hat{\nabla}^C_{\tilde{B}X_0^C}\tilde{B}Y_0^C, \tilde{B}Z_0^C) + \hat{g}^C(\tilde{B}Y_0^C, \hat{\nabla}^C_{\tilde{B}X_0^C}\tilde{B}Z_0^C)\\
&+(\hat{\eta}_0(\tilde{B}Z_0))^V(\hat{g}(\tilde{B}\phi X_0, \tilde{B}Y_0))^C\\
&+(\hat{\eta}_0(\tilde{B}Z_0))^C(\hat{g}(\tilde{B}\phi X_0, \tilde{B}Y_0))^V + (\hat{\eta}_0(\tilde{B}Y_0))^V(\hat{g}(\tilde{B}\phi X_0, \tilde{B}Z_0))^C\\
&+(\hat{\eta}_0(\tilde{B}Y_0))^C(\hat{g}(\tilde{B}\phi X_0, \tilde{B}Z_0))^V\\
=&(\tilde{B}X_0^C)\hat{g}^C(\tilde{B}Y_0^C, \tilde{B}Z_0^C) + (\hat{\eta}_0(\tilde{B}Z_0))^V(\hat{g}(\tilde{B}\phi X_0, \tilde{B}Y_0))^C\\
&+(\hat{\eta}_0(\tilde{B}Z_0))^C(\hat{g}(\tilde{B}\phi X_0, \tilde{B}Y_0))^V + (\hat{\eta}_0(\tilde{B}Y_0))^V(\hat{g}(\tilde{B}\phi X_0, \tilde{B}Z_0))^C\\
&+(\hat{\eta}_0(\tilde{B}Y_0))^C(\hat{g}(\tilde{B}\phi X_0, \tilde{B}Z_0))^V.
\end{aligned}$$

On solving, we get

$$\begin{aligned}
\hat{g}^C(\bar{\nabla}^C_{\tilde{B}X_0^C}\tilde{B}Y_0^C, \tilde{B}Z_0^C) &= (\hat{\eta}_0(\tilde{B}Z_0))^V(\hat{g}(\tilde{B}\phi X_0, \tilde{B}Y_0))^C\\
&+ (\hat{\eta}_0(\tilde{B}Z_0))^C(\hat{g}(\tilde{B}\phi X_0, \tilde{B}Y_0))^V\\
&+ (\hat{\eta}_0(\tilde{B}Y_0))^V(\hat{g}(\tilde{B}\phi X_0, \tilde{B}Z_0))^C\\
&+ (\hat{\eta}_0(\tilde{B}Y_0))^C(\hat{g}(\tilde{B}\phi X_0, \tilde{B}Z_0))^V.
\end{aligned} \tag{3.4}$$

Hence, we state the following:

Theorem 3.4. *Let $\bar{\nabla}$ be a QSSC with respect to $\hat{\nabla}$ in $(M, \hat{g})$ that fulfills equations (2.4) and (2.5). Then the QSSC $\bar{\nabla}^C$ on a Sasakian manifold with respect to $\hat{\nabla}$ in $(TM, \hat{g}^C)$ is represented by (3.4).*

Now, applying the complete lifts on (2.4) and with the help of (2.6), we infer

$$\begin{aligned}
(\bar{\nabla}_{BX}BY)^{\bar{C}} =&(\hat{\nabla}_{BX}BY)^{\bar{C}} - (\hat{\eta}_0(BX)(B\phi Y_0))^{\bar{C}} + (\hat{g}(B\phi X_0, BY)\hat{\xi})^{\bar{C}},\\
(\bar{\nabla}_{BX}BY)^{\bar{C}} =&(\hat{\nabla}_{BX}BY)^{\bar{C}} - (\hat{\eta}_0(BX)^{\bar{C}}(B\phi Y_0)^{\bar{V}} + \hat{\eta}_0(BX)^{\bar{V}}(B\phi Y_0)^{\bar{C}}) + (\hat{g}(B\phi X_0, BY))^{\bar{C}}\hat{\xi})^{\bar{V}} + (\hat{g}(B\phi X_0, BY))^{\bar{V}}\hat{\xi})^{\bar{C}},\\
\bar{\nabla}^C_{\tilde{B}X_0^C}\tilde{B}Y_0^C =&\hat{\nabla}^C_{\tilde{B}X_0^C}\tilde{B}Y_0^C - \hat{\eta_0}^C(\tilde{B}X_0^C)(\tilde{B}\phi Y_0^V) + \hat{g}^C(\tilde{B}(\phi X_0)^C, \tilde{B}Y_0^C))\hat{\xi})^{\bar{V}} + \hat{g}^C(\tilde{B}(\phi X_0)^V, \tilde{B}Y_0^C))\hat{\xi})^{\bar{C}},
\end{aligned}$$

for arbitrary vector fields X_0 and Y_0 in S. Hence, from the equation (2.7) and the equation (3.1), we get

$$\begin{aligned}
(B(\mathring{\nabla}_X Y) + m(X_0, Y_0)N)^{\bar{C}} =&(B(\nabla_X Y) + h(X_0, Y_0)N)^C - \hat{\eta_0}^C(\tilde{B}X_0^C)(\tilde{B}(\phi Y_0)^V) - \hat{\eta_0}^V(\tilde{B}X_0^C)(\tilde{B}(\phi Y_0)^C)\\
&+\hat{g}^C(\tilde{B}(\phi X_0)^C, \tilde{B}Y_0^C))(\tilde{B}\hat{\xi})^V + \lambda^V N^{\bar{V}}) + \hat{g}^C(\tilde{B}(\phi X_0)^V, \tilde{B}Y_0^C))(\tilde{B}\hat{\xi})^C + \lambda^C N^{\bar{C}} + \lambda^C N^V),
\end{aligned}$$

$$\tilde{B}(\mathring{\nabla}_X Y)^C = \tilde{B}(\nabla_X Y)^C - \hat{\eta}_0{}^C(\tilde{B}X_0^C)(\tilde{B}(\phi Y_0)^V) - \hat{\eta}_0{}^V(\tilde{B}X_0^C)(\tilde{B}(\phi Y_0)^C) + \hat{g}^C(\tilde{B}(\phi X_0)^C, \tilde{B}Y_0^C))(\tilde{B}\hat{\xi})^V + \hat{g}^C(\tilde{B}(\phi X_0)^V, \tilde{B}Y_0^C))(\tilde{B}\hat{\xi})^C. \tag{3.5}$$

$$m^V(X_0^C, Y_0^C)N^{\bar{C}} + m^C(X_0^C, Y_0^C)N^{\bar{V}} = h^V(X_0^C, Y_0^C)N^{\bar{C}} + h^C(X_0^C, Y_0^C)N^{\bar{V}}) + \lambda^V \hat{g}^C(\tilde{B}(\phi X_0)^C, \tilde{B}Y_0^C)N^{\bar{V}}) + \lambda^c \hat{g}^C(\tilde{B}(\phi X_0)^C, \tilde{B}Y_0^C)N^{\bar{C}}) + \lambda^C \hat{g}^C(\tilde{B}(\phi X_0)^C, \tilde{B}Y_0^C)N^{\bar{V}}). \tag{3.6}$$

$$(\mathring{\nabla}_X Y)^C = (\nabla_X Y)^C - \eta_0^C(X_0^C)(\phi Y_0)^V - \eta_0^V(X_0^C)(\phi Y_0)^C + \tilde{g}((\phi X_0)^C, Y_0^C)\xi^V + \tilde{g}((\phi X_0)^V, Y_0^C)\xi^C, \mathring{\nabla}^C_{X_0^C} Y_0^C \tag{3.7}$$

$$= \nabla^C_{X_0^C} Y_0^C - \eta_0^C(X_0^C)(\phi Y_0)^V - \eta_0^V(X_0^C)(\phi Y_0)^C + \tilde{g}((\phi X_0)^C, Y_0^C)\xi^V + \tilde{g}((\phi X_0)^V, Y_0^C)\xi^C \tag{3.8}$$

We have

$$\mathring{\nabla}^C_{X_0^C} Y_0^C - \mathring{\nabla}^C_{Y_0^C} X_0^C - [X_0^C, Y_0^C] = \eta_0^C(Y_0^C)(\phi X_0)^V + \eta_0^V(Y_0^C)(\phi X_0)^C - \eta_0^C(X_0^C)(\phi Y_0)^V - \eta_0^V(X_0^C)(\phi Y_0)^C$$

Similarly,

$$\tilde{g}(\mathring{\nabla}^C_{X_0^C} Y_0^C, Z_0^C) = X_0^C(\tilde{g}(Y_0^C, Z_0^C)) + (\eta_0(Z_0))^V \tilde{g}((\phi X_0)^C, Y_0^C) + (\eta_0(Z_0))^C \tilde{g}((\phi X_0)^V, Y_0^C) + (\eta_0(Y_0))^V \tilde{g}((\phi X_0)^C, Z_0^C) + (\eta_0(Y_0))^C \tilde{g}((\phi X_0)^C, Z_0^V),$$

$$\mathring{\nabla}^C_{X_0^C}\tilde{g})(Y_0^C, Z_0^C) = (\eta_0(Z_0))^V \tilde{g}((\phi X_0)^C, Y_0^C) + (\eta_0(Z_0))^C \tilde{g}((\phi X_0)^V, Y_0^C) + (\eta_0(Y_0))^V \tilde{g}((\phi X_0)^C, Z_0^C) + (\eta_0(Y_0))^C \tilde{g}((\phi X_0)^V, Z_0^C). \tag{3.9}$$

Hence, we state the following:

Theorem 3.5. *Let $\mathring{\nabla}$ be a QSSC with respect to ∇ in (S, g). Then the QSSC $\mathring{\nabla}^C$ on a Sasakian manifold with respect to ∇^C in $(TS, \tilde{g})$ is represented by (3.9).*

The QSSC $\mathring{\nabla}^C$ on $(TS, \tilde{g})$ is defined as

$$\mathring{\nabla}^C_{X_0^C} Y_0^C = \nabla^C_{X_0^C} Y_0^C - \eta_0^C(X_0^C)(\phi Y_0)^V - \eta_0^V(X_0^C)(\phi Y_0)^C + \tilde{g}((\phi X_0)^C, Y_0^C)\xi^V + \tilde{g}((\phi X_0)^V, Y_0^C)\xi^C.$$

On applying the complete lifts of (3.1), we get

$$\bar{\nabla}^C_{\tilde{B}X_0^C} \tilde{B}Y_0^C = \tilde{B}(\mathring{\nabla}^C_{X_0^C} Y_0^C) + m^V(X_0^C, Y_0^C)N^{\bar{C}} + m^C(X_0^C, Y_0^C)N^{\bar{V}}.$$

From the equation (3.6), we acquire

$$m^V(X_0^C, Y_0^C) = h^V(X_0^C, Y_0^C) + \lambda^C \hat{g}^C(\tilde{B}(\phi X_0)^V, \tilde{B}Y_0^C)$$
$$m^C(X_0^C, Y_0^C) = h^V(X_0^C, Y_0^C) + \lambda^V \hat{g}^C(\tilde{B}(\phi X_0)^C, \tilde{B}Y_0^C) + \lambda^C \hat{g}^C(\tilde{B}(\phi X_0)^V, \tilde{B}Y_0^C)N^{\bar{V}}).$$

Thus, TS is totally umbilical iff

$$m^V(\tilde{X}_0, \tilde{Y}_0) = \delta \tilde{g}(\tilde{X}_0, \tilde{Y}_0),$$
$$m^C(\tilde{X}_0, \tilde{Y}_0) = \mu \tilde{g}(\tilde{X}_0, \tilde{Y}_0),$$

$\forall X_0, Y_0 \in \Im_0^1(S)$, where δ and μ are differentiable functions. If $\delta = \mu = 0$, then TS is totally geodesic.
Hence, we state the following:

Theorem 3.6. *TS is totally umbilical corresponding to the QSSC $\mathring{\nabla}^C$ on a Sasakian manifold iff it is totally umbilical or totally geodesic with respect to ∇^C.*

4. Conclusion

We introduced and studied a Sasakian manifold immersed with a QSSM connection to the tangent bundle and some fundamental results are obtained of it. Certain theorems on geometrical properties like totally umbilical, totally geodesic on a Sasakian manifold on the tangent bundle are proved.

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