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Generalized Ricci-Recurrent Weyl Manifolds

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ABSTRACT

This present paper is concerned with the study of the generalized Ricci-recurrent Weyl manifolds. First, we obtain a sufficient condition for the generalized Ricci-recurrent Weyl manifold admitting harmonic conformal curvature tensor to be a quasi-Einstein Weyl manifold. Also, we give an example of a generalized Ricci-recurrent Weyl manifold. Then, we prove that a generalized Ricci-recurrent Weyl manifold satisfying the Codazzi type of Ricci tensor is an Einstein Weyl manifold if and only if its scalar curvature is a prolonged covariant constant. Moreover, we prove that a generalized Ricci-recurrent Weyl manifold with a generalized concircularly symmetric tensor is an Einstein-Weyl manifold if and only if its scalar curvature is prolonged covariant constant.

Keywords: *Generalized Ricci-recurrent Weyl manifold, quasi-Einstein Weyl manifold, harmonic conformal curvature tensor, Einstein Weyl manifold, concircular curvature tensor.*

AMS Subject Classification (2020): *Primary: 53A30 ; Secondary: 53C25.*

1. Introduction

Weyl geometry was introduced by Hermann Weyl [24]. He generalized Riemannian geometry by introducing scale freedom of the underlying metric and formulated unified field theory in 1918. Weyl geometry was taken up explicitly in different research fields of theoretical physics during the second half of the 20th century. A Weyl manifold is a conformal manifold equipped with a torsion free connection preserving the conformal structure, called a Weyl connection.

Scalar curvature for an Einstein manifold of dimension $n > 2$ is constant. But, the scalar curvature need not to be constant for the Einstein-Weyl manifold. Pedersen and Tod [23] have shown that scalar curvature is an analytic function for a suitable choice of metric and local coordinates.

There are a number of remarkable studies of Einstein-Weyl structure under various conditions [7, 11, 12]. The Einstein–Weyl condition plays a key role in physics, the pure Einstein theory being too strong as a system model for various physical questions. Moreover, quasi-Einstein manifolds have been studied by valuable authors recently.

The idea of Ricci-recurrent manifold was introduced by Patterson [21]. Then, Ricci-recurrent manifolds have been studied by many authors [4, 15, 22]. In [6], De, Guha and Kamilya introduced and studied the generalized Ricci-recurrent Riemannian manifolds. In [14], authors have obtained some results for a generalized Ricci-recurrent Riemannian manifold to be a quasi-Einstein manifold. In [8], authors have studied quasi-Einstein Weyl manifolds. In [26], conformally symmetric generalized Ricci-recurrent manifolds have been studied, and proved that such a manifold is a quasi-Einstein manifold. In [13], authors have shown that a conformally flat generalized Ricci-recurrent pseudo-Riemannian manifold is an Einstein manifold.

In this work, we consider the generalized Ricci-recurrent Weyl manifolds. We first obtain a sufficient condition for a generalized Ricci-recurrent Weyl manifold with $\dot{\nabla}_h C^h_{ijk} = 0$, to be a quasi-Einstein Weyl manifold. We also construct a non-trivial example of generalized Ricci-recurrent Weyl manifold. Then, we prove that a generalized Ricci-recurrent Weyl manifold satisfying Codazzi type of Ricci tensor is an Einstein-Weyl manifold if and only if its scalar curvature is prolonged covariant constant. Finally, we prove that a

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generalized Ricci-recurrent Weyl manifold with a generalized concircularly symmetric tensor is an Einstein-Weyl manifold if and only if its scalar curvature is prolonged covariant constant.

2. Preliminaries

A differentiable manifold of dimension n having a conformal class $C[g]$ of metrics and a torsion-free connection ∇ preserving $C[g]$ is called a Weyl manifold [17, 27] which will be denoted by $W_n(g,T)$ where $g \in C[g]$ and T is a 1-form satisfying the compatibility condition

$$\nabla g = 2(g \otimes T). \tag{2.1}$$

Under the conformal re-scaling (renormalization)

$$\bar{g} = \lambda^2 g \quad (\lambda > 0) \tag{2.2}$$

of the metric tensor g, T is transformed by the rule

$$\bar{T} = T + d(\ln \lambda). \tag{2.3}$$

It is well-known that the pair $(\bar{g}, \bar{T})$ generates the same Weyl manifold. The process of passing from (g,T) to $(\bar{g}, \bar{T})$ is called a gauge transformation.

The curvature tensor, the covariant curvature tensor, the Ricci tensor, and the scalar curvature of $W_n(g,T)$ are, respectively, defined by [3, 23]

$$(\nabla_k \nabla_l - \nabla_l \nabla_k)\, v^p = v^j\, R^p_{jkl}\ , \tag{2.4}$$

$$R_{hjkl} = g_{ph}\, R^p_{jkl}\ , \tag{2.5}$$

$$R_{ij} = R^h_{ijh} = g^{kh}\, R_{kijh}\ , \tag{2.6}$$

$$R = g^{ij}\, R_{ij}. \tag{2.7}$$

From (2.4) it follows that

$$R^p_{jkl} = \partial_k \Gamma^p_{jl} - \partial_l \Gamma^p_{jk} + \Gamma^p_{hk}\Gamma^h_{jl} - \Gamma^p_{hl}\Gamma^h_{jk}\ \ ,\ \ \partial_k = \frac{\partial}{\partial x^k}$$

where Γ^h_{ik} are the coefficients of the Weyl connection ∇ given by

$$\Gamma^h_{ik} = \left\{ \begin{array}{c} h \\ ik \end{array} \right\} - (\,\delta^h_i\, T_k + \delta^h_k\, T_i - g^{hm}\, g_{ik}\, T_m\,), \tag{2.8}$$

in which $\left\{ \begin{smallmatrix} h \\ ik \end{smallmatrix} \right\}$ are the coefficients of the Levi-Civita connection.

By straightforward calculations it is easy to see that the antisymmetric part of R_{ij}, $R_{[ij]}$, has the property

$$R_{[ij]} = n \nabla_{[i} T_{j]}. \tag{2.9}$$

Definition 2.1. An object A defined on $W_n(g,T)$ is called a satellite of weight $\{p\}$ of the tensor g_{ij}, if it admits a transformation of the form

$$\bar{A} = \lambda^p A \tag{2.10}$$

under the renormalization (2.2) of the metric g [9, 27].

Definition 2.2. The prolonged covariant derivative of a satellite A of weight $\{p\}$ is defined by [9]

$$\dot{\nabla}_k\, A = \nabla_k\, A - p\, T_k\, A\,. \tag{2.11}$$

We note that the prolonged covariant derivative preserves the weight.

Writing (2.1) in local coordinates and expanding it, we find that

$$\dot{\nabla}_k g_{ij} = \partial_k g_{ij} - g_{hj}\, \Gamma^h_{ik} - g_{ih}\, \Gamma^h_{jk} - 2\, T_k\, g_{ij} = 0. \tag{2.12}$$

Definition 2.3. A Weyl manifold is said to be an Einstein-Weyl manifold [10, 23] if the symmetric part of the Ricci tensor is proportional to the metric tensor $g \in C[g]$, and hence we have

$$R_{(ij)} = \frac{R}{n} g_{ij}. \tag{2.13}$$

Definition 2.4. A Weyl manifold $W_n(g,T)$ is a quasi Einstein-Weyl manifold, if the symmetric part $R_{(ij)}$ of the Ricci tensor R_{ij} satisfies the condition [8]

$$R_{(ij)} = \alpha g_{(ij)} + \beta A_i A_j \tag{2.14}$$

where α and β are scalars of weight -2, A_i is a nonzero 1-form of weight 1, which is normalized by the condition

$$g^{ij} A_i A_j = 1.$$

A_i is called the associated 1-form, and α and β are called associated scalars.

Definition 2.5. A non-flat Weyl manifold $W_n(g,T)$ is called Ricci-recurrent if its Ricci tensor R_{ij} satisfies the condition

$$\dot{\nabla}_k R_{ij} = A_k R_{ij} \tag{2.15}$$

where A is non-zero 1-form.

Definition 2.6. $W_n(g,T)$ is called the generalized Ricci-recurrent Weyl manifold, if its Ricci tensor R_{ij} satisfies the condition

$$\dot{\nabla}_k R_{ij} = A_k R_{ij} + B_k g_{ij}, \tag{2.16}$$

where A and B are two non-zero 1-forms of weight 0 and -2, respectively [2]. If the associated 1-form B becomes zero, the manifold reduces to a Ricci-recurrent manifold.

It is easy to see that the antisymmetric part and symmetric part of R_{ij} of the generalized Ricci-recurrent Weyl manifold $W_n(g,T)$ are in the following forms, respectively,

$$\dot{\nabla}_k R_{[ij]} = A_k R_{[ij]}, \tag{2.17}$$

$$\dot{\nabla}_k R_{(ij)} = A_k R_{(ij)} + B_k g_{ij} \, . \tag{2.18}$$

The conformal curvature tensor C^h_{ijk} of $W_n(g,T)$ is given by [16]

$$\begin{aligned} C^h_{ijk} \; = \; & R^h_{ijk} + \frac{2}{n(n-2)} \Big(\delta^h_k R_{[ij]} - \delta^h_j R_{[ik]} - g_{ik} g^{hm} R_{[mj]} + g_{ij} g^{hm} R_{[mk]} \\ & -(n-2)\delta^h_i R_{[kj]} \Big) - \frac{1}{n-2} \Big(\delta^h_k R_{ij} - \delta^h_j R_{ik} - g_{ik} g^{hm} R_{mj} + g_{ij} g^{hm} R_{mk} \Big) \\ & + \frac{R}{(n-1)(n-2)} \left(g_{ij} \delta^h_k - g_{ik} \delta^h_j \right) , \end{aligned} \tag{2.19}$$

and the tensor L_{ij} as

$$L_{ij} = -\frac{R_{ij}}{n-2} + \frac{2}{n(n-2)} R_{[ij]} + \frac{R \, g_{ij}}{2(n-1)(n-2)} \, . \tag{2.20}$$

3. Generalized Ricci-Recurrent Weyl Manifolds Admitting Harmonic Conformal Curvature Tensor

In [14], authors proved that a conformally flat generalized Ricci-recurrent Riemannian manifold is a quasi-Einstein manifold, provided the basic vector fields are co-directional.

In this section, we consider a generalized Ricci-recurrent Weyl manifold admitting harmonic conformal curvature tensor, i.e., $\dot{\nabla}_h C^h_{ijk} = 0$. We examine the condition for a generalized Ricci-recurrent Weyl manifold with harmonic conformal curvature tensor to be a quasi-Einstein Weyl manifold.

In [5], Çivi and Arsan obtained the following relations for a Weyl manifold:

By substituting the tensor L_{ij} with weight $\{0\}$ into (2.19) for a Weyl manifold $W_n(g,T)$ $(n>3)$, the conformal curvature tensor is given by

$$C^h_{ijk} = R^h_{ijk} + \delta^h_k\, L_{ij} - \delta^h_j\, L_{ik} + L^h_k\, g_{ij} - L^h_j\, g_{ik} - 2\delta^h_i\, L_{[jk]}\ , \tag{3.1}$$

where

$$L^h_k = g^{lh} L_{lk},$$

and

$$L_{[jk]} = \frac{1}{n} R_{[kj]} = \nabla_{[k} T_{j]}. \tag{3.2}$$

Also, in [5], authors obtained the following expression for the conformal curvature tensor of $W_n(g,T)$ $(n>3)$.

$$\dot{\nabla}_h C^h_{ijk} = (n-3)(\dot{\nabla}_j L_{ik} - \dot{\nabla}_k L_{ij}), \tag{3.3}$$

$$\dot{\nabla}_k L_{[ji]} + \dot{\nabla}_j L_{[ik]} + \dot{\nabla}_i L_{[kj]} = 0, \tag{3.4}$$

and

$$\begin{aligned}\dot{\nabla}_j L_{ik} - \dot{\nabla}_k L_{ij} \;=\;& \dot{\nabla}_j \Big(\frac{-R_{ik}}{n-2} + \frac{2}{n(n-2)} R_{[ik]} + \frac{R g_{ik}}{2(n-1)(n-2)} \Big) \\ & - \dot{\nabla}_k \Big(\frac{-R_{ij}}{n-2} + \frac{2}{n(n-2)} R_{[ij]} + \frac{R g_{ij}}{2(n-1)(n-2)} \Big).\end{aligned} \tag{3.5}$$

So, by using the above relations, we can state the following theorem.

Theorem 3.1. *A generalized Ricci-recurrent Weyl manifold of dimension >3 admitting harmonic conformal curvature tensor is a quasi-Einstein Weyl manifold if the condition*

$$g^{jp} A_p \dot{\nabla}_i (\nabla_{[k} T_{j]}) + g^{jp} A_k \dot{\nabla}_p (\nabla_{[i} T_{j]}) = 0$$

holds true.

Proof. Assume that the generalized Ricci-recurrent Weyl manifold $W_n(g,T)$ $(n>3)$ has the harmonic conformal curvature tensor. Then, in view of (3.3), it can be easily obtained that

$$\dot{\nabla}_j L_{ik} - \dot{\nabla}_k L_{ij} = 0,$$

and from (3.5), it follows that

$$\dot{\nabla}_k R_{ij} - \dot{\nabla}_j R_{ik} - \frac{2}{n}(\dot{\nabla}_k R_{[ij]} - \dot{\nabla}_j R_{[ik]}) + \frac{1}{2(n-1)} \Big[g_{ik}(\dot{\nabla}_j R) - g_{ij}(\dot{\nabla}_k R) \Big] = 0.$$

Hence considering the antisymmetric part and symmetric part of R_{ij}, and using (2.9), it is obtained

$$\begin{aligned}\dot{\nabla}_k R_{(ij)} - \dot{\nabla}_j R_{(ik)} \;=\;& -(n-2)\dot{\nabla}_k(\nabla_{[i} T_{j]}) + (n-2)\dot{\nabla}_j(\nabla_{[i} T_{k]}) \\ & - \frac{1}{2(n-1)} \Big[g_{ik}(\dot{\nabla}_j R) - g_{ij}(\dot{\nabla}_k R) \Big].\end{aligned} \tag{3.6}$$

Suppose now that $W_n(g,T)$ $(n>3)$ is a generalized Ricci recurrent Weyl manifold. Then, (2.16) must hold. Multiplying (2.16) by g^{ij} and summing up, and using the fact that $g^{ij}R_{ij}=R$, we obtain

$$\dot{\nabla}_k R = A_k R + nB_k. \tag{3.7}$$

Replacing (2.16) and (3.7) in (3.6), we obtain

$$\begin{aligned} A_k R_{(ij)} + B_k g_{ij} - A_j R_{(ik)} - B_j g_{ik} &= -(n-2)\dot{\nabla}_k(\nabla_{[i}T_{j]}) + (n-2)\dot{\nabla}_j(\nabla_{[i}T_{k]}) \\ &\quad - \frac{1}{2(n-1)}\Big[g_{ik}(A_j R + nB_j) - g_{ij}(A_k R + nB_k)\Big]. \end{aligned} \tag{3.8}$$

Multiplying (3.8) by g^{ij} and using the fact that $g^{ij}(\nabla_{[i}T_{j]})=0$, we get

$$g^{ji}A_j R_{(ik)} = \frac{1}{2}(A_k R - nB_k) + (n-1)B_k - (n-2)g^{ij}\dot{\nabla}_j(\nabla_{[i}T_{k]}). \tag{3.9}$$

Multiplying (3.8) by $g^{jp}A_p$ and using $A_pA^p = |A_p|^2 = \mu$, we obtain

$$\begin{aligned} A_k g^{jp}A_p R_{(ij)} &+ A_i B_k - \mu R_{(ik)} - g^{jp}A_p B_j g_{ik} \\ &= (2-n)\Big[g^{jp}A_p\dot{\nabla}_k(\nabla_{[i}T_{j]}) - g^{jp}A_p\dot{\nabla}_j(\nabla_{[i}T_{k]})\Big] \\ &- \frac{1}{2(n-1)}\Big[\mu g_{ik}R + ng^{jp}A_pB_jg_{ik} - A_iA_kR - nA_iB_k\Big]. \end{aligned} \tag{3.10}$$

Inserting (3.9) into (3.10) and making the necessary arrangements, we obtain

$$\begin{aligned} R_{(ik)} &= \Big[\frac{1}{2\mu}A_iR + \Big(\frac{n-2}{2\mu}\Big)B_i + \frac{(n-2)}{\mu}g^{jp}\dot{\nabla}_p(\nabla_{[i}T_{j]})\Big]A_k + \frac{1}{\mu}A_iB_k \\ &- \frac{1}{\mu}g^{jp}A_pB_jg_{ik} + \frac{(n-2)}{\mu}\Big[g^{jp}A_p\dot{\nabla}_k(\nabla_{[i}T_{j]}) - g^{jp}A_p\dot{\nabla}_j(\nabla_{[i}T_{k]})\Big] \\ &+ \frac{1}{2\mu(n-1)}\Big[\mu g_{ik}R + ng^{jp}A_pB_jg_{ik} - A_iA_kR - nA_iB_k\Big]. \end{aligned} \tag{3.11}$$

By interchanging i and k in equation (3.11) and subtracting these two equations, we get a relation between two basic vector fields A_k and B_k as $B_k = \lambda A_k$, where λ is a non-zero scalar. Then from (3.11), we get

$$\begin{aligned} R_{(ik)} &= \Big[\frac{3n\lambda - 2\lambda + R}{2(n-1)}\Big]g_{ik} + \Big[\frac{(n^2-2n)\lambda + (n-2)R}{2(n-1)\mu}\Big]A_iA_k \\ &+ \frac{(n-2)}{\mu}\Big[g^{jp}A_p\dot{\nabla}_k(\nabla_{[i}T_{j]}) + g^{jp}A_k\dot{\nabla}_p(\nabla_{[i}T_{j]}) - g^{jp}A_p\dot{\nabla}_j(\nabla_{[i}T_{k]})\Big]. \end{aligned} \tag{3.12}$$

Using (3.2) and (3.4) in (3.12), we obtain

$$R_{(ik)} = \Big[\frac{3n\lambda - 2\lambda + R}{2(n-1)}\Big]g_{ik} + \Big[\frac{(n^2-2n)\lambda + (n-2)R}{2(n-1)\mu}\Big]A_iA_k + \frac{(n-2)}{\mu}\varphi_{ik}, \tag{3.13}$$

where

$$\varphi_{ik} = g^{jp}A_p\dot{\nabla}_i(\nabla_{[k}T_{j]}) + g^{jp}A_k\dot{\nabla}_p(\nabla_{[i}T_{j]}). \tag{3.14}$$

If $\varphi_{ik}=0$ for a generalized Ricci-recurrent Weyl manifold $W_n(g,T)$ $(n>3)$, from (3.13), we get

$$R_{(ik)} = \Big[\frac{3n\lambda - 2\lambda + R}{2(n-1)}\Big]g_{ik} + \Big[\frac{(n^2-2n)\lambda + (n-2)R}{2(n-1)\mu}\Big]A_iA_k.$$

So, it shows us that $W_n(g,T)$ $(n>3)$ is a quasi-Einstein Weyl manifold under the following condition

$$g^{jp}A_p\dot{\nabla}_i(\nabla_{[k}T_{j]}) + g^{jp}A_k\dot{\nabla}_p(\nabla_{[i}T_{j]}) = 0.$$

□

4. An Example of a Generalized Ricci-Recurrent Weyl Manifold

We consider a four-dimensional manifold $W_4(g,T)$ with a metric by

$$ds^2 = g_{ij}dx^i dx^j = e^{2x^1}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2]$$

and a 1-form $T = e^{x^1}dx^1$. Then, the only non-vanishing Weyl connection coefficients are

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \Gamma^4_{14} = \Gamma^4_{41} = 1 - e^{x^1}, \;\; \Gamma^1_{22} = \Gamma^1_{33} = \Gamma^1_{44} = -1 + e^{x^1}.$$

It is easy to see that $W_4(g,T)$ is a Weyl manifold with the Weyl connection satisfying the condition $\dot{\nabla}_k g_{ij} = 2\,T_k\,g_{ij}$. Hence, a straightforward calculation leads to the following expression for the nonzero components of the Ricci tensor:

$$R_{11} = -3e^{x^1}, \; R_{22} = R_{33} = R_{44} = 2 - 5e^{x^1} + 2e^{2x^1}.$$

The scalar curvature R and the components of the symmetric parts of the Ricci tensor and their prolonged covariant derivatives are:

$$R = 6(1 + e^{-2x^1} - 3e^{-x^1}),$$

$$R_{(11)} = -3e^{x^1}, \; R_{(22)} = R_{(33)} = R_{(44)} = 2 - 5e^{x^1} + 2e^{2x^1},$$

$$\dot{\nabla}_1 R_{(11)} = 3e^{x^1} - 6e^{2x^1}, \;\; \dot{\nabla}_1 R_{(22)} = \dot{\nabla}_1 R_{(33)} = \dot{\nabla}_1 R_{(44)} = -4 + 9e^{x^1} - 10e^{2x^1} + 4e^{3x^1},$$
$$\dot{\nabla}_2 R_{(12)} = \dot{\nabla}_3 R_{(13)} = \dot{\nabla}_4 R_{(14)} = 2(-1 + 2e^{x^1} - 2e^{2x^1} + e^{3x^1}).$$

We shall verify that our $W_4(g,T)$ is a generalized Ricci-recurrent Weyl manifold. To verify that the manifold $W_4(g,T)$ is a generalized Ricci-recurrent Weyl manifold, let us choose the 1-forms as follows:

$$A_i = \begin{cases} \dfrac{-2 + 3e^{x^1} - 2e^{2x^1} + 2e^{3x^1}}{1 - e^{x^1} + e^{2x^1}} & , \; i = 1 \\ 0 & , \; \text{otherwise} \end{cases}$$

$$B_i = \begin{cases} \dfrac{3e^{x^1}(-1 + e^{2x^1})}{1 - e^{x^1} + e^{2x^1}} & , \; i = 1 \\ 0 & , \; \text{otherwise} \end{cases}$$

at any point in $W_4(g,T)$. In our $W_4(g,T)$, it is seen that the following equations are satisfied:

$$\dot{\nabla}_1 R_{(11)} = A_1 R_{(11)} + B_1 g_{11},$$
$$\dot{\nabla}_1 R_{(22)} = A_1 R_{(22)} + B_1 g_{22},$$
$$\dot{\nabla}_1 R_{(33)} = A_1 R_{(33)} + B_1 g_{33},$$
$$\dot{\nabla}_1 R_{(44)} = A_1 R_{(44)} + B_1 g_{44}.$$

Thus, $W_4(g,T)$ is a generalized Ricci-recurrent Weyl manifold which is neither Ricci symmetric nor Ricci recurrent.

5. Generalized Ricci-Recurrent Weyl Manifolds with Codazzi Type Ricci Tensor

In this section, we consider generalized Ricci-recurrent Weyl manifolds satisfying the Codazzi type of Ricci tensor and prove the following theorem.

Theorem 5.1. *A necessary and sufficient condition for a generalized Ricci-recurrent Weyl manifold having Codazzi type of Ricci tensor to be an Einstein-Weyl manifold is that its scalar curvature be a prolonged covariant constant.*

Proof. Let us suppose that a Weyl manifold is under consideration, having the Codazzi type of Ricci tensor. Then, we have

$$\dot{\nabla}_k R_{(ij)} = \dot{\nabla}_j R_{(ik)}. \tag{5.1}$$

Multiplying (5.1) by g^{ij}, we get

$$\dot{\nabla}_k R = \dot{\nabla}_j R^j_k. \tag{5.2}$$

In [19], it is proved that a Weyl manifold satisfies the identity

$$\dot{\nabla}_j R^j_k = \frac{1}{2}\dot{\nabla}_k R + 2g^{ij}\dot{\nabla}_j(\nabla_{[i}T_{k]}). \tag{5.3}$$

Hence, from (5.2) and (5.3), for a Weyl manifold having the Codazzi type of Ricci tensor, we obtain

$$\dot{\nabla}_k R = 4g^{ij}\dot{\nabla}_j(\nabla_{[i}T_{k]}). \tag{5.4}$$

Consider now that generalized Ricci-recurrent Weyl manifold $W_n(g,T)$ with Codazzi type of Ricci tensor. Then, from (2.18) and (5.1), we have

$$A_k R_{(ij)} + B_k g_{ij} - A_j R_{(ik)} - B_j g_{ik} = 0. \tag{5.5}$$

Transvecting (5.5) by g^{ik}, we find

$$A_k g^{ik} R_{(ij)} = A_j R + (n-1)B_j. \tag{5.6}$$

On the other hand, multiplying (5.5) by $g^{pj}A_p$, we get

$$A_k g^{pj} A_p R_{(ij)} + g^{pj}A_p B_k g_{ij} - g^{pj}A_p A_j R_{(ik)} \; - \; g^{pj}A_p B_j g_{ik} = 0. \tag{5.7}$$

From (5.6) and (5.7), we obtain

$$\mu R_{(ik)} = A_k\Big[A_i R + (n-1)B_i\Big] + A_i B_k - g^{pj}A_p B_j g_{ik} \tag{5.8}$$

where

$$g^{pj}A_p A_j = \mu.$$

Putting (5.4) into (3.7), we obtain

$$B_k = \frac{4}{n}g^{ij}\dot{\nabla}_j(\nabla_{[i}T_{k]}) - \frac{1}{n}A_k R. \tag{5.9}$$

From (5.8) and (5.9), we get

$$\begin{aligned}\mu R_{(ik)} \; &= \; A_i A_k R + (n-1)A_k\Big(\frac{4}{n}g^{mj}\dot{\nabla}_j(\nabla_{[m}T_{i]}) - \frac{1}{n}A_i R\Big) \\ &+ \; A_i\Big(\frac{4}{n}g^{mj}\dot{\nabla}_j(\nabla_{[m}T_{k]}) - \frac{1}{n}A_k R\Big) - g^{pj}A_p\Big(\frac{4}{n}g^{lm}\dot{\nabla}_m(\nabla_{[l}T_{j]}) - \frac{1}{n}A_j R\Big)g_{ik}.\end{aligned} \tag{5.10}$$

In view of (5.4) and (5.10), we obtain

$$R_{(ik)} = \Big(\frac{R}{n} - \frac{g^{jp}A_p}{n\mu}\dot{\nabla}_j R\Big)g_{ik} + \frac{n-1}{n\mu}A_k\dot{\nabla}_i R + \frac{1}{n\mu}A_i\dot{\nabla}_k R. \tag{5.11}$$

If R is a prolonged covariant constant, i.e., if

$$\dot{\nabla}_k R = \nabla_k R + 2RT_k = 0 \tag{5.12}$$

then, from (5.11), we obtain

$$R_{(ik)} = \Big(\frac{R}{n} - \frac{g^{jp}A_p}{n\mu}\dot{\nabla}_j R\Big) g_{ik} \tag{5.13}$$

which shows us that the generalized Ricci-recurrent Weyl manifold $W_n(g,T)$ with Codazzi type of Ricci tensor is the Einstein-Weyl manifold.

Conversely, assume that the generalized Ricci-recurrent Weyl manifold $W_n(g,T)$ with Codazzi type of Ricci tensor is the Einstein-Weyl manifold. From (2.13) and (5.1), we obtain

$$\big(\dot{\nabla}_k R\big)g_{ij} - \big(\dot{\nabla}_j R\big)g_{ik} = 0. \tag{5.14}$$

Multiplying (5.14) by g^{ij} and using $g^{ij}g_{ik} = \delta^j_k$, we find

$$(n-1)\dot{\nabla}_k R = 0.$$

For $n > 1$, we have

$$\dot{\nabla}_k R = 0$$

which shows that the scalar curvature R of $W_n(g,T)$ is a prolonged covariant constant. □

6. Generalized Ricci-Recurrent Weyl Manifolds with Generalized Concircularly Symmetric Tensor

In general, a geodesic circle does not transform into a geodesic circle by the conformal transformation of the fundamental tensor g_{ij}. The transformation which preserves geodesic circles was first introduced by Yano [25]. In [18], as a generalization of a geodesic circle in a Riemannian manifold, by using prolonged covariant differentiation authors proved the following statements.

A conformal mapping of a Weyl manifold upon another Weyl manifold is called generalized concircular mapping if it preserves the generalized circles [18].

A $(1,3)$ type tensor Z which remains invariant under generalized concircular mapping of $W_n(g,T)$ is given by [18]

$$Z^p_{ijl} = R^p_{ijl} - \frac{R}{n(n-1)}(\delta^p_l g_{ij} - \delta^p_j g_{il}). \tag{6.1}$$

where R^p_{ijl} is the Weyl curvature tensor and R, the scalar curvature.

In [1], authors obtained some results about generalized concircularly flat Weyl manifolds.

So, by using the above expressions, we can state the following theorem.

Theorem 6.1. *A generalized Ricci-recurrent Weyl manifold $W_n(g,T)$ with a generalized concircularly symmetric tensor is an Einstein-Weyl manifold if and only if its scalar curvature is prolonged covariant constant.*

Proof. Let the generalized Ricci-recurrent Weyl manifold $W_n(g,T)$ be generalized concircularly symmetric manifold. Then, according to (6.1), we have

$$\dot{\nabla}_k Z^p_{ijl} = \dot{\nabla}_k R^p_{ijl} - \frac{\dot{\nabla}_k R}{n(n-1)}(\delta^p_l g_{ij} - \delta^p_j g_{il}) = 0. \tag{6.2}$$

Contraction on the indices p and l gives

$$\dot{\nabla}_k Z_{ij} = \dot{\nabla}_k R_{ij} - \frac{\dot{\nabla}_k R}{n} g_{ij} = 0 \tag{6.3}$$

or, equivalently,

$$\dot{\nabla}_k R_{(ij)} + \dot{\nabla}_k R_{[ij]} - \frac{\dot{\nabla}_k R}{n} g_{ij} = 0 \tag{6.4}$$

Since $W_n(g,T)$ is generalized Ricci-recurrent, using (2.17) and (2.18), we get

$$A_k R_{(ij)} + B_k g_{ij} + A_k R_{[ij]} - \frac{A_k R + nB_k}{n} g_{ij} = 0. \tag{6.5}$$

In view of (2.9) and (6.5), we obtain

$$R_{(ij)} - \frac{R}{n} g_{ij} = -R_{[ij]} = -n\nabla_{[i}T_{j]}. \tag{6.6}$$

Then, (6.6) reduces to $R_{(ij)} - \frac{R}{n} g_{ij} = 0$ and $\nabla_{[i}T_{j]} = 0$. In this case, $W_n(g,T)$ is an Einstein-Weyl manifold and T is locally a gradient i.e., $T_{j,i} - T_{i,j} = 0$. Therefore, $W_n(g,T)$ is generalized concircular to an Einstein manifold.

On the other hand, in [20], Özdeger proved that an Einstein-Weyl manifold for $(n > 2)$ satisfies the equation

$$\frac{1}{n}(\dot{\nabla}_j R) - 2g^{ki}\dot{\nabla}_k(\nabla_{[i}T_{j]}) = 0. \tag{6.7}$$

Since T is locally a gradient, from (6.7), we obtain $\dot{\nabla}_j R = \nabla_j R + 2RT_j = 0$ which means that R is prolonged covariant constant. □

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Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] Arsan, G. G., Yildirim G. C.: *Generalized circles and their conformal mapping in a subspace of a Weyl space*. Acta Mathematica Scientia. **25** (2), 331-339 (2005).
[2] Canfes, E. Ö.: *On generalizedrecurrent Weyl spaces and Wong's conjecture*, Differential Geometry and Dynamical Systems. **8**, 34–42 (2006).
[3] Canfes, E. Ö, Özdeğer, A.: *Some applications of prolonged covariant differentiation in Weyl spaces*, J.Geom. **60**, 7–16 (1997).
[4] Chaki, M. C.: *Some theorems on recurrent and Ricci-recurrent spaces*, Rendiconti del Seminario Matematico della Università di Padova. **26**, 168–176 (1956).
[5] Çivi, G., Arsan, G.G.: *On Weyl manifolds with harmonic conformal curvature tensor*, Analele Stiintifice ale Universitatii Al I Cuza din Iasi - Matematica 1 (F2), 327–335 (2016).
[6] De, U. C., Guha, N., Kamilya, G.: *On generalized Ricci-recurrent manifolds*, Tensor N.S., **56**, 312–317 (1995).
[7] Ghosh, A.: *Einstein-Weyl Structure and Contact Geometry*. Results Math 77, 58 (2022).
[8] Gül I., Canfes, E. Ö.: *On quasi-Einstein Weyl manifolds*, Int. J. Geom. Methods Mod. Phys. **14** (9), 1750–1822 (2017).
[9] Hlavaty, V.: *Theorie d'immersion d'une W_m dans W_n*, Ann. Soc. Polon. Math. **21**, 196–206 (1949).
[10] Hitchin, N. J.: Complex Manifolds and Einstein Equations, in Twistor Geometry and Non-linear Systems (Primorsko, 1980), Lecture Notes in Math. 970, Springer, Berlin - New York, 73—99 (1982).
[11] Honda, N., Nakata, F.: *Minitwistor spaces, Severi varieties, and Einstein–Weyl structure*. Ann Glob Anal Geom. **39**, 293–323 (2011).
[12] Jones, P. E., Tod, K. P.: *Minitwistor spaces and Einstein-Weyl spaces*, Class. Quantum Grav. 2, 4, 565-577 (1985).
[13] Loo, T. H., De, A.: *A classification of conformally flat generalized Ricci recurrent pseudo-Riemannian manifolds*, Int. J. Geom. Methods Mod. Phys. **19**(2), 2250023 (2022).
[14] Mallick, S., De, A., De, U. C.: *On generalized Ricci recurrent manifolds with applications to relativity*, Proceedings of the National Academy of Sciences, India Section A: Physical Sciences, Springer, 143–152 (2013).
[15] Matsumoto, M.: *On Riemannian spaces with recurrent projective curvature*, Tensor N. S. 11–18 (1968).
[16] Miron, R.: *Mouvements confermes dans les espaces W_n et N_n*, Tensor N.S. **19**, 33–41 (1968).
[17] Norden, A.: Affinely Connected Spaces. Moscow, Nauka, (in Russian) (1976).
[18] Özdeger, A., Şentürk Z.: *Generalized cirles in a Weyl space and their conformal mapping*, Publ. Math. Debrecen **60**(1-2), 75–87 (2002).
[19] Özdeger, A.: *Conformal and generalized concircular mappings of Einstein-Weyl manifolds*. Acta Math. Scientia Ser. B Engl. Ed. **30** (5), 1739–1745 (2010).

[20] Özdeger, A.: *Generalized Einstein Tensor for a Weyl Manifold and Its Applications*, Acta Math. Sinica, English Series Feb., **29** (2), 373—382 (2013).
[21] Patterson, E. M.: *Some Theorems on Ricci-Recurrent Spaces*, Journal of the London Mathematical Society **2**7 (3), 287–295 (1952).
[22] Prakash, N.: *A note on Ricci-recurrent and recurrent spaces*, Bull. Cal Math. Society 1–7 (1962).
[23] Pedersen, H., Tod, K. P.: *Three-dimensional Einstein-Weyl Geometry*, Adv. Math. **97**, 74–109 (1993).
[24] Weyl, H.: Gravitation und Elektrizltdt, S.-B. Preuss. Akad. Wiss. Berlin, p. 465 (1918) (Translated in the principle of relativity, Dover Books, New York).
[25] Yano, K.: *Concircular geometry I. Concircular transformations*. Proc. Imp. Acad. Tokyo **16**, 195—200 (1940).
[26] Yılmaz, H. B.: *On conformally symmetric generalized Ricci-recurrent manifolds with applications in general relativity*, Bulletin of Mathematical Analysis and Applications **13**, 39–50 (2021).
[27] Zlatanov, G., Tsareva, B.: *On the geometry of nets in the n-dimensional space of Weyl*, J. Geom. **38** (1/2), 182–197 (1990).

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