Journal of Computer and Education Research
October 2023 Volume 11 Issue 22
http://dergipark.org.tr/jcer
(ISSN:2148-2896)

Research Article

Didactic Praxeologies Employed by Mathematics Teachers in Teaching the Inverse Function

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Article Info

Received: *16 September 2023*
Accepted: *23 October 2023*

Keywords: Anthropological theory of didactics, praxeology, teachers, inverse function, curriculum

10.18009/jcer.1361502

Publication Language: English

Abstract

This study investigates the praxeologies teachers use about the inverse function in the teaching process when the curriculum is changed. A case study, one of the qualitative research methods, was used in the study. The participants of the study were three experienced mathematics teachers. The data were collected by recording the teaching process of the teachers with a video camera and a voice recorder. The praxeological analysis method of the Anthropological Theory of Didactics (ATD) was used in the data analysis. The findings of the study show that teachers use two different praxeologies in the inverse function. The first one is praxeology based on informal mapping with the effect of the dominant definition of the concept of function in the curriculum, and this praxeology was used to introduce the concept of inverse function. The other praxeology, which shows the monoid structure more clearly, emerged due to both a necessity and the necessity to exhibit an approach appropriate to the curriculum in more complex tasks and was shaped as a mixed praxeology. It was determined that teachers did not structure both praxeologies well and made sudden transitions from one praxeology to another.

To cite this article: Gök, M. & Erdoğan, A. (2023). Didactic praxeologies employed by mathematics teachers in teaching the inverse function. *Journal of Computer and Education Research*, *11* (22), 1089-1112. https://doi.org/10.18009/jcer.1361502

Introduction

The various definitions of mathematical concepts contain clues about how related concepts can be taught in instructional settings. One of the most illustrative examples of this is observed within the concept of function. Numerous definitions have been presented throughout the historical development of the concept of function. Among these definitions, the Eulerian, Dirichlet, and Bourbaki definitions stand out (Cha, 1999). In different periods of curricula, one or more of these definitions come to the forefront, and this approach directly impacts the teaching of the concept of function as well as associated concepts like inverse functions and composition of functions (Gök, Erdoğan, Özdemir Erdoğan, 2019). The diverse definitions of the concept of function provide avenues for employing various methods in teaching their related sub-dimensions, ultimately bringing flexibility and richness to instructional situations. However, this flexibility and richness necessitate the establishment of a coherent organization for teaching concepts related to the function definition featured in the curriculum. Research by Erdogan (2014) on functions demonstrates explicitly that achieving the desired flexibility and richness goals of the curriculum may prove difficult, even with the teacher's experience and effort, if a consistent organization is not provided. Additionally, neglecting the connections between university- and high school-level mathematics (Zazkis & Leiken, 2010) would further hinder the creation of a coherent organization in teaching practices. In this context, a coherent organization entails the meaningful integration of the concept of function with both other topics and its internal sub-dimensions. To this end, this study concentrates on the relationship between the dimensions inherent in the function itself and the concept of inverse functions.

The concepts of composition of functions and inverse function are directly related to the concept of function itself. Regardless of the adopted definition for the function, these two concepts hold critical importance in comprehending, utilizing, and establishing relationships with other concepts within the realm of the function (Even, 1990; Wasserman, 2017; Ikram, Purwanto, Parta & Susanto, 2020; Weber, Mejía-Ramos, Fukawa-Connelly & Wasserman, 2020). For instance, Weber et al. (2020) propose two distinct approaches to teaching the concept of function and its inverse. The first approach reflects the Bourbaki definition, which emphasizes that a function is a particular relation between two sets; the second approach defines it as ordered pairs that satisfy the univalence criterion, regardless of the set. In the first approach, if $f=(F, A, B)$ is a function, then its inverse function (if it exists) is denoted as $f^{-1}=(F^{-1}, B, A)$. Here, F^{-1} is the inverse of the F relation. In the second approach, if (x,y_1) and (x, y_2) are in the function f, then $y_1=y_2$. The inverse relation denoted as f^{-1} is an inverse function for f if and only if f^{-1} is a function (Weber et al., 2020).

Exploring how the concepts of composition of functions and inverse functions are addressed in instructional settings, as well as how they are associated with specific function definitions, is believed to offer a new perspective on research concerning the topic of functions. This study focuses on the approaches teachers adopt in teaching the concept of inverse functions within a specific curriculum framework (the Ministry of National Education [MoNE], 2013). It delves into the analysis of teachers' strategies for teaching the concept of inverse functions and examines the mathematical tasks related to inverse functions that teachers present to their students.

The Concept of Function in Curricula in Turkey

In Turkey, while many concepts related to the concept of function were covered in the 9th grade under the 2005 curriculum (such as the concept of function, types of functions, and operations with functions), the 2013 curriculum change resulted in the fragmentation of these concepts to be taught in the 9th and 10th grades. Subsequently, with the 2018 revision, they were updated to be taught in the 10th and 11th grades (MoNE, 2005, 2013, 2018). As highlighted earlier, these changes can be interpreted as a quest for consistent and successful organization for teaching the concept of function. Among these changes, the most critical one, in terms of program consistency, was the 2013 curriculum change, where the topic was initially fragmented for teaching. During the significant 2013 change, in the 9th grade, the curriculum covered the definition of the concept of function, various representations, basic function graphs, and injective and surjective functions. In the 10th grade, it included symmetries and algebraic properties of functions, the concept of composition of functions and inverse functions, and applications related to functions. In contrast to the 2005 curriculum, the 2013 change introduced the graphing of basic functions and symmetry transformations, enhancing the understanding of the concept of function. This approach encourages the exploration of new solution methods in concepts related to the concept of function. This illustrates that the 2013 curriculum change can potentially influence teachers' practices in teaching functions by promoting changes in their approaches.

When examining the curriculum changes concerning the concept of function, it can be observed that the 2005 curriculum did not provide a specific definition for the concept of function, but it referred to the Dirichlet-Bourbaki definition in its explanations (Gök et al., 2019). On the other hand, the 2013 curriculum change limited the scope of studies related to the concept of function to the set of real numbers. This curriculum defined the concept of function as "A relation that associates each element of one set with one and only one element of another set" (MoNE, 2013). It can be noted that while the Bourbaki definition was not wholly abandoned, there was a shift towards the Dirichlet definition in this context.

In the 2013 curriculum changes, it is thought that the conceptual axis change of functions and the inclusion of graphical approaches (basic graphs and symmetry transformations) in the curriculum affect the teaching of the concepts related to the concept of function. One of these is the concept of inverse function. In the curricula, the objectives related to the inverse function were expressed similarly as "Finds the inverse of an injective and surjective function" (MoNE, 2005) and "Finds the inverse of a given function by determining the necessary and sufficient conditions for a function to have an inverse according to the composite operation" (MoNE, 2013). However, with the 2013 curriculum change, it is understood that the inverse of a function is expected to be taught as an algebraic structure under the operation of composition on the set of real numbers referring to its monoid structure. This structure provides union and inefficient element properties according to the operation of composition on a set G ($<G,\circ>$ structure) (Fraleigh, 2014). However, it is unclear what approach will be preferred in teaching the inverse of a function in the context of the monoid structure, which function definition it is associated with, and to what extent it is supported by graphical approaches.

In terms of teaching the concept, the main question is how the new definitional approach influences teachers' teaching approaches to the concept of a function and what kind of explanations and learning tasks they realize acquiring the concept of inverse function in line with this new definition. In order to investigate the answers to these questions, the Anthropological Theory of Didactics (ATD) was used as the theoretical framework.

Analysis of Mathematical Tasks in Light of the Anthropological Theory of Didactics

The information nourishing teachers' instructional actions is like the hidden part of an iceberg we cannot see. This information might have been shaped in various periods and contexts (high school, university, department meetings, etc.). This situation indicates that teachers' actions related to teaching a relevant concept cannot be random; instead, there might be a specific logical network underlying these actions. Indeed, Chevallard (2006) pointed out that there is a coherence of meaning behind a purposeful human action like mathematics. The ATD and its praxeological analysis model have significant potential to uncover these actions and the logical network behind them.

The ATD is based on the idea that mathematical actions, like all human actions, are systematic and can be explained under specific components. In this regard, Chevallard and Bosch (2020) express that the dictionary meaning of praxeology is the study of human actions and behaviors, and they explain this concept as the basic unit for analyzing human actions on a large scale in the ATD. For example, these actions can be a daily activity expressed as going from home to school or a mathematical activity expressed as graphing the linear function f(x)=ax+b defined in real numbers.

In theory, like all human actions, mathematical actions are modeled by praxeologies in institutions (Chevallard, 2006) and can only survive through institutions (Chevallard, 2019; Chevallard & Bosch, 2020). In education, every classroom and every subject (e.g., mathematics) can be described as an institution. In this sense, mathematics modeled by praxeologies can be produced, taught, applied, and disseminated in social institutions (Garcia, Pérez, Higueras & Casabó, 2006). Thus, knowing praxeology is equivalent to knowing the knowledge it contains. This situation directs us to understand what the components of praxeology are.

A praxeology consists of four components: type of tasks (*Type de Tache*: T), technique (*Technique*: τ), technology (*Technologie*: θ), and theory (*Théorie*: Θ) (Chevallard, 2006, 2019; Chevallard & Bosch, 2020). While the first two constitute a "practice part" (know-how) known as the praxis block, the last two reflect a "knowledge part" (know-that) known as the logos block, which expresses logical explanations of why this is valid (Chevallard & Bosch, 2020). In more detail, a type of task consists of a specific set of tasks (e.g., those solved with the same technique). There are mathematical methods that can be used to solve each type of task. In theory, these are called techniques (Chevallard, 2006). Technology includes functions such as explaining, proving, and even designing the technique (Chevallard & Sensevy, 2014). On the other hand, theory is a set of general models, concepts, and simple assumptions (axioms) that validate technology, enabling the organization of praxeological elements as a whole (Bosch, 2015). It is stated that praxeologies did not emerge suddenly, but emerged as a result of processes that continue with complex dynamics, requiring analysis of what is happening in different institutions that create the knowledge to be taught, curriculum and curriculum reforms (Barquero, Jessen, Ruiz-Hidalgo & Goldin, 2023).

From an instructional perspective, mathematical knowledge in an educational institution is divided into mathematical and didactic praxeologies (Artigue & Winsløw, 2010). Mathematical praxeologies can be characterized as the set of existing praxeologies related to any mathematical knowledge. On the other hand, didactic praxeologies refer to the use of any praxeology for instructional purposes (Artigue & Winsløw, 2010; Gellert, Barbé & Espinoza, 2013). Chevallard (1998) stated that only certain praxeologies that meet certain constraints can be used in an institution, and all praxeologies that can be given in this context are shaped by these constraints. In such praxeologies, the praxis block is specified as didactic types of tasks and techniques, and the logos block as the didactic technological-theoretical environment (Barbe´, Bosch, Espinoza & Gascón, 2005).

Some of the Mathematical Praxeologies related to teaching mathematical concepts are transformed into Didactic Praxeologies by teachers' choices. Chevallard (2007) emphasized that in an environment involving learning situations, not only the content but also the manner of delivering that content is significant. In this regard, praxeological analysis provides an effective way to analyze teacher actions. These analyses make visible the ideas beyond teacher actions (Pansell, 2023). In other words, they provide insight into how teachers analyze the curriculum and why they choose specific approaches when teaching particular information. However, whether these teacher actions form a consistent didactic structure is uncertain. Additionally, challenges may arise in transforming mathematical praxeologies related to a piece of information into didactic praxeologies by teachers (Chevallard, 1997, 2022).

When fundamental changes are made to a mathematical concept, establishing a consistent organization for that concept can become even more challenging. This study focuses on teacher actions during a transitional period characterized by substantial changes related to a mathematical concept, specifically the function topic. The study aims to uncover Didactic Praxeologies used by teachers in the instructional process concerning the inverse function, which is considered challenging from a didactic perspective and holds importance in structuring the concept of function. In line with this goal, the study seeks answers to the following questions.

- What praxeologies do mathematics teachers use regarding the concept of inverse function?
- How holistic and consistent do the teachers use the praxeologies regarding the inverse function in terms of the components of the praxeological model?

Method

Research Design

This study, which examines the didactic praxeologies of teachers regarding the inverse function, employs the qualitative research method of a case study. Merriam and Tisdell (2015) define a case study as an in-depth description and analysis of a limited system. In this context, the limited system refers to the process of significant changes in the function topic within the curriculum (Gök et al., 2019; MoNE, 2013) and the actions of teachers toward teaching the inverse function during this process.

Participants

The study involves observing the lessons of three mathematics teachers working in different schools as participant observers. The criterion sampling method within purposeful sampling was used to select these teachers to obtain rich data (Büyüköztürk, Çakmak, Akgün, Karadeniz & Demirel, 2017). In this sense, the criterion for selecting teachers is having at least ten years of professional experience to ensure their awareness of limitations in the curriculum related to the function topic and to encourage them to incorporate more didactic praxeologies. The characteristics of these teachers and their schools are presented in Table 1.

Table 1. Information about teachers and their schools

Teacher	Gender	Experience	Graduation	School	Economic Level	Coursebook
Burak	Male	14	Education	Science High School	Middle and low	Coursebook A
Arda	Male	15	Science	Anatolian High School	Middle and low	Coursebook B
Tuna	Male	19	Science	Anatolian High School	Middle and low	Coursebook B

Burak is a graduate of the Faculty of Education, while the other teachers have graduated from the Faculty of Science and became teachers after completing a certain period of pedagogical formation training. Burak and Arda use smart boards and whiteboards during their lessons, while Tuna only uses a whiteboard. Burak uses Coursebook A as the textbook, while the others use Coursebook B. Coursebook A includes subheadings for teaching functions, such as Symmetries of Functions, Operations in Functions, Composition of Functions, Inverse of a Function, Composition of a Function and its Inverse, and Applications Related to Functions. On the other hand, Coursebook B covers subheadings for teaching functions, including Symmetry Transformations in Functions, Operations in Functions, Inverse of a Function, Composition of Functions, and Applications with Functions. Even though teachers utilize textbooks during the instructional process, they have provided their unique solutions when solving tasks. The schools where the teachers work are known as successful institutions in their respective cities, even though most students come from low and middle-income families.

Data Collection and Analysis

In this study, the focus was directed towards the types of tasks, techniques, and the rationale behind their validity that teachers employ during the instructional process of the inverse function, as well as explanations for their applicability (Chevallard, 2006, 2019; Chevallard & Bosch, 2020). The data for this study were gathered through video recording and audio recording devices during lessons where teachers covered the inverse functions, along with researcher observations. Within these observations, the tasks, techniques, and technological explanations employed by teachers in teaching inverse functions were noted. These annotations guided the identification of segments for analysis from the video recordings. The analyses were centered not only on the typical lesson progression but also on instances that revealed how teachers established their didactic organization and whether this organization was consistent (Strømskag & Chevallard, 2023). Consequently, the study aimed to uncover how teachers introduce the concept of inverse functions, how praxeology evolves, and the actions of teachers in moments of instructional challenge. Within this context, the investigation delved into the nature of teachers' praxeologies, the shifts in their praxeologies across different tasks, and the alignment of these shifts within the praxeological framework.

In the data analysis, employing the praxeological analysis framework, the types of tasks teachers utilize in teaching the concept of inverse functions, how they navigate these tasks to resolution, and the reasons behind their choice of solutions were determined. The elucidation of how teachers' actions in inverse function instruction align with the praxeological analysis model is expounded in Table 2:

Table 2. Analyzing teachers' actions related to inverse function with praxeological analysis

Types of Tasks	Techniques	Technology	Theory
Identifying the tasks used by teachers in teaching inverse function. Categorizing these tasks into types of task. Example: If f(x)=ax+b in R, what is $f^{-1}(x)$?	Identifying the techniques used in inverse function-related types of task. Analysing alternative techniques. Such as mapping and univalence. Example: If $(f\circ f^{-1})(x)=I(x)$, then $f^{-1}(x)=(x-b)/a$	Analyzing the explanations offered when designing or applying techniques. Example: The composition of a function and its inverse yields the identity function. If f is a bijection, then f(x)=y and $f^{-1}(y)=x$.	Identifying more general statements, if available, that justify the technology. Example: Monoid structure.

Teachers' actions related to the inverse function were analyzed regarding the types of tasks, techniques, technology, and theory components of the praxeological analysis. Wasserman (2017) stated that there are mapping, univalence, graphical approach, and algebraic structure approaches in tasks related to inverse function. These approaches are presented below in detail through two sample tasks in terms of praxeological aspects.

t_1: Given a function f defined for real numbers, where f(x)= 2x+5, what is $f^{-1}(x)$?

MP 1 (τ_1, informal mapping): Since f is injective and surjective if f(x) = y, then $f^{-1}(y)=x$. If f(x)=y, then y=2x+5, and hence x=(y-5)/2. Since $x=f^{-1}(y)$, then $f^{-1}(y)=(y-5)/2$ and, in other words, $f^{-1}(x)=(x-5)/2$.

θ_1: f is injective and surjective, if f: A→B, then f^{-1}: B→A, and f(x)=y if and only if $f^{-1}(y)=x$, as in the Dirichlet-Bourbaki definition.

MP 2 (τ_2, formal mapping): Since f is invertible under the composition operation, $f\circ f^{-1}=f^{-1}\circ f=I$. Then, $f\circ f^{-1}(x)=I(x)$. Hence, $(f(f^{-1}(x))=x$; $2f^{-1}(x)+5=x$; $f^{-1}(x)=(x-5)/2$.

θ_2: Inverse element property of functions defined in R under the composition operation involves the right-inverse of a function, composition operation, and monoid structure with axioms.

MP 3 (τ_3, symmetry transformations): Since f is bijective, the graph of f and f^{-1} is symmetric with respect to the line y=x. This means that if $(y,x)\in f^{-1}$, then $(x,y)\in f$.

θ_3: Bijective function, linear function, symmetry transformations with respect to the line y=x.

MP 4: τ_4 (univalence): If $f=\{(x,y)|\ y=2x+5 \text{ and } x,y\in R\}$, then $f^{-1}=\{(y,x)|\ (x,y)\in R\}$, assuming the univalence condition is met.

θ_4: Univalence

MP 0 (τ_0, primitive solution): Solving the task related to the concept of inverse function only in the context of the composition operation.

θ_0: Conditions for the composition of two functions, axioms, and monoid structure.

Θ: The group of invertible functions (f[x]°) guarantees a single solution.

In this study, data triangulation (participant observation, video, and audio recording) was ensured to increase the validity of the study. In the data analysis, the processes were explained in detail, and the analyses were subject to scrutiny by two experts. It can be stated that such situations increase the reliability of the study as they ensure that the results obtained in the analyses are consistent.

Findings

In this section, the types of tasks used by each teacher were identified, and then the praxeological components of these tasks and the relationships between them were examined.

Burak's Praxeologies

After explaining many types of tasks in different representations related to the function composition in the order of the textbook he used, Burak moved on to the inverse function (Figure 1).

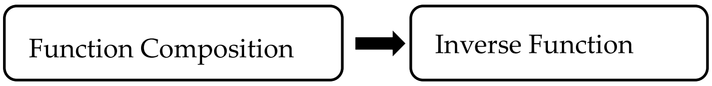


Figure 1. Burak's organization of the sub-dimensions of functions

The fact that Burak handled the function composition before the inverse function in the teaching process suggests that he may want to explain the inverse function through the composition operation. In order to find out whether the teacher used this kind of praxeology, firstly, the types of tasks and techniques he used in the teaching process were determined in Table 3.

Table 3. The types of tasks and techniques Burak used in the teaching process

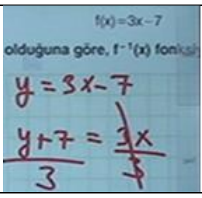
TT	Type of Task (TT) Statement	Number	Technique
T_1	Finding the inverse image of a value in a function in schema representation	5	τ_1
T_2	Find the inverse image of a value in a list representation function	2	τ_4
T_3	Finding the inverse image of a value in a function in algebraic representation (AR)	22	τ_1
T_4	Finding the relationship between the graph of the function and the graph of its inverse	1	τ_3
T_5	Finding the inverse of a function given as AR	15	τ_1
T_6	Finding the composition of a function given as AR and its inverse	12	τ_2
T_7	Finding the inverse of the composition function for functions in AR	8	τ_1
T_8	Finding one of the components and functions while the other is known in AR	16	τ_1&τ_2
T_9	Complex tasks (combinations of T_3, T_5, and T_8)	5	τ_1&τ_2

Burak included 86 tasks in nine types of tasks related to inverse function. The first types of tasks are related to finding the inverse image in different representations of the function (T_1-T_4). These were used as a transition to the type of task T_5. It is seen that the teacher carried out an algebraic representation-weighted instruction in the following types of tasks and included more complex types of tasks (T_8-T_9) in the later stages of the teaching process. It is likely that the techniques that the teacher can use in these types of tasks are mapping or univalence. Besides, explanations about why the technique used is valid are also crucial for the integrity of the praxeological organization.

Although the teacher used four different techniques in the types of tasks related to the inverse function in Table 3, it is understood that he used two techniques intensively. While an informal technique based on the mapping rule (τ_1) was used in the types of tasks (T_1, T_3, T_5) involving the inverse image of a certain value in different representations, it was determined that this evolved into a technique (τ_1&τ_2) that emphasized the monoid structure through the composite operation in more complex types of tasks. Apart from these, the type of task T_2 was solved using univalence, and the type of task T_4 was solved using symmetry transformations. From this point of view, it can be stated that Burak used the mapping rule extensively in the solution of the types of tasks and utilized the monoid structure to a certain extent when necessary.

A more detailed examination of Burak's mapping rule approach and composition operation approach to the inverse function is vital for understanding the teacher's didactic praxeologies on the inverse function. Therefore, firstly, the teacher's praxeologies in a task in which he used the mapping rule approach are given in Table 4. Since the teacher taught functions mainly through algebraic representations, we focused on this type of task.

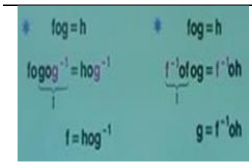
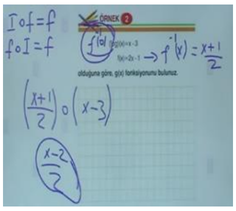
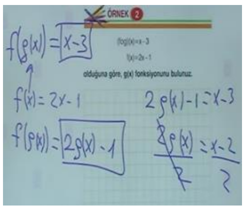
Table 4. Burak's didactic praxeologies on the inverse function

Task	Solution of the Task	Praxeological Analysis
	B: *OK, guys, this is y. y=3x-7. I aim to isolate x, and after isolating x, write y instead of x and x instead of y. y=3x-7, (y+7)/3=3x/3, x=(y+7)/3, $f^{-1}(x)$= (x+7)/3. This is the inverse of f. That's it.*	$t_{5,1}$ (T_5) $\tau_{5,1}$ (τ_1) $\theta_{5,1}$ (Partially θ_1) *isolating x… is the inverse of f.* [inverse function informal]

Burak states the task of finding the inverse of a function ($t_{5,1}$) as a mapping between x and y. A deeper look at this mapping rule reveals that it is some how related to the definition of function. Here, the teacher considers the expression of y in terms of x as a function (informal definition of function) and, therefore, states the expression of x in terms of y as an inverse function. This shows that the teacher did not randomly design his praxeology about the inverse function and constructed it based on the mapping rule definition of the function (See MP 1). However, when the solution is analyzed, it is seen that many steps are skipped, such as not explaining that the function is injective and surjective, not examining the range of definitions, and transforming variable substitution into an algorithm that needs to be memorized. In particular, the technique was applied without referring to the rule of $x=f^{-1}(y)$ if and only if $y=f(x)$, which has an important place in the mapping rule. This situation shows that didactic praxeology was applied without a knowledge block. Therefore, it can be suggested that the praxeological organization was incompletely structured in this task. It was observed that the teacher applied the praxeologies in other tasks similarly incompletely.

On the other hand, the sequence of topics in the teaching process gave the impression that the inverse function would be explained by associating it with the monoid structure through the composition operation. However, this was not observed in the first types of tasks. Although this kind of praxeology was implied in the type of task T_6, it was not revealed, and it was briefly explained just before the more complex type of task T_8. Table 5 analyses these explanations and one task afterward.

Table 5. Burak's approach in the transition to the type of task T_8

Task	Solution of the Task	Praxeological Analysis
	*B: "If we need f, we combine it with g^{-1} on the right side of the equation, allowing both g and g^{-1} to subtract. If g is needed, we write the inverse of f on the left side of the equation; the fs cancel each other out, leaving us with g. This formula proves to be quite practical and versatile."*	θ_2 (partially) $t_{8,1}$ (T_8)
	S4: Sir, I did not understand.	$\tau_{8,1}$ (τ_1 missing)
	*B: … Let's find the inverse of f, and it is (x+1)/2, isn't it? I'll combine f^{-1} from this side of the equation.*	θ_1 (implicit)
	S6: Why, sir?	
	B: I need g(x). To subtract f, I have to write f^{-1} next to f. With f^{-1}, f is subtracted and g(x) remains.	
	$f^{-1}o(fog)(x)=(x+1)/2\ o(x-3)=(x-2)/2$	$\tau_{8,1}$ (τ_2 missing)
	S8: Sir, why didn't we write those two down?	
	B: These two make I. Whatever function you combine I with, you will find the function itself. Iof=f and foI=f.	θ_2 (partially)
	[Students do not understand]	
	B: I do it the other way round. f(g(x))=x-3, right? f(x)=2x-1, right? What about f(g(x))? Notice that I have to write g(x) instead of x. Then f(g(x))=2g(x)-1. f(g(x))=x-3 These must be equal to each other. If 2g(x)-1=x-3, then 2g(x)=x-2 hence g(x)=(x-2)/2.	$\tau_{8,2}$ (τ_0) θ_0 (composition, a=c if a=b and b=c, equation)
	S7: It is easier this way.	
	B: In fact, you did not understand the other first method; if you had understood it, you would have found it easier.	

Regarding the type of task T_8, Burak presented a praxeology that is a mixture of MP 1 and MP 2 instead of using MP 2. The main difficulties here are that the teacher set up both MP 1 and MP 2 incompletely and quickly transitioned to this praxeology despite changing the praxeology. On the other hand, it is understood from the teacher's words, "*This formula* *proves to be quite practical and versatile*" and "*In fact, you did not understand the other first method; if you had understood it, you would have found it easier*" that he wanted to come up with MP 2. However, the incomplete construction of the technology of MP 2 makes this problematic. The fact that the teacher then solved this task with a different technique from the technological explanations known to the students shows that when the technology is well established, praxeology can be used more functionally for students' understanding. In other tasks, Burak generally used praxeology, a mixture of MP 1 and MP 2. The relationship between the praxeologies used by Burak in the types of tasks related to the concept of inverse function is given in Figure 2.

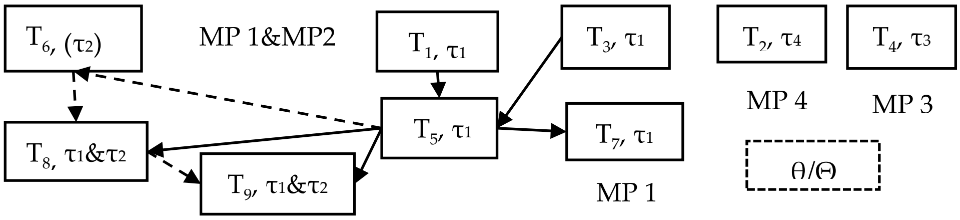


Figure 2. The praxeologies Burak used in the inverse function.

Burak used two different praxeologies extensively in the inverse function teaching. MP 1, based on the mapping rule, which is present in most the type of task and which the teacher cannot give up, emerged in relation to the established definition of the function in the curriculum (Dirichlet-Bourbaki definition). This praxeology was generally used in simple tasks, and its technological dimension was partially included. In relatively more complex tasks, instead of MP 2, which reveals the formal understanding of the monoid structure, there is an abrupt transition to a poorly structured mixture of MP 1 and MP 2. Although Burak imposed this praxeology during the teaching process, difficulties in teaching arose due to the lack of sufficient explanation and the abrupt transition.

Arda's Praxeologies

Arda included the concept of inverse function first and then the function composition in the teaching process with the order in the textbook he used.

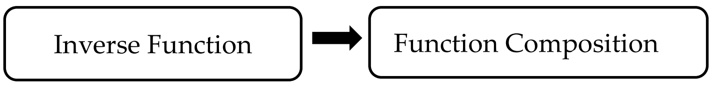


Figure 3. Arda's organization of the sub-dimensions of functions

The fact that the inverse function is given before the function composition shows that the concept of inverse function will not be taught through the composition operation, contrary to the learning objective related to the inverse function in the curriculum. Arda used

46 tasks in teaching inverse functions. Table 6 shows the type of task and techniques in which these tasks took place.

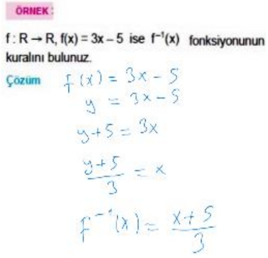
Table 6. The types of tasks and techniques Arda used in teaching inverse function.

TT	Type of Task Statement	Number	Technique
T_1	Finding the inverse image of a value in a function in schema representation	3	τ_1
T_2	Finding the inverse image of a value in a function in AR	6	τ_1
T_3	Finding the relationship between the graph of the function and the graph of its inverse	2	τ_3
T_4	Finding the inverse of a function given as AR	6	τ_1
T_5	Finding one of the components and functions while the other is known in AR	9	τ_1 & τ_2
T_6	Finding the composition of at least two functions given as AR	7	τ_0
T_7	Finding the image of a certain value in the composition of two functions in AR	2	τ_0
T_8	Finding the image and inverse image of certain values in the graphs of functions	11	τ_1 &τ_3

Arda used five techniques in his teaching, including eight the types of tasks related to inverse function. Arda's introduction to the inverse function without giving any information about the composition operation shows that the teacher will build his praxeology on mapping. To this end, in the type of task (T_1, T_2, T_4), he generally analyzed with the technique based on the matching rule (τ_1). It was determined that mapping and symmetry transformations were used in inverse function tasks involving function graphs. In AR, a mixed technique (τ_1&τ_2) was used in the complex type of task (T_5) involving compounds. Since the teacher had not yet explained the composition operation as a necessity, the teacher explained how to apply the composition operation before the type of task T_5. It can be stated that using a mixed technique in this type of task instead of using the τ_2 technique, which clearly shows the monoid structure, is related to the definition of the concept of function in the curriculum (Dirichlet-Bourbaki definition).

This made Arda's praxeologies used in the types of tasks T_2 and T_5 important. Firstly, Table 7 analyzes how Arda constructed the inverse of a function in the type of task T_2. This will allow an attempt to understand how the established definition of the function is employed in finding its inverse.

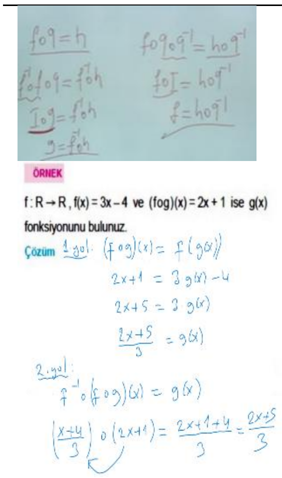
Table 7. Arda's didactic praxeologies about inverse function rule

Task	Solution of the Task	Praxeological Analysis
	*A: If this function y=f(x) is 1-1 and surjective, what is the inverse of f? We denote it by f^{-1}. Look, I match y with x. What is the inverse? I match x with y. f(x)=y and $f^{-1}(y)$=x, okay? f(x)=3x-5, f(x)=y, y=3x-5, if you cancel out x in this relation, the expression you get is the inverse of f.* *y+5=3x; (y+5)/3=x; $f^{-1}(x)$=(x+5)/3.*	$t_{2,1}$ (T_2) $\tau_{2,1}$ (τ_1 missing) $\theta_{2,1}$ (θ_1 partially) *Look, I match y with x.* [function informal definition] *What is the inverse? I match x with y.* [inverse function informal]

Arda realized the inverse function rule with mapping-based praxeology (MP 1). He clearly emphasizes that the expression of y in terms of x indicates a function, and when the expression of x in terms of y is obtained in this equation, the inverse of the function is acquired. From this point of view, it is understood that the meaning that the teacher attributed to the definition of the function provides a technique for the inverse of the function. On the other hand, there are some deficiencies in applying this technique. For example, although the teacher pointed out that he did this operation if the function is injective and surjective, he did not make any verification regarding this. In addition, y=f(x) if and only if x=f^{-1}(x), he skipped the steps of changing variables in the rule. This situation shows that the teacher provided incomplete technology related to his technique. Similarly, it was determined that the teacher used MP 1 in relatively simple tasks.

It is important to understand the praxeological organization that Arda will follow in more complex tasks (T_5) involving composition. In Table 8, the teacher's explanations on this subject and the process of solving a task are examined.

Table 8. Arda's approach in transition to the type of task T_5

Task	Solution of the Task	Praxeological Analysis
	*A: If the question is about g, the composition of f^{-1} is written on both sides of the equation. Then, fog=h; f^{-1}ofog= f^{-1}oh; Iog= f^{-1}oh. "I" has no effect. g= f^{-1} oh. In such a question where f is asked, we would subtract g. Then, fog=h; fogog^{-1}=hog^{-1}; f= hog^{-1}. You know, in composition operations, it is important which side you write: the right or left side of the composition, OK?* *S12: (fog)(x)=f(g(x)); 2x+1=3g(x)-4; 2x+5=3g(x); g(x)=(2x+5)/3.*[Students solved another way] *A: Now, let's do it the second way. (fog)(x) is given, isn't it? What is asked for? g(x). Then, what happens if f^{-1} is included in the process from this side, guys? fs are subtracted, and g(x) remains. f^{-1}o(fog)(x) equals= g(x), right? What is f^{-1}? (x+4)/3. What is fog? 2x+1. (x+4)/3 o (2x+1)=(2x+5)/3. We will take this and write it where we see x here.*	θ_2 (partially) $t_{5,1}$ (T_5) $\tau_{5,1}$ (τ_0) θ_0 (function composition) $\tau_{5,2}$ (τ_1&τ_2 missing) θ_1 (surjective) θ_2 (partially) θ_0

Arda presented a technological explanation partially suitable for MP 2 praxeology in the type of task T_5. Although the left or right composition of functions, and unit function of the algebraic structure are emphasized here, it is observed that there is no reference to the composition property. In this context, S12 solved the task ($t_{5,1}$) with the praxeology shaped on the basis of the operation of composition without any explanation. Afterward, Arda started with MP 2 but quickly applied MP 1 in the inverse rule. From this point, it was determined that the teacher used a mixed praxeology (MP 1 & MP2), which was not well structured for more complex tasks. In addition, his direct introduction to the inverse function without sufficient explanation about the operation of compositing and his sudden switching between the praxeologies made the learning process difficult. The praxeologies that Arda included in the teaching process about inverse function are given in Figure 4.

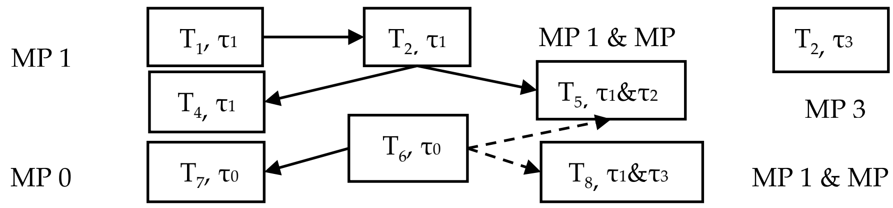


Figure 4. The praxeologies Arda used in the inverse function.

Arda built the concept of inverse function on mapping praxeology without associating it with the operation of composition (MP 1). It was observed that the teacher's perception of function and inverse function as a mapping rule between two variables was effective in such a choice. In more complex tasks (T_5), this praxeology was found to evolve into a mixture of MP 1 and MP 2 suddenly and without being well structured. In this transition process, Arda introduced the composite operation and its properties with short explanations as needed. As can be seen in Figure 5, he included tasks related to the operation of composition after the type of task T_5. This sudden praxeological change shows that the teacher did not organize the praxeologies related to inverse function consistently. On the other hand, in both praxeologies, the teacher mentioned technological explanations about why the techniques are valid in a very limited way. This shows that Arda constructed the didactic praxeologies in the inverse function incompletely, especially in terms of knowledge block.

Tuna's Praxeologies

Tuna, with the order in the coursebook he used, first included the concept of inverse function and then the function composition in the teaching process.

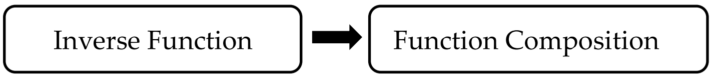


Figure 5. Tuna's organization of the sub-dimensions of functions

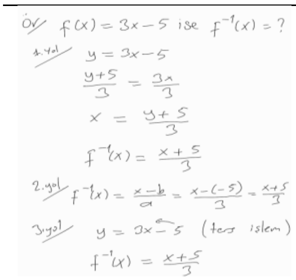
The fact that the inverse function is given before the function composition shows that the inverse function will not be taught through the composition operation. The teacher used 32 tasks in the teaching process related to inverse function. Table 9 shows the types of tasks and techniques in which these tasks took place.

Table 9. The types of tasks and techniques Tuna in teaching inverse function

TT	Type of Task Statement	Number	Technique
T_1	Finding the inverse of a function in AR	12	τ_1
T_2	Finding the inverse image of a certain value in a function in AR	4	τ_1
T_3	Finding the composition of at least two functions given as AR	6	τ_0
T_4	Finding one of the components and functions while the other is known in AR	3	τ_1&τ_2
T_5	Finding the image of a certain value in the conjunction of two functions given as AR	4	τ_0
T_6	Complex tasks in AR involving composition of functions and inverse functions	3	τ_1&τ_2

Tuna's introduction to the concept of inverse function without the operation of composition shows that he will use mapping-based praxeology. Tuna did not include any preparatory task and directly introduced the inverse function with the type of task involving finding the inverse rule of the function with algebraic representation. He analyzed only the tasks related to the inverse function with algebraic representation. It was determined that the teacher used a praxeology based on informal mapping (τ_1) in the first and simple types of tasks involving inverse function (T_1 and T_2) and a mixed praxeology (τ_1&τ_2) in more complex type of task (T4). In order to reveal the teacher's praxeology about the inverse function, the praxeologies applied in these tasks (except T_3 and T_5) should be analyzed. Therefore, Tuna's informal mapping praxeology, in which he constructed the concept of an inverse function, is analyzed in Table 10.

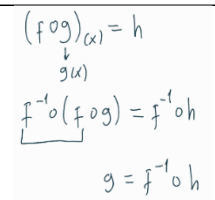
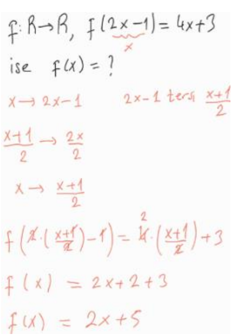
Table 10. Tuna's didactic praxeologies on inverse function rule

Task	Solution of the Task	Praxeological Analysis
	*T: If f(x)=3x-5, what is $f^{-1}(x)$? I will solve the question with three methods. First, I will operate until x is subtracted.* *y=3x-5; (y+5)/3=x; $f^{-1}(x)$=(x+5)/3* *Here, we were doing the switching, writing the inverse of the function, right? $f^{-1}(x)$=(x+5)/3. Only this y is replaced by x. Nothing else.*	$t_{1,1}$ (T_1) $\tau_{1,1}$ (τ_1 missing) $\theta_{1,1}$ (θ_1 partially) *until x is subtracted …* [informal inverse function], *doing the switching* [variable switching]

In Table 10, Tuna explained the same technique in three ways. While applying this technique, the function was expressed as a mapping between x and y. Then, it was pointed out that the inverse function would be obtained when the variable x is written in y in the equation. This approach shows that the concept of inverse function is realized in line with the mapping-based praxeology (See MP 1). The emergence of this praxeology is strongly influenced by the Dirichlet-Bourbaki definition. In applying the technique, situations such as the function must be injective and surjective and the mapping rules are not explained. This situation shows that the technique is applied without technological explanations.

It is important to determine which praxeology Tuna will use in the type of task T_4 involving composition of function and inverse functions. In Table 11, the teacher's explanations and the analysis of one task are given.

Table 11. Tuna's approach in transition to the type of task T_4

Task	Solution of the Task	Praxeological Analysis
	*T: How do we find the function g? Here, what did I add on both sides so that g is subtracted?* *S1: Inverse of f.* *T: So, f^{1}o(fog)=f^{1}oh? And what is the result of this?* *S1: Unit function.* *T: g=f^{1}oh, right? So, we found g. Let's solve a problem. If f: R→R,*	$\theta_{4,1}$ (θ_2 partially)
	*f(2x-1)=4x+3, then* f(x)=? *S7: We equalize to one.* *T: Guys, I don't think f(x) will be a number..* *S3: x* *T: In order for this to be x, guys, you need to find the inverse of this function (2x-1) and write it instead of x. So let's think like this.* *x→2x-I don't say they are equal because they can't be equal.* *x+1→2x; x→(x+1)/2. I found the inverse of this function, OK? I will take this and write it instead of x.* *f(2(x+1)/2 -1)=4(x+1)/2+3; f(x)=2x+2+3; f(x)=2x+5*	$t_{4,1}$ (T_4) $\tau_{4,1}$ (τ_1&τ_2 missing) τ_1 (missing) θ_1 partially θ_0

Tuna used a mixed technique consisting of MP 1 and MP 2 instead of MP 2, which emphasized the formal meaning of the monoid structure in the type of task T_4. However, praxeology was applied without sufficient explanation. For example, the inverse of a function must be injective and surjective to denote a function, and the properties and axioms of the composition operation were not explained. When such technological explanations are not given, it will not be understood what to do after one step and why the action is done. This situation is clearly visible in the students' responses. From this point of view, it can be stated that when the technology component in praxeology is presented incompletely and poorly structured, it may cause disruptions in the teaching process. The relationships between the praxeologies Tuna used in teaching the inverse function are given in Figure 6.

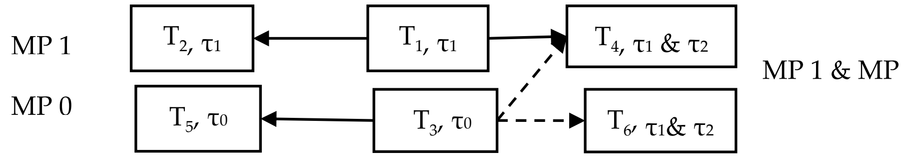


Figure 6. The praxeologies Tuna used in the inverse function.

Tuna used two praxeologies related to the inverse function. MP 1, based on the informal mapping rule, was partially used for introducing the concept and for relatively simple tasks. This praxeology was indirectly related to the definition of the function. A mixed praxeology, a mixture of MP 1 and MP 2, was used with a very limited block of knowledge for more complex tasks. More specifically, it was found that the praxeologies introduced by the teacher about the inverse function were not well structured, and there were abrupt transitions between praxeologies without sufficient explanation.

Discussion

In this study, which uncovers the praxeologies employed by teachers concerning the concept of inverse functions within the instructional process and investigates the coherence of these praxeologies, it was determined that teachers generally utilize two distinct praxeologies. One of these praxeologies emerges as the praxeology of informal mapping (MP 1) during the introduction of the inverse function within the curriculum, while the other takes shape as a combination of praxeologies involving both informal and formal mapping, highlighting the monoid structure in more complex tasks (MP 1 & MP 2).

Firstly, it was found that praxeology based on informal mapping emerged under the influence of the established definition of function in the curriculum. The basic idea is based on the fact that teachers think of the function as an expression of the variable y in terms of x. The inverse of the function is obtained when the variable x is acquired in terms of y in some way. Due to its structure, this praxeology was used alone in introducing the inverse function and in relatively simple tasks. On the other hand, it is understood that this praxeology does not coincide with the learning objective (MoNE, 2013) in the curriculum, which is to teach the inverse function through composition. Such situations are more likely to occur, especially in periods of radical changes in the curriculum. In support of this, it was stated that there was a lack in the logos regarding the concept of integrability in a textbook written in Norway when the program was changed (Topphol, 2023). It is known that curricula contain many constraints that affect teacher actions (Barbe´ et al., 2005). With the assumption that changes in the curriculum can affect the praxeological organization, it can be argued that teachers are likely to face situations they have never encountered before during curriculum changes, and they should be open to change (Chevallard, 2022).

Secondly, it was determined that teachers preferred a mixed praxeology with the properties of MP 1 and MP 2 in more complex tasks instead of using MP 2 based on formal mapping that clearly shows the monoid structure. It is anticipated that the fact that the structure of MP 2 was more abstract and that the teachers had introduced the inverse function through MP 1 was effective in not preferring MP 2. In mathematical terms, there can be many praxeologies for teaching a concept. Teachers may prefer one of them or a combination of them due to various constraints and obligations. In Turkey, although the curriculum requires teaching the inverse of a function through composition (MoNE, 2013), teachers taught the concept of inverse function through mapping. However, in complex problems, they revealed a mixed praxeology for teaching the inverse function through composition. Therefore, the influence of the approach towards inverse functions within the curriculum is observed in the emergence of this praxeology. Besides, the fact that only MP 1 is difficult to apply in complex problems may have been effective. Similar to the results obtained in this study, it is reported that in France, in the tasks related to analyzing the monotonicity of one function from the monotonicity of another function (excluding derivatives), there are at least two algebraic techniques (based on the definition of monotonicity) and at least two functional techniques (based on the concept of composition of functions or symmetry transformations). It was reported that teachers used a mixture of semi-algebraic and semi-functional techniques and did so by introducing a technique to the extent they could understand and in a way they could teach because they could not fully understand the curriculum's expectations (Erdogan, 2014).

Teachers' utilization of coursebooks in function teaching may have caused incomplete construction of praxeology. The preparation of textbooks with a popular approach and without considering the didactic organization may have prevented teachers from presenting a consistent praxeology about the inverse function. In support of this, it was pointed out that teachers who make use of coursebooks with insufficiently structured knowledge blocks in the teaching process find it difficult to deal with advanced tasks (Putra, 2020). The praxeological organization needs to be considered holistically. This raises the problem that the coherence and components of a praxeology should be well structured. This study determined that teachers applied both praxeologies partially and without providing sufficient explanations; that is, they were not well structured. Additionally, it was indicated that sudden transitions were made from one praxeology to another, making it difficult to understand the praxeology constructed in the learning process. As a matter of fact, it was observed that the teachers who encountered such difficulties had conflicts about using the praxeologies that the students know and that the curriculum foresees in complex tasks related to inverse function. For example, Burak used two different praxeologies in such a complex task. One was understood by the students because it was a praxeology that the students knew before, and the other was not understood since it was prescribed by the curriculum but presented by the teacher without being well structured. The teacher tried to use the praxeology suggested by the curriculum by saying, *"In fact, you did not understand the other first method; if you had understood it, you would have found it easier."* From this point of view, it can be asserted that the teacher tried to structure the praxeology in the curriculum, but the incomplete presentation of praxeology made it challenging to understand praxeology.

Conclusion

This study analyzed the praxeologies of three teachers about the inverse function. The teachers put forward praxeology based on mapping in the concept of inverse function with the effect of the Dirichlet-Bourbaki definition of the concept of function in the curriculum. However, in more complex tasks where this praxeology was inadequate (tasks involving the composition of function and inverse function), they suddenly changed and switched to a mixed praxeology stipulated by the curriculum. While the teachers constructed both praxeologies, it was determined that they had deficiencies in terms of praxeological components, especially in terms of knowledge block.

This study has illuminated that within the realm of inverse functions, the emergence of praxeologies chosen by teachers for instructing the sub-dimensions of a concept is influenced by various aspects. These aspects encompass the inclusion of a particular concept definition in the curriculum and the pedagogical approach undertaken to impart instruction on the sub-dimensions of the concept. Furthermore, it is noteworthy that when extended to diverse conceptual contexts, this modest-scale study holds promise for a more comprehensive comprehension of teacher actions. In addition, it may be recommended to conduct studies on why teachers structure their praxeology incompletely, what are the conditions that cause this, and how these can be overcome.

Acknowledgment

This work has been supported by Anadolu University Scientific Research Projects Coordination Unit under grant number 1408E364. This article is extracted from doctorate dissertation of the first author entitled "Ecological problems related the function topic in mathematics curriculum (2013) and teachers' praxeology", supervised by Prof. Dr. Abdulkadir Erdoğan (Ph.D. Dissertation, Anadolu University, Eskişehir, Turkey, 2018)

Ethical Committee Permission Information

It has been confirmed by the researcher that the data used in this study dates back to before 2020.

Author Contribution Statement

Mustafa GÖK: *Conceptualization, literature review, methodology, implementation, data analysis, translation, and writing.*

Abdulkadir ERDOĞAN: *Conceptualization, methodology, editing, and commenting.*

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