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FOUR AXIOMS FOR A THEORY OF RHYTHMIC SETS AND THEIR IMPLICATIONS

Abstract

In a recent article (Lugos Abarca, 2023) an equation was proposed that allows us to know the number of measures that a song has μ_{mar} from the musical variables of tempo T, song duration t and time signature β. Also, it was found that that by solving the equation μ_{mar} for the variable t yields a formula capable of expressing the duration in minutes of any rhythmic figure. Proceeding with this line of research, four axioms are presented whose purpose is to function as a basis for the construction of a set theory for rhythmic figures, during this process the consequences of the third axiom that establishes the non-commutativity in the sum of certain sets that have the same elements but with different order are studied, and whose most relevant consequence is to introduce the theorem that determines the existence of different types of empty sets.

Keywords: set theory, axioms, rhythmic figures, commutativity, empty set.

1. Introduction

In 1923 the composer and musical theorist Arnold Schoenberg proposed the twelve-tone technique, better known as dodecaphonism (Perle, 1972; Maor, 2020; Schoenberg, 2014), which revolutionized music theory to a certain extent and generated controversy due to its characteristic sonority. After a while, Allen Forte formalized the pitch class set theory (Forte, 1974), which studies musical notes under a numerical approach with the purpose of composing melodies based on the sonority generated by intervallic relationships.

This musical language applied to atonal Composition apparently makes its effort in studying exclusively the musical notes and not the other elements that complement them (Schuijer, 2008), such situation has led other researchers and music theorists to study musical rhythm from a mathematical perspective, where they discovered that the best mathematical field to perform this task is geometry (Demaine et al., 2009; Gómez-Martín, 2022; Toussaint, 2005; Toussaint, 2019; Tymoczko, 2011; Hsü & Hsü, 1990; Chahine & Montiel, 2015).

Yet, the use of mathematics applied to rhythm are entirely for creative purposes (Lovemore et al., 2021; Mehta et al., 2016), accordingly to have new forms of composition and not in a mathematically pure context, alternatively, given the recent research conducted (Lugos Abarca, 2023) opens the opportunity to study rhythmic figures through another branch of mathematics which are number theory and set theory now with a totally pure approach (Mall et al., 2016; Rahn, 1979; Mora, 2012), in other words, without the need to be applied to musical creation or interpretation.

2. Axiom No.1: Axiom of Rhythmic Figures

The first axiom proposes the following:

Any rhythmic figure could be expressed in terms of time t with the equation:

$$t = \frac{\beta}{T} \tag{2.1}$$

Such that T is the tempo of the song and β is the pulse of a rhythmic figure.

Let study how this axiom is deduced.

To begin with, the task of mathematically defining the rhythmic figures was carried out in greater depth in a recent investigation (Lugos Abarca, 2023). However, how to construct an equation for rhythmic figures will be briefly reviewed. Begin by eliminating the variable t from the equation μ_{mar}, giving the following expression

$$t = \frac{\mu_{mar}}{T}\beta \tag{2.2}$$

Where μ_{mar} is the total number of bars in a song.

So, expression 2.2 gives us the duration value of a rhythmic figure in units of minutes. This equation could be simplified because under this context, rhythmic figures are performed within a single measure (Honing, 2013), therefore always $\mu_{mar} = 1\ min^2$, this suggests that 2.2 could be rewritten as:

$$t = \frac{\beta}{T} \tag{2.3}$$

Consequently 2.2 is reduced to two variables: T is the tempo of the song and β the pulses of a rhythmic figure. By means of the pulse or the value corresponding to each rhythmic figure (Herrera, 2022; Camiruaga, 2000) it is possible to generalize equation 2.3 for a specific figure:

$$\begin{gathered} t_0 = \frac{1}{T},\ \ t_1 = \frac{2}{T},\ \ t_2 = \frac{3}{T},\ \ t_3 = \frac{4}{T} \\ t_2^1 = \frac{1}{T2},\ \ t_3^1 = \frac{1}{T3},\ \ t_4^1 = \frac{1}{T4},\ \ t_5^1 = \frac{1}{T5} \end{gathered} \tag{2.4}$$

Where t_0 is the quarter note, t_1 is the half note, t_2 is the half note with augmented dot, t_3 is the whole note, t_2^1 is the 8th note, t_3^1 is the triplet note, t_4^1 is the 16th note and t_5^1 is the quintuplet note.

3. Axiom No.2: Axiom of the Rhythmic Limit

The second axiom states that:

Let $\psi^\dagger$ be the set representing a musical measure, it is determined that its elements will be only rhythmic figures t, whose result of summing these figures will be equal to the value of the time signature of the set $\psi^\dagger$.

$$Si \dagger \in \psi^\dagger = n \longrightarrow \sum t \in \psi^\dagger = n \tag{3.1}$$

Let explain this axiom:

Having numerically defined the rhythmic figures, the set $\psi^\dagger$ is introduced, which represents a musical measure and whose symbol † indicates the numerator of the time signatura. So $\psi^\dagger$ is a musical measure whose time signature is $\frac{1}{4}$, ψ^2 is a musical measure whose time signature is $\frac{2}{4}$, ψ^3 is a musical measure whose time signature is $\frac{3}{4}$ and ψ^4 is a musical measure whose time signature is $\frac{4}{4}$.

Musically, the time signature will act as a limit to the total value of digits that could be used within a musical measure (Schönberg, 1994; Jones & Pearson Jr, 2013). To illustrate, be the set ψ^4 its elements may have the following combinatorics:

$$\psi^4 = (t_1 + t_{0-0} + t_{0-1}) \tag{3.2}$$

In score set 3.2 is:

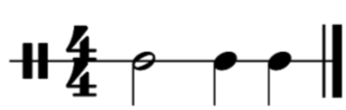

Figure 3.1: One half note and two quarter notes in score.

Another way would be:

$$\psi^{4} = (t_{2} + t_{0}) \tag{3.3}$$

It follows that:

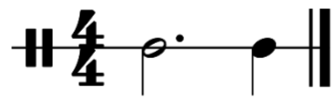

Figure 3.2: One half note with augmented dot and one quarter note in score.

As described, it does not matter which figure is being played within the measure, the condition is that the sum total of the values or pulses of the rhythmic figures is equal to the pulse of the time signature of the set (Schmeling, 2011), which could be expressed mathematically as follows:

$$Si \dagger \in \psi^{\dagger} = n \longrightarrow \sum t \in \psi^{\dagger} = n \tag{3.4}$$

Therefore, based on this definition, it is established that:

$$\psi^{1} = t_{0},\ \ \psi^{2} = t_{1},\ \ \psi^{3} = t_{2},\ \ \psi^{4} = t_{3} \tag{3.5}$$

Similarly, the subdivision of rhythmic figures (Schoenberg, 2016) mathematically could be represented as subsets:

$$t_{5}^{1} \subset t_{4}^{1} \subset t_{3}^{1} \subset t_{2}^{1} \subset t_{0} \subset t_{1} \subset t_{2} \subset t_{3} \tag{3.6}$$

Also, musical measures may have the following properties:

$$\psi^{1} \subset \psi^{2} \subset \psi^{3} \subset \psi^{4} \tag{3.7}$$

4. Axiom No.3: Axiom of Quasi Commutativity

The third axiom proposes that:

Given two sets $\psi^{\dagger}$ whose elements are the same figures t, but different from each other and with different order, it is established that by musical property the sum of both sets does not commute:

$$(t_{2}^{1} + t_{4-0}^{1} + t_{4-1}^{1}) \neq (t_{4-0}^{1} + t_{2}^{1} + t_{4-1}^{1}) \tag{4.1}$$

Let analyze this axiom in detail. The way in which the order of the rhythmic figures could be combined within a set $\psi^{\dagger}$ depends on both the time signature and the value of the rhythmic figures (Burns, 2010). Conversely, the fact that a set has a different order with respect to its elements, even having the same figures, interpretatively they are not the same (Pearsall, 1997), to clarify consider the following set:

$$\psi^{1} = (t_{2}^{1} + t_{4-0}^{1} + t_{4-1}^{1}) \tag{4.2}$$

For instance, sum-within-set notation is used respecting the same notation used by Cannas and Andreatta in their paper (Cannas & Andreatta, 2018) to express sets of musical notes.

Rhythmic set 4.2 represents musically:

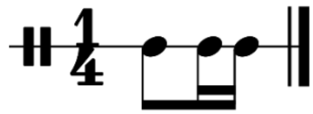

Figure 4.1: One 8th note and two 16th notes in score.

On the other hand, with the same elements but in a different order, another set is constructed with the following form:

$$\psi^1 = (t^1_{4-0} + t^1_2 + t^1_{4-1}) \qquad (4.3)$$

This suggests that:

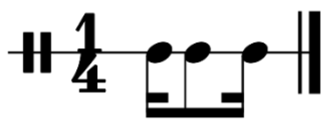

Figure 4.2: One 8th note and two 16th notes with different order.

When comparing both figures, it is possible to easily observe that the sets are not the same, since musically their performance is distinguishable between each one. For this reason, it is considered that under this context the commutative property does not pertain to them (Zaldívar, 2014), as a result:

$$(t^1_2 + t^1_{4-0} + t^1_{4-1}) \neq (t^1_{4-0} + t^1_2 + t^1_{4-1}) \qquad (4.4)$$

Instead, this phenomenon does not always occur when two sets with the same elements have different order, to illustrate, let look at the following set:

$$\psi^1 = (t^1_{4-0} + t^1_{4-1}) \qquad (4.5)$$

In score 4.5 is:

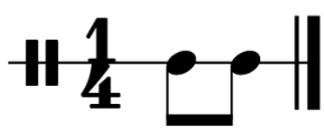

Figure 4.3: Two 8th notes in score.

And if the order of 4.5 is changed, it follows that:

$$\psi^1 = (t^1_{4-1} + t^1_{4-0}) \qquad (4.6)$$

Which is clearly identical to Figure 4.3:

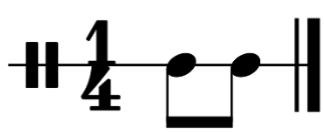

Figure 4.4: Two 8th notes in score.

When observing Figures 4.3 and 4.4, it could be seen that musically both are performed in the same way, so, the commutative property is preserved in these two sets, hence:

$$(t_{4-0}^{1}+t_{4-1}^{1})=(t_{4-1}^{1}+t_{4-0}^{1}) \tag{4.7}$$

Thus, the non-commutative property between two sets $\psi^{\dagger}$ occurs if and only if two sets have the same elements, but between them there is at least one different figure and in turn they have different order.

5. Axiom No. 4: Axiom of Musical Silence

The fourth axiom indicates that:

All musical silences are defined with $\widetilde{t_i}\ /\ \beta=0\ b$, however, although all silences are worth $\widetilde{t_i}=0\ min$, by musical property none are equal to each other.

$$\widetilde{t_i}\neq\widetilde{t_j}\ /\ \widetilde{t_i},\widetilde{t_j}=0 \tag{5.1}$$

To clarify the meaning of this axiom, first let assign to each musical rest a notation which is related to its temporal duration. Let be $\widetilde{t_1}$ quarter note rest, $\widetilde{t_2}$ half note rest, $\widetilde{t_3}$ dotted half note rest, $\widetilde{t_4}$ whole note rest, furthermore, $\widetilde{t_2^1}$ eighth note rest, $\widetilde{t_3^1}$ eighth note triplet rest, $\widetilde{t_4^1}$ sixteenth note rest.

Musical silence could be understood as the lack of pulse due to the fact that the interpretation of musical silence is assumed as the absence of a rhythmic figure (Schoenberg, 2016; Kania, 2010; Margulis, 2007). Therefore, it is stated that there is no pulse, so, musical silence could be considered to be mathematically represented when $\beta=0\ \mathrm{b}$, instead, a paradox arises at the moment of remembering the existence that there are different musical silences, since although all silences are worth $\tilde{t}=0$, the fact that each of them lasts different time causes that they are not equal, let see an example, $\widetilde{t_1}$ in score it follows that:

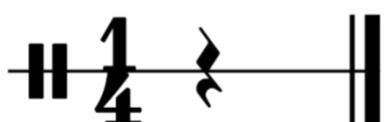

Figure 5.1: quarter note rest.

Whereas, silence $\widetilde{t_4}$ represents musically:

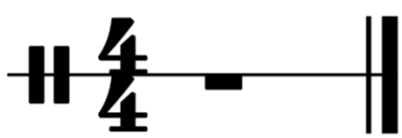

Figure 5.2: whole note rest.

Interpretatively, the rests in Figures 5.1 and 5.2 are not equal because they last different time, but both not having pulse, that is $\beta=0\ \mathrm{b}$, causes that the two rests are worth $\tilde{t}=0\ min$, from a mathematical point of view, this could be taken into account as a paradox, since it is being determined that musically $\widetilde{t_1}\neq\widetilde{t_4}$ but that mathematically both are worth 0, therefore, $\widetilde{t_1}=\widetilde{t_4}$, i.e., $\widetilde{t_1}\neq\widetilde{t_4}\Leftrightarrow\widetilde{t_1}=\widetilde{t_4}$, a form reminiscent of

Russell's paradox (Ferreirós, 2000). Conversely, by introducing musical characteristics to a mathematical context, it could be argued that it is correct to ignore the paradox and consider this phenomenon as part of the mathematical-musical nature, so:

$$\widetilde{t_\iota} \neq \widetilde{t_J} \;/\; \widetilde{t_\iota}, \widetilde{t_J} = 0 \tag{5.2}$$

Another consequence of this axiom appears as a logical deduction that dictates the existence of different zeros, since if $\widetilde{t_1} \neq \widetilde{t_2} \neq \widetilde{t_3} \neq \widetilde{t_4}$ that means $0 \neq 0 \neq 0 \neq 0$, in other words, there are four different kinds of zeros, as a result the conclusion is:

$$\forall \tilde{t} = 0 \; \exists 0 \neq 0 \tag{5.3}$$

6. Lemma No.1: Numbers Belonging to the Empty Set

The previously proposed axiom no. 3 stands out by itself in determining the noncommutative and commutative property for sets $\psi^{\dagger}$ within specific cases. Yet, the consequences of this axiom could become symbolic and controversial for axiomatic set theory and even in number theory (Lopez Mateos, 2017; Mora, 2012). In order to understand the reason for this conclusion, this topic will be further explored.

Let return to sets 4.2 and 4.3.

$$\psi^1 = (t_2^1 + t_4^1 + t_4^1) \tag{6.1}$$
$$\psi^1 = (t_4^1 + t_2^1 + t_4^1)$$

Let solve each rhythmic figure considering that $T = 100\ bpm$, as a result:

$$\psi^1 = (t_2^1 + t_4^1 + t_4^1) = \left(\frac{1}{100 \times 2} + \frac{1}{100 \times 4} + \frac{1}{100 \times 4}\right) = \frac{1}{100}\ min \tag{6.2}$$
$$\psi^1 = (t_4^1 + t_2^1 + t_4^1) = \left(\frac{1}{100 \times 4} + \frac{1}{100 \times 2} + \frac{1}{100 \times 4}\right) = \frac{1}{100}\ min$$

Therefore:

$$\psi^1 = \frac{1}{100}\ min \quad \psi^1 = \frac{1}{100}\ min \tag{6.3}$$

In the first instance, it is possible to start from the obvious fact that:

$$\psi^1 = \psi^1 \tag{6.4}$$

Consequently, according to axiom no. 3 it is known that:

$$(t_2^1 + t_4^1 + t_4^1) \neq (t_4^1 + t_2^1 + t_4^1) \tag{6.5}$$

Thus, following such an axiom, hence:

$$\psi^1 \neq \psi^1 \tag{6.6}$$

This would be represented numerically as:

(6.7)

$$\frac{1}{100}\ min \neq \frac{1}{100}\ min$$

If the definition of an empty set is conveniently introduced (Suppes, 1972):

$$\emptyset = (x: x \neq x) \tag{6.8}$$

And considering the result obtained in 6.6, it is possible to propose the hypothesis that:

$$\emptyset = \left(\frac{1}{100}\ min\right) \tag{6.9}$$

Thus the argument for writing the following lemma it follows that:

$$\left(t_i + t_j\right) \neq \left(t_j + t_i\right) \longrightarrow \forall\psi^{\dagger} \in \mathbb{R}^{+} \exists\ \psi^{\dagger} \neq \psi^{\dagger} \tag{6.10}$$

7. Theorem No.1: Different Empty Sets

Starting from lemma 6.10, this concept could be further expanded, since, if operating in a similar way in sets 6.2 and 6.3, but now being $T = 110\ bpm$ consequently:

$$\emptyset = \left(\frac{1}{110} min\right) \tag{7.1}$$

And such that:

$$\frac{1}{110} min \neq \frac{1}{100} min \tag{7.2}$$

It may be considered that:

$$\emptyset \neq \emptyset \tag{7.3}$$

For this reason, the hypothesis just proposed within this section is proved in advance ■, being that such a theorem is completed as follows:

$$\left(t_i + t_j\right) \neq \left(t_j + t_i\right) \longrightarrow (\forall\psi^{\dagger} \in \mathbb{R}^{+} \exists\ \psi^{\dagger} \neq \psi^{\dagger}) \therefore \psi^{\dagger} \in \emptyset \longrightarrow \emptyset \neq \emptyset \tag{7.4}$$

8. Corollary No.1: Obtaining Numbers by means of Algebra of Empty Sets

Let take sets 6.9 and 7.1:

$$\emptyset_i = \left(\frac{1}{100}\ min\right),\ \ \emptyset_j = \left(\frac{1}{110} min\right) \tag{8.1}$$

If operating a difference of two sets (Suppes, 1972), two results would be obtained according to the order of the sets, this suggests that:

Result No. 1:

$$\emptyset_i \setminus \emptyset_j = \left(\frac{1}{100}\ min\right) \tag{8.2}$$

Result No. 2:

(8.3)

$$\emptyset_j \setminus \emptyset_i = \left(\frac{1}{110} min\right)$$

Which means that the intersection of two empty sets results in a number, this suggests that:

$$n = \langle \emptyset_i = \{n\} \setminus \emptyset_j = \{m\} \rangle \tag{8.4}$$

And as a result

$$m = \langle \emptyset_j = \{m\} \setminus \emptyset_i = \{n\} \rangle$$

Yet, if instead of operating on a difference of two sets, an intersection is operated on, a real empty set would be solved, therefore:

$$\emptyset = \langle \emptyset_i = \{n\} \cap \emptyset_j = \{m\} \rangle \tag{8.5}$$

9. Conclusion

In this work four axioms have been constructed following the basic rules of music theory with respect to rhythmic figures, as well as mathematics under the concept of set, consequently, a lemma, a theorem and a corollary were established respectively.

It should be noted that, the initial objective of this article was to propose only the mathematical bases for the development of a more studied and rigorous theory of rhythmic sets that, in future works could be complemented to the Pitch-class set theory. Yet, as it was observed, the action of axiomatizing only the rhythmic figures as independent musical parameters generates conclusions that fall into the ambiguity of the term paradox, since it is inevitable not to consider that the consequences of these four axioms stand out by themselves and that if they are accepted, they could affect the way in which the theory of numbers and sets is conceived (Mora, 2012; Suppes, 1972).

To begin with, the first two axioms do not show any conflict for mathematics because they are based on musical properties belonging to the figures that satisfy the mathematical rules. Alternatively, it is the third and fourth axioms that could generate some controversy from a mathematical point of view, because to think that the void has as an element a real number and that there are also different empty sets as different zeros, is in every sense of the word: absurd.

And it is here where it is possible to stop and reflect on two valid conclusions: the first is to consider these results as meaningless paradoxes and discard everything worked within this research. The second option is to analyze it not from a pure mathematical point of view, but from a mathematical and musical perspective emphasizing the difference between both branches. This suggests that, in the mathematical-musical, the existence of different empty sets and zeros is manifested in an intuitive and logical way, while, in conventional mathematics this does not occur.

To make this distinction, perhaps, would avoid the problems that could be caused within pure mathematics if these ideas are introduced, since, although the mathematical concepts are respected to represent the musical phenomenology, it does not mean that their consequences are direct to mathematics or that they should be taken into account for that field. Instead, they affect only musical mathematics, which should not be related to pure mathematics or at least not its consequences.

Given the above, this work could be summarized in three objectives. The first is to propose four axioms for rhythmic figures from their musical properties, the second is to study the consequences of these axioms, that is, their theorems and corollaries, and finally, to make the distinction between mathematics and music-mathematics.

Ethical Statement of the Research

In this study, all the rules specified in the "Directive on Scientific Research and Publication Ethics of Higher Education Institutions" were followed. None of the actions specified under the second section of the Directive, "Actions Contrary to Scientific Research and Publication Ethics", were carried out.

In addition, according to ULAKBIM TR Index 2020 criteria, there was no need for any data collection requiring ethics committee approval in the study.

Declaration of the Contribution Rate of the Researchers to the Article:

1st author contribution rate: 100%

Conflict of Interest Statement:

There are no personal or financial conflicts of interest between the authors.

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