Yuzuncu Yil University
Journal of the Institute of Natural & Applied Sciences
https://dergipark.org.tr/en/pub/yyufbed

Research Article

On the Unit Group of the Integral Group Ring $\mathbb{Z}(S_3 \times C_3)$ [#]

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Article Info

Received: 18.09.2023
Accepted: 18.12.2023
Online April 2024

DOI:10.53433/yyufbed.1361776

Keywords
Complex representation,
Cyclic group,
Direct product group,
Group rings,
Integral group rings,
Symmetric group

Abstract: Describing the group of units in the integral group ring is a famous and classical open problem. Let S_3 and C_3 be the symmetric group of order 6 and a cyclic group of order 3, respectively. In this paper, a description of the units of the integral group ring $\mathbb{Z}(S_3 \times C_3)$ of the direct product group $S_3 \times C_3$ concerning a complex representation of degree two is given. As a result, a part of the conjecture which is introduced in (Low, 2008) and related to group rings over a complex integral domain is resolved using representation theory.

$\mathbb{Z}(S_3 \times C_3)$ İntegral Grup Halkasındaki Birimsel Elemanlar Grubu Üzerine

Makale Bilgileri

Geliş: 18.09.2023
Kabul: 18.12.2023
Online Nisan 2024

DOI:10.53433/yyufbed.1361776

Anahtar Kelimeler
Devirli grup,
Direkt çarpım grubu,
Grup halkaları,
İntegral grup halkaları,
Kompleks temsil,
Simetrik grup

Öz: Verilen bir sonlu grubun integral grup halkasındaki birimsel elemanların grubunu belirlemek çoğu grup için meşhur ve klasik bir açık problemdir. S_3 ve C_3 sırasıyla 6 mertebeli simetrik grup ve 3 mertebeli bir devirli grup olsun. Bu makalede, iki dereceli bir kompleks temsile göre $S_3 \times C_3$ direkt çarpım grubunun $\mathbb{Z}(S_3 \times C_3)$ integral grup halkasının birimsel elemanlarının yapısı verilmektedir. Sonuç olarak, kompleks bir tamlık bölgesi üzerinde tanımlı grup halkalarına ilişkin (Low, 2008)' de sunulan konjektürün bir kısmı, temsil teorisi kullanılarak çözülmüştür.

1. Introduction

We denote the integral group ring of a given finite group G over the ring of integers by $\mathbb{Z}G$. Its elements are all finite sums of the form $\sum_{g\in G} \alpha_g g$ where $\alpha_g \in \mathbb{Z}$. The ring epimorphism defined by

[#] This paper has been generated from the corresponding author's Ph.D. thesis conducted under the supervision of Assoc. Prof. Dr. İsmail Hakkı DENİZLER.

$\epsilon: \mathbb{Z}G \longrightarrow \mathbb{Z}$, $\epsilon(g) = 1$, is called the *augmentation map*. The kernel of ϵ is the *augmentation ideal* $\Delta_{\mathbb{Z}}(G) = \{g - 1: g \in G\}$.

The group of units of $\mathbb{Z}G$ is denoted by $U(\mathbb{Z}G)$ and the group of units (of augmentation one) is denoted by $U_1(\mathbb{Z}G)$. Clearly, $\pm U_1(\mathbb{Z}G) = U(\mathbb{Z}G)$ and the set $\pm G$ are the *trivial units* in $U(\mathbb{Z}G)$.

Describing units of integral group rings is a difficult and open classical problem. Over the years, it has drawn the attention of those working in the areas of algebra, number theory, and algebraic topology. Most descriptions of $U(\mathbb{Z}G)$ in the mathematical literature either give an explicit description of unit group, the general structure of $U(\mathbb{Z}G)$, or a subgroup of finite index of the unit group $U(\mathbb{Z}G)$. Interested readers can read more about group rings in (Jespers & del Rio A, 2016) and (Milies & Sehgal, 2002). Additional information can be found in Sehgal's comprehensive study (Sehgal, 1993) on the unit problem in integral group rings.

Units, as well as other special elements in a group ring such as idempotent, nilpotent, etc., are useful elements to obtain novel results in group ring theory. Küsmüş (2020) introduced some results related to units which are generated from idempotent elements of group rings. Hanoymak & Küsmüş (2023) studied some units which are correlated with nilpotent elements in group rings. Jespers & Parmenter (1993) described $U(\mathbb{Z}G)$ for groups of order 16. Jespers & Parmenter (1992) described $U(\mathbb{Z}G)$ where G is the dihedral group of order twelve and for $G = D_8 \times C_2$. Kelebek & Bilgin (2014) gave the structure of $U(\mathbb{Z}(C_n \times V))$ where V is a Klein-4 group. Bilgin, Küsmüş & Low (2016) determined the unit group of the integral group ring of

$$T \times C_2 = \langle a, b, x: a^6 = x^2 = 1, a^3 = b^2, bab^{-1} = a^{-1}, ax = xa, bx = xb\rangle$$

as a semidirect product of finitely generated free groups.

In (Low, 1998 and 2008), a general algebraic framework was developed to study $U(\mathbb{Z}G^*)$, where $G^* = G \times C_p$, where p is prime and G is a finite group. Low (2008) asserts that an implicit characterization of $U(\mathbb{Z}(G \times C_p))$ depends on an understanding of the structure of $U(RG)$, where $R = \mathbb{Z}[\zeta]$ is a complex integral domain, for prime $p \geq 3$ and ζ is a primitive pth root of unity.

Eisele et al. (2015) have introduced a general (but implicit) characterization of the unit group of $\mathbb{Z}G$ for some finite groups up to commensurability, using Wedderburn decompositions and idempotent elements from $\mathbb{Q}Ge$. Their method is based on exploring an isomorphism between components of Wedderburn decompositions of $\mathbb{Q}G$ and $\mathbb{Q}Ge$, where $e \in \mathbb{Q}Ge_i$ is a non-central idempotent element (Eisele et al., 2015). In our paper, we do not utilize idempotent elements as in (Eisele et al., 2015). We focus on the open problem which is related to the unit group of integral group ring $\mathbb{Z}(G \times C_p)$, using its ideals where G is a symmetric group of order 6 and $p = 3$ found in (Low, 2008). In particular, we characterize the unit group of the integral group ring of

$$S_3^* := S_3 \times C_3 = \langle a, b, x: a^3 = b^2 = x^3 = 1, bab^{-1} = a^{-1}, ax = xa, bx = xb\rangle$$

in terms of a decomposition of ideals of $\mathbb{Z}S_3^*$. To do this, we utilize linear extensions of a specific complex representation of degree two of the group S_3^* to some ideals of its integral group ring and correlate units in $\mathbb{Z}S_3^*$ with the units in some matrix rings.

2. Material and Methods

We first recall the theorem found in (Jespers, 1995) and (Jespers & Parmenter, 1992).

Theorem 2.1. In $U_1(\mathbb{Z}S_3)$, S_3 has a torsion-free normal complement which is generated by bicyclic units as $U_1(\mathbb{Z}S_3) = V \rtimes S_3$ such that $V = \langle u_{b,a}, u_{ba,a}, u_{ba^2,a}\rangle$, where

$$u_{b,a} = 1 + (1 - b)a(1 + b),$$
$$u_{ba,a} = 1 + (1 - ba)a(1 + ba),$$
$$u_{ba^2,a} = 1 + (1 - ba^2)a(1 + ba^2).$$

Since $S_3^* := S_3 \times C_3 = \langle a, b, x : a^3 = b^2 = x^3 = 1, bab^{-1} = a^{-1}, ax = xa, bx = xb \rangle$,
we have

$$\mathbb{Z}S_3^* \simeq (\mathbb{Z}S_3)C_3 = \{c_0 + c_1 x + c_2 x^2 : c_i \in \mathbb{Z}S_3\},$$

$$\mathbb{Z}S_3^* \simeq (\mathbb{Z}C_3)S_3 = \{c_0' + c_1' a + c_2' a^2 + c_3' b + c_4' ba + c_5' ba^2 : c_i' \in \mathbb{Z}C_3\},$$

$$\mathbb{Z}S_3^* \simeq \mathbb{Z}[(C_3 \times C_3) \rtimes C_2] \simeq \{c_1'' + c_1'' b : c_i'' \in \mathbb{Z}(C_3 \times C_3)\}.$$

Now, let π_g denote the natural projection defined over S_3^* with $\pi(g) = 1$. We can linearly extend π_g to the integral group ring of S_3^*. Here, we can define the ring epimorphisms as $\pi_x : \mathbb{Z}S_3^* \longrightarrow \mathbb{Z}S_3$,

$$\pi_x(c_0 + c_1 x + c_2 x^2) = c_0 + c_1 + c_2$$

and

$$\pi_a : \mathbb{Z}S_3^* \simeq (\mathbb{Z}C_3)S_3 \longrightarrow \mathbb{Z}\langle b, x \rangle \simeq \mathbb{Z}(C_2 \times C_3) \simeq (\mathbb{Z}C_3)C_2$$

such that

$$\pi_a\left(\sum_{i=0}^{2} \alpha_i a^i + \beta_i b a^i\right) = \sum_{i=0}^{2} \alpha_i + \beta_i b,$$

where $c_i \in \mathbb{Z}S_3$ and $\alpha_i, \beta_i \in \mathbb{Z}\langle x \rangle \simeq \mathbb{Z}C_3$, respectively. Thus we observe that

$$Ker(\pi_x) = \mathbb{Z}S_3(1 - x) \oplus \mathbb{Z}S_3(1 - x^2)$$

and

$$Ker(\pi_a) = \mathbb{Z}\langle b, x \rangle(1 - a) \oplus \mathbb{Z}\langle b, x \rangle(1 - a^2).$$

Restricting π_x to $Ker(\pi_a)$, the image of an element of the form

$$\gamma = (\gamma_0 + \gamma_1 x + \gamma_2 x^2)(1 - a) + (\delta_0 + \delta_1 x + \delta_2 x^2)(1 - a^2)$$

is

$$\pi_x(\gamma) = \sum_{i=0}^{2} \gamma_i(1 - a) + \sum_{i=0}^{2} \delta_i(1 - a^2),$$

where $\gamma_i, \delta_i \in \mathbb{Z}\langle b \rangle$. Thus, kernel of this restriction (denoted by K^x) is

$$K^x = \{k_1\alpha_{11} + k_2\alpha_{12} + k_3\alpha_{21} + k_4\alpha_{22} : k_i \in \mathbb{Z}\langle b \rangle\} \subseteq Ker(\pi_x)$$

where $\alpha_{ij} = (1 - x^i)(1 - a^j)$ for $i, j \in \{1,2\}$. Note that this is a direct sum. That is,

$$K^x = \prod_j \prod_i \mathbb{Z}\langle b \rangle \alpha_{ij}.$$

3. Results

Proposition 3.1. Let K^a be the kernel of π_a, restricted to $Ker(\pi_x)$. Then, $K^a = K^x$.

Proof. For an element $w \in K^a$, we know that $w \in Ker(\pi_x)$ and $\pi_a(w) = 0$. This implies that $\pi_x(w) = 0$ and so $w \in K^x$. Therefore, $K^a \subseteq K^x$. The converse of this inclusion is similarly shown.

Note that these kernels yield a noncommutative $\mathbb{Z}\langle b\rangle$-algebra which we denote by $J_4 = \langle \alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}\rangle_{\mathbb{Z}\langle b\rangle}$. In J_4, the following hold:

$$\alpha_{11}^2 = 4\alpha_{11} - 2\alpha_{12} - 2\alpha_{21} + \alpha_{22}, \quad (1)$$

$$\alpha_{12}^2 = -2\alpha_{11} + 4\alpha_{12} + \alpha_{21} - 2\alpha_{22}, \quad (2)$$

$$\alpha_{21}^2 = -2\alpha_{11} + \alpha_{12} + 4\alpha_{21} - 2\alpha_{22}, \quad (3)$$

$$\alpha_{22}^2 = \alpha_{11} - 2\alpha_{12} - 2\alpha_{21} + 4\alpha_{22}, \quad (4)$$

$$\alpha_{11}\alpha_{12} = \alpha_{12}\alpha_{11} = 2\alpha_{11} + 2\alpha_{12} - \alpha_{21} - \alpha_{22}, \quad (5)$$

$$\alpha_{11}\alpha_{21} = \alpha_{21}\alpha_{11} = 2\alpha_{11} - \alpha_{12} + 2\alpha_{21} - \alpha_{22}, \quad (6)$$

$$\alpha_{11}\alpha_{22} = \alpha_{22}\alpha_{11} = \alpha_{11} + \alpha_{12} + \alpha_{21} + \alpha_{22}, \quad (7)$$

$$\alpha_{12}\alpha_{21} = \alpha_{21}\alpha_{12} = \alpha_{11} + \alpha_{12} + \alpha_{21} + \alpha_{22}, \quad (8)$$

$$\alpha_{12}\alpha_{22} = \alpha_{22}\alpha_{12} = -\alpha_{11} + 2\alpha_{12} - \alpha_{21} + 2\alpha_{22}, \quad (9)$$

$$\alpha_{21}\alpha_{22} = \alpha_{22}\alpha_{21} = -\alpha_{11} - \alpha_{12} + 2\alpha_{21} + 2\alpha_{22} \quad (10)$$

and

$$b\alpha_{11} = \alpha_{12}b,\ \ b\alpha_{12} = \alpha_{11}b,\ \ b\alpha_{22} = \alpha_{21}b,\ \ b\alpha_{21} = \alpha_{22}b. \quad (11)$$

Theorem 3.1. Let J_2 denote $\mathbb{Z}\langle b\rangle(1-a) \oplus \mathbb{Z}\langle b\rangle(1-a^2)$. Then,

$$U_1(1+J_2) \rtimes \langle b\rangle \simeq V \rtimes \langle a\rangle \rtimes \langle b\rangle$$

where V is a torsion-free normal complement of S_3.

Proof. We know that $J_2 = \mathbb{Z}\langle b\rangle(1-a) \oplus \mathbb{Z}\langle b\rangle(1-a^2)$ is the kernel of the map $\pi_a\colon \mathbb{Z}S_3 \longrightarrow \mathbb{Z}\langle b\rangle$ by $a \mapsto 1$. It is known that if I is an augmentation ideal, then the augmentation of I is 0. Hence, the unit group which is generated from I lies in $1 + I$, due to the fact that the augmentation of a unit has to be one. Thus, $U_1(1+J_2) = (1+J_2) \cap U_1(\mathbb{Z}S_3^*)$ is the kernel of π_a (restricted to the unit groups) with the exact sequence

$$1 \to U_1(1+J_2) \xrightarrow{i} U_1(\mathbb{Z}S_3) \xrightarrow{\pi_a} U_1(\mathbb{Z}\langle b\rangle) \to 1.$$

Since $U_1(\mathbb{Z}\langle b\rangle) \hookrightarrow U_1(\mathbb{Z}S_3)$ is an embedding, the sequence splits as

$$U_1(\mathbb{Z}S_3) = U_1(1+J_2) \rtimes U_1(\mathbb{Z}\langle b\rangle).$$

As $U_1(\mathbb{Z}\langle b\rangle) = \langle b\rangle$ and $U_1(\mathbb{Z}S_3) = V \rtimes S_3$ from Theorem 2.1, we see that

$$U_1(1+J_2) \rtimes \langle b\rangle \simeq V \rtimes \langle a\rangle \rtimes \langle b\rangle.$$

Theorem 3.2. Let J_3 denote $\mathbb{Z}\langle b\rangle(1-x) \oplus \mathbb{Z}\langle b\rangle(1-x^2)$. Then,

$$U_1(1+J_3) = \langle x\rangle \simeq C_3.$$

Proof. Let $J_3 = \mathbb{Z}\langle b\rangle(1-x) \oplus \mathbb{Z}\langle b\rangle(1-x^2)$. In particular,
$$J_3 = \{(k_0+k_1b)(1-x) + (k_2+k_3b)(1-x^2): k_i \in \mathbb{Z}\}.$$

Notice that $\langle b,x\rangle \simeq \langle bx: o(bx) = 6\rangle \simeq C_6$. Then, J_3 is the kernel of the ring epimorphism $\pi_x: \mathbb{Z}\langle bx\rangle \longrightarrow \mathbb{Z}\langle b\rangle$, given by

$$\pi_x(\sum_{i=0}^{5} c_i\big(bx)^i\big) = (c_1+c_3+c_5) + (c_0+c_2+c_4)b.$$

Using the linearity of π_x, we have $\tilde{\pi}_x: U_1(\mathbb{Z}\langle bx\rangle) \longrightarrow U_1(\mathbb{Z}\langle b\rangle)$ and

$$Ker(\tilde{\pi}_x) = U_1(1+J_3) = (1+J_3) \cap U_1(\mathbb{Z}C_6).$$

It follows that

$$1 \to U_1(1+J_3) \overset{i}{\to} U_1(\mathbb{Z}\langle bx\rangle) \overset{\tilde{\pi}_x}{\to} U_1(\mathbb{Z}\langle b\rangle) \to 1$$

and

$$U_1(\mathbb{Z}\langle bx\rangle) = U_1(1+J_3) \times U_1(\mathbb{Z}\langle b\rangle).$$

It is well known that the unit groups of integral group rings of abelian groups of order 2,3,4 and 6 have to be trivial (Sehgal, 1993). Hence, $U_1(\mathbb{Z}\langle bx\rangle) \simeq U_1(\mathbb{Z}C_6) = \langle bx\rangle$ and

$$U_1(\mathbb{Z}\langle b\rangle) \simeq U_1(\mathbb{Z}C_2) = \langle b\rangle.$$

Thus, $C_6 \simeq U_1(1+J_3) \times C_2$ and we conclude that $U_1(1+J_3) = \langle x\rangle \simeq C_3$.

Theorem 3.3. Let $J_4 = \langle \alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}\rangle_{\mathbb{Z}\langle b\rangle}$ where $\alpha_{ij} = \big(1-x^i\big)\big(1-a^j\big)$, as described in equations (1)-(11). Then,

$$U_1(1+J_4) = \{1 + c_0W_1 + c_1W_2: c_0, c_1 \in \mathbb{Z}\}$$

where $W_1 = (1-b)(1-x)(a^2-a)$ and $W_2 = (1-b)(1-x^2)(a^2-a)$.

Proof. We know that $U_1(1+J_4)$ is implicitly composed of units of the form

$$u = 1 + (k_0+l_0b)\alpha_{11} + (k_1+l_1b)\alpha_{12} + (k_2+l_2b)\alpha_{21} + (k_3+l_3b)\alpha_{22}.$$

By applying a linear extension of the projection operator

$$\pi_b(\alpha) = \begin{cases} 1, & \alpha = b \\ \alpha, & \alpha \neq b \end{cases}$$

to J_4, it is possible to separate this ideal into two distinct parts. Using the short exact sequence

$$0 \to Ker(\pi_b) \overset{i}{\to} J_4 \overset{\pi_b}{\to} Im(\pi_b) \to 0,$$

note that

$$Im(\pi_b) = \{c_0\alpha_{11} + c_1\alpha_{12} + c_2\alpha_{21} + c_3\alpha_{22} : c_i \in \mathbb{Z}\}.$$

When we move the above sequence to the subgroup $U_1(1+J_4)$ of units, we obtain the sequence

$$1 \to 1 + Ker(\pi_b) \xrightarrow{i} U_1(1+J_4) \xrightarrow{\pi_b} 1 + Im(\pi_b) \to 1.$$

We claim that $1 + Im(\pi_b)$ has no nontrivial units. To see this, consider $\rho(1 + Im(\pi_b))$, where $\rho: S_3^* \longrightarrow GL(2, \langle\omega\rangle)$ is a complex representation of degree two such that

$$\rho(a) = \begin{bmatrix} \omega & 0 \\ 0 & \omega^2 \end{bmatrix}, \rho(b) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \rho(x) = \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix} = \omega I_2$$

and ω is the primitive 3rd root of unity. Restriction of ρ to ideal J_4 gives the following representations:

$$\rho(\alpha_{11}) = \begin{bmatrix} 2(1-\omega) - (1-\omega^2) & 0 \\ 0 & (1-\omega) + (1-\omega^2) \end{bmatrix},$$

$$\rho(\alpha_{12}) = \begin{bmatrix} (1-\omega) + (1-\omega^2) & 0 \\ 0 & 2(1-\omega) - (1-\omega^2) \end{bmatrix},$$

$$\rho(\alpha_{21}) = \begin{bmatrix} (1-\omega) + (1-\omega^2) & 0 \\ 0 & -(1-\omega) + 2(1-\omega^2) \end{bmatrix},$$

$$\rho(\alpha_{21}) = \begin{bmatrix} (1-\omega) + (1-\omega^2) & 0 \\ 0 & -(1-\omega) + 2(1-\omega^2) \end{bmatrix},$$

$$\rho(\alpha_{22}) = \begin{bmatrix} -(1-\omega) + 2(1-\omega^2) & 0 \\ 0 & (1-\omega) + (1-\omega^2) \end{bmatrix}.$$

As a linear operator, extending ρ to integral group ring $\mathbb{Z}S_3^*$ yields the representation of a unit in $1 + Im(\pi_b)$ as

$$\rho(u) = \begin{bmatrix} u_1 & 0 \\ 0 & u_2 \end{bmatrix}$$

with inverse

$$\rho(u)^{-1} = \begin{bmatrix} u_1^{-1} & 0 \\ 0 & u_2^{-1} \end{bmatrix}$$

where

$$u_1 = 1 + (2c_0 + c_1 + c_2 - c_3)(1-\omega) + (-c_0 + c_1 + c_2 + 2c_3)(1-\omega^2),$$

$$u_2 = 1 + (c_0 + 2c_1 - c_2 + c_3)(1-\omega) + (c_0 - c_1 + 2c_2 + c_3)(1-\omega^2).$$

As $Det(\rho(u)) = u_1u_2 \in U_1(\mathbb{Z}\langle\omega\rangle)$ and $U_1(\mathbb{Z}\langle\omega\rangle) \simeq U_1(\mathbb{Z}C_3) \simeq C_3$, we see that both u_1 and u_2 must be trivial units. Moreover, notice that $u_1 = u_2 = 1$. Thus, we conclude that

$$U_1(1 + J_4) = \left(1 + Ker(\pi_b)\right) \cap U_1(1 + J_4).$$

Recall that

$$Ker(\pi_b) = \{(1 - b)(d_0\alpha_{11} + d_1\alpha_{12} + d_2\alpha_{21} + d_3\alpha_{22}): d_i \in \mathbb{Z}\}.$$

Since $v \in (1 + Ker(\pi_b)) \cap U_1(1 + J_4)$, the representation $\rho(v)$ is obtained as

$$\rho(v) = \begin{bmatrix} 1 + v_1 & -v_2 \\ -v_1 & 1 + v_2 \end{bmatrix}$$

where

$$v_1 = (2d_0 + d_1 + d_2 - d_3)(1 - \omega) + (-d_0 + d_1 + d_2 + 2d_3)(1 - \omega^2)$$

and

$$v_2 = (d_0 + 2d_1 - d_2 + d_3)(1 - \omega) + (d_0 - d_1 + 2d_2 + d_3)(1 - \omega^2).$$

Observe that $Det(\rho(v)) = 1 + v_1 + v_2$ is invertible if and only if

$$Det\left(\rho(u)\right) = 1 + (1 - \omega)(3d_0 + 3d_1) + (1 - \omega^2)(3d_2 + 3d_3) \in U_1(\mathbb{Z}\langle\omega\rangle) = \langle\omega\rangle.$$

Hence, we deduce that $d_0 + d_1 = 0$ and $d_2 + d_3 = 0$. This shows us that v can be written in the form $v = 1 + d_0W_1 + d_2W_2$.

Since W_1 and W_2 are nilpotent elements of nilpotency index two and $W_1W_2 = W_2W_1 = 0$, the subgroup $U_1(1 + J_4)$ consists of units of form $v = 1 + d_0W_1 + d_2W_2$ with the inverse $v^{-1} = 1 - d_0W_1 - d_2W_2$. It is clear that $U_1(1 + J_4)$ is torsion-free since

$$u^n = 1 + nc_0W_1 + nc_1W_2$$

for any $n \in \mathbb{Z}$. Thus, $U_1(1 + J_4) = \{1 + c_0W_1 + c_1W_2: c_0, c_1 \in \mathbb{Z}\}$.

Corollary 3.1. $U(\mathbb{Z}S_3^*) = (U_1(1 + J_4) \times C_3) \rtimes (V \rtimes S_3)$ where $C_3 = \langle x\rangle$, V is a torsion-free normal complement of S_3 and $U_1(1 + J_4) = \{1 + c_0W_1 + c_1W_2: c_0, c_1 \in \mathbb{Z}\}$ such that W_1 and W_2 are as obtained in Theorem 3.3.

Proof. We construct the following commutative diagram of exact sequences:

$$\begin{array}{ccccc} J_4 & \xrightarrow{i} & Z\langle b, x\rangle[(1 - a) \oplus (1 - a^2)] & \xrightarrow{\pi_x} & J_2 \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Z}S_3[(1 - x) \oplus (1 - x^2)] & \xrightarrow{i} & \mathbb{Z}S_3^* & \xrightarrow{\pi_x} & \mathbb{Z}S_3 \\ \downarrow & & \downarrow & & \downarrow \\ J_3 & \xrightarrow{i} & \mathbb{Z}\langle b, x\rangle & \xrightarrow{\pi_x} & \mathbb{Z}\langle b\rangle \end{array}$$

Since the inverses of the projections π_a and π_x are embeddings, the sequences split in all of the rows and columns. This implies that

$$\mathbb{Z}S_3^* = [\mathbb{Z}S_3(1 - x) \oplus \mathbb{Z}S_3(1 - x^2)] \rtimes \mathbb{Z}S_3,$$

$$\mathbb{Z}S_3^* = [\mathbb{Z}\langle b, x\rangle(1 - a) \oplus \mathbb{Z}\langle b, x\rangle(1 - a^2)] \rtimes \mathbb{Z}\langle b, x\rangle,$$

$$\mathbb{Z}S_3 = J_2 \rtimes \mathbb{Z}\langle b\rangle,$$

$$J_4 \rtimes J_3 = \mathbb{Z}S_3(1-x) \oplus \mathbb{Z}S_3(1-x^2),$$

$$J_4 \rtimes J_2 = \mathbb{Z}\langle b,x\rangle(1-a) \oplus \mathbb{Z}\langle b,x\rangle(1-a^2),$$

$$\mathbb{Z}\langle b,x\rangle = J_3 \times \mathbb{Z}\langle b\rangle.$$

Moving these sequences and split extensions to the unit groups level, we have

$$\begin{array}{ccccc} U(1+J_4) & \overset{i}{\to} & U(1+Z\langle b,x\rangle[(1-a)\oplus(1-a^2)]) & \overset{\pi_x}{\to} & U(1+J_2) \\ \downarrow & & \downarrow & & \downarrow \\ U(1+\mathbb{Z}S_3[(1-x)\oplus(1-x^2)]) & \overset{i}{\to} & U(\mathbb{Z}S_3^*) & \overset{\pi_x}{\to} & U(\mathbb{Z}S_3) \\ \downarrow & & \downarrow & & \downarrow \\ U(1+J_3) & \overset{i}{\to} & U(\mathbb{Z}\langle b,x\rangle) & \overset{\pi_x}{\to} & U(\mathbb{Z}\langle b\rangle) \end{array}$$

We can conclude that

$$U(\mathbb{Z}S_3^*) = U(1+J_4) \rtimes U(1+J_3) \rtimes U(1+J_2) \rtimes U(\mathbb{Z}\langle b\rangle).$$

Using Theorems 3.2. and 3.3., we obtain

$$U(\mathbb{Z}S_3^*) = (U_1(1+J_4) \times C_3) \rtimes (V \rtimes S_3).$$

4. Discussion and Conclusion

Keeping in mind that S_3 has a torsion-free normal complement which is generated by bicyclic units and every $\mathbb{Z}$-module has a complex representation of finite degree, we have reduced the problem of describing the unit group of $\mathbb{Z}S_3^*$ to the problem of detecting units in matrix rings. We have moved a complex representation of S_3 of degree 2 to a matrix ring whose entries are from the complex integral domain $\mathbb{Z}[\omega]$ where $\omega^3 = 1$. It is clear that $\mathbb{Z}[\zeta] = \mathbb{Z}$ when $\zeta^2 = 1$ and so we have characterization of the unit group of integral group ring $\mathbb{Z}(G \times C_2)$ in terms of the unit group in $\mathbb{Z}G$ for a given finite group G and a cyclic group C_2 of order 2 as in (Low, 2008). However, giving an explicit description of $U(\mathbb{Z}(G \times C_p))$ through ideals of $\mathbb{Z}(G \times C_p)$ and its complex representations over $\mathbb{Z}[\theta]$ is still an open problem when $p \geq 3$ and θ is primitive pth root of unity. In this paper, we have given an approach to this problem for $p = 3$ and proved that $U(ZS_3^*)$ consists of trivial units, torsion-free units and torsion-free normal complement of S_3, using representation theory of finite groups.

The problem of establishing an explicit characterization of the unit group in the integral group ring for a given (as a general form) direct product group still remains an open problem. In this regard, we believe that the topic holds significant interest for researchers working on group rings and the representation theory of finite groups. This serves as a substantial source of motivation for future research endeavors.

Acknowledgements

The corresponding author of this paper would like to express gratitude to Dear Mr. DENİZLER for his encouragement regarding this article, as well as (Küsmüş, 2020) and (Hanoymak & Küsmüş, 2023).

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