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NOTE ON APPROXIMATION OF TRUNCATED BASKAKOV OPERATORS

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ABSTRACT. In this paper, we firstly introduce nonlinear truncated Baskakov operators on compact intervals and obtain some direct theorems. Also, we give the approximation of fuzzy numbers by truncated nonlinear Baskakov operators.

1. INTRODUCTION

Zadeh introduced the ideas of fuzzy sets, in [1]. Fuzzy numbers represented by proper intervals are a significant subject with numerous applications in a wide range of disciplines. Furthermore, it is well recognized that working with fuzzy numbers can be complicated due to the complicated approaches in which their membership function shapes are represented. Therefore, using trapezoidal or triangular fuzzy members to approximate fuzzy numbers, many studies have recently been published (see [13]-[19]).

The core topic of Korovkin type is the approximation of a continuous function by a series of linear positive operators (see [20],[21]). Bede et al. [6] have recently proposed nonlinear positive operators in place of linear positive operators. Although the Korovkin theorem fails for these nonlinear operators, they behave similarly to linear operators in terms of approximation.

The purpose of the paper is to use called max-product Truncated Baskakov operator which is given in the book [6] by applying continuous membership functions to approximate fuzzy integers. These operators additionally maintain the quasi-concavity in a manner analogous to the specific state of the unit interval will be given. These results turn out to be particularly useful for fuzzy numbers since they will enable us to construct fuzzy numbers with the same support in a straightforward manner. Additionally, these operators provide a good order of approximation for the (non-degenerate) segment core.

Baskakov [3] demonstrated the positive and linear operators, which are typically associated to functions that are bounded and uniformly continuous to $\nu \in C[0, +\infty]$ and specified

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by

$$V_n(\nu)(\theta) = (1+\theta)^{-n} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \theta^k (1+\theta)^{-k} \nu\left(\frac{k}{n}\right), \ \forall n \in \mathbb{N}. \tag{1.1}$$

It is known that the pointwise approximation result (see [4]) exists as:

$$| V_n(\nu)(\theta) - \nu(\theta) | \leq C\omega_2^{\varphi}\left(\nu; \sqrt{\theta(1+\theta)/n}\right), \ \theta \in [0,\infty), n \in \mathbb{N},$$

where $\varphi(\theta) = \sqrt{\theta(1+\theta)}$ and $I = [0,\infty)$. In this case,
$I_h = [h^2/(1-h)^2, +\infty)$, $h \leq \delta < 1$. Additionally, $V_n(\nu)$ satisfies the convexity and monotonicity of the function ν on $[0,+\infty)$ (see [5]).

The truncated Baskakov operators are identified by

$$U_n(\nu)(\theta) = (1+\theta)^{-n} \sum_{k=0}^{n} \binom{n+k-1}{k} \theta^k (1+\theta)^{-k} \nu\left(\frac{k}{n}\right), \ \nu \in C[0,1].$$

Truncated Baskakov operator of max product kind $\nu : [0,1] \to \mathbb{R}$ are identified by (see [6])

$$U_t^{(M)}(\nu)(\theta) = \frac{\bigvee_{k=0}^{t} b_{t,k}(\theta)\nu\left(\frac{k}{t}\right)}{\bigvee_{k=0}^{t} b_{t,k}(x)}, \ \theta \in [0,1], t \in \mathbb{N}, \ t \geq 1,$$

where $b_{t,k}(\theta) = \binom{t+k-1}{k}\theta^k(1+\theta)^{-t-k}$, $t \geq 1$, $\theta \in [0,1]$. For any function ν, $U_t^{(M)}(\nu)(\theta)$ is positive, continuous on $[0,1]$ and provides $U_t^{(M)}(\nu)(0) = \nu(0)$ for all $t \in \mathbb{N}, t \geq 2$ (see in [6], Lemma 4.2.1). In [7], authors showed that the uniform approximation order in the entire class $C_+([0,1])$ of positive continuous functions on $[0,1]$ cannot be developed, so there is a function $\nu \in C_+([0,1])$ that the approximation order by the truncated max-product Baskakov operator is $C\omega_1(\nu, 1/\sqrt{t})$. The fundamentally better order of approximation $\omega_1(\nu, 1/t)$ was attained for some functional subclasses, such as the nondecreasing concave functions. Finally, some shape preserving properties were proved. In this work, by demonstrating that the uniform approximation order of the truncated Baskakov operators of max-product kind is the same as in the specific case of the unit interval, we expand their definition to an arbitrary compact interval. Then it is shown that these operators maintain the quasi-concavity. These solutions show out to be particularly suited in the approximation of fuzzy numbers since they enable us to construct fuzzy numbers with the same support in a straightforward manner.

2. Preliminaries

Definition 2.1. *([10],[22])(**fuzzy numbers**) Let u be a fuzzy subset of $\mathbb{R}$ with membership function $\mu_u(\theta) : \mathbb{R} \longrightarrow [0,1]$. Then u is called a fuzzy number if:*

(1) u is normal, i.e. $\exists \theta_0 \in \mathbb{R}$ such that $\mu_u(\theta_0) = 1$;

(2) u is fuzzy convex, i.e. $\mu_u(\lambda\theta + (1-\lambda)\vartheta) \geq \min\{\mu_u(\theta), \mu_u(\vartheta)\}$;

(3) μ_u is upper semicontinuous; and

(4) Let (cl) be the closure operator .i.e. $supp(u)$ is compact and,
$supp(u) = cl\{x \in \mathbb{R} \mid \mu_u(\theta) > 0\}$. Then $supp(u)$ is bounded.

For any fuzzy number u there is $t_1, t_2, t_3, t_4 \in \mathbb{R}$ and $l_u, r_u : \mathbb{R} \to [0,1]$ such that we describe a membership function μ_u as follows:

$$\mu_u(\theta) = \begin{cases} 0 & if \quad \theta < t_1, \\ l_u(\theta) & if \quad t_1 \le \theta \le t_2, \\ 1 & if \quad t_2 \le \theta \le t_3, \\ r_u(\theta) & if \quad t_3 \le \theta \le t_4, \\ 0 & if \quad t_4 < \theta. \end{cases}$$

Here, the left side of a fuzzy number u is $l_u : [t_1, t_2] \to [0,1]$ and, the right side of a fuzzy number u is $r_u : [t_3, t_4] \to [0,1]$, also l_u is nondecreasing, r_u is nonincreasing.

Another type of fuzzy number representation is the α-cut, often known as the LU parametric representation. Using this, we have that the fuzzy number u is given by a pair of functions (u^-, u^+) where $u^-, u^+ : [0,1] \to \mathbb{R}$ provide the next qualifications:

i. u^- is nondecreasing,
ii. u^+ is nonincreasing,
iii. $u^-(1) \le u^+(1)$.

For $u = (u^-, u^+)$, we have $core(u) = [u^-(1), u^+(1)]$ and $supp(u) = [u^-(0), u^+(0)]$. The following well-known relations provide an important connection between a fuzzy number's membership function and its parametric representation:

$$u^-(\alpha) = \inf\{\theta \in \mathbb{R} : u(\theta) \ge \alpha\},$$

$$u^+(\alpha) = \sup\{\theta \in \mathbb{R} : u(\theta) \ge \alpha\}, \alpha \in (0,1],$$

and

$$[u^-(0), u^+(0)] = cl(\{\theta \in \mathbb{R} : u(x) > 0\}),$$

where cl denotes the closure operator. Additionally, if u is continuous with $supp(u) = [a,b]$ and $core(u) = [c,d]$, then it can be proved that

$$u(u^-(\alpha)) = \alpha, \forall \alpha \in [a,c],$$

$$u(u^+(\alpha)) = \alpha, \forall \alpha \in [d,b].$$

Therefore, the end points of the intervals

$$[u]_\alpha = [u^-(\alpha), u^+(\alpha)], \forall \alpha \in [0,1],$$

determine a fuzzy number $u \in \mathbb{R}_F$. Hence, by considering

$$\{(u^-(\alpha), u^+(\alpha)) \mid 0 \le \alpha \le 1\},$$

we can describe a fuzzy number $u \in \mathbb{R}_F$ and write $u = (u^-, u^+)$.

3. Truncated Baskakov Operators defined on compact intervals

Throughout this paper, we indicate the continuous function space identified on interval I by $C(I)$ and the positive continuous function space identified on interval I by $C_+(I)$. From the result of Weierstrass theorem (see [2]), $P(\theta)$ converges to continuous function $\nu(\theta)$ in the interval $[0,1]$, we just have to move functions from $[0,1]$ to an arbitrary interval $[\eta, \zeta]$. In fact, let the continuous function $g : [0,1] \to \mathbb{R}$ and the function $\nu(\theta)$ is continuouson $[\eta, \zeta]$, we put $g(\vartheta) = \nu(\eta + (\zeta - \eta)\vartheta)$.

If u is a continuous fuzzy number with $supp(u) = [\eta, \zeta]$, $a < b$ and $core(u) = [c,d]$, $c < d$ then we identify

$$\widetilde{U}_t(u)(\theta) = \begin{cases} 0, \ \theta \notin [\eta, \zeta] \\ U_t(u; [\eta, \zeta]) = \sum_{k=0}^{t} b_{t,k}(\theta) u\left(\eta + (\zeta - \eta)\frac{k}{t}\right), \theta \in [\eta, \zeta]. \end{cases}$$

where $b_{t,k}(\theta) = \binom{t+k-1}{k}\left(\frac{\theta-\eta}{\zeta-\eta}\right)^k \left(\frac{\zeta-2\eta+\theta}{\zeta-\eta}\right)^{-t-k}$.

$$U_t^{(M)}(\nu;[\eta,\zeta])(\theta)=\frac{\bigvee_{k=0}^{t}b_{t,k}(\theta)\nu\left(\eta+(\zeta-\eta)\frac{k}{t}\right)}{\bigvee_{k=0}^{t}b_{t,k}(\theta)},\ \theta\in[\eta,\zeta]$$

where $b_{t,k}(\theta)=\binom{t+k-1}{k}\left(\frac{\theta-\eta}{\zeta-\eta}\right)^k(\frac{\zeta-2\eta+\theta}{\zeta-\eta})^{-t-k}$.

It is known that $\sum_{k=0}^{t}b_{t,k}(\theta)=1$ for all $\theta\in[\eta,\zeta]$ so we get that $\bigvee_{k=0}^{t}b_{t,k}(\theta)>0$. Hence, we get that $U_t^{(M)}(\nu;[\eta,\zeta])$ is well defined.

We obtain that for each $\nu\in C_+([\eta,\zeta])$, $U_t^{(M)}(\nu;[\eta,\zeta])\in C_+([\eta,\zeta])$, because the continuous function is the maximum of a limited number of continuous functions. Here, we will show that $U_t^{(M)}:C_+([\eta,\zeta])\to C_+([\eta,\zeta])$ retains the quasi-concavity. Also the uniform approximation order of $U_t^{(M)}$ will be the same as that of the linear Baskakov operator. We first require the results and definitions given below.

Theorem 3.1. i. *([7])Let the function $\nu:[0,1]\to\mathbb{R}_+$ is continuous then we obtain*

$$|\,U_t^{(M)}(\nu;[0,1])(\theta)-\nu(\theta)\,|\le 24\omega_1\left(\nu;\frac{1}{\sqrt{t+1}}\right)_{[0,1]},\ t\in\mathbb{N},\ t\ge 2,\ \theta\in[0,1].$$

ii. *([7]) Let the function $\nu:[0,1]\to\mathbb{R}_+$ is concave then we obtain*

$$|\,U_t^{(M)}(\nu;[0,1])(\theta)-\nu(\theta)\,|\le 2\omega_1\left(\nu;\frac{1}{t}\right)_{[0,1]},\ t\in\mathbb{N},\ \theta\in[0,1].$$

Theorem 3.2. *Let the function $\nu:[0,1]\to\mathbb{R}_+$ and fix $t\in\mathbb{N}$, $t\ge 2$. Assume that there exists $c\in[0,1]$ such that ν is nondecreasing on $[0,c]$ and nonincreasing on $[c,1]$. Then, there exists $c'\in[0,1]$ such that $U_t^{(M)}(\nu)$ is nondecreasing on $[0,c']$ and nonincreasing on $[c',1]$. In addition, we have $|\,c-c'\,|\le 1/(t+1)$ and $|\,U_t^{(M)}(\nu)(c)-\nu(c)\,|\le\omega_1(\nu;1/(t+1))$.*

Proof. Let $c\in[0,1]$ and $j_c\in\{0,1,\cdots,t-2\}$ be such that $\left[\frac{j_c}{t-1},\frac{j_c+1}{t-1}\right]$. We investigate the monotonicity on each interval of the form $\left[\frac{j}{t-1},\frac{j+1}{t-1}\right]$, $j\in\{0,1,\cdots,t-2\}$. Therefore we can specify the monotoncity of $U_t^{(M)}(\nu)$ on $[0,1]$ using the continuity of $U_t^{(M)}(\nu)$ Let take arbitrary $j\in\{0,1,\cdots,j_c-1\}$ and $\theta\in\left[\frac{j}{t-1},\frac{j+1}{t-1}\right]$. By the monotonicity of ν, it follows that $\nu\left(\frac{j}{t}\right)\ge\nu\left(\frac{j-1}{t}\right)\ge\cdots\ge\nu(0)$. From [7](proof of Lemma 3.2) the following claims are true:

$$\text{If } j\le k\le k+1\le t \text{ then} 1\ge m_{k,t,j}(\theta)\ge m_{k+1,t,j}(\theta), \tag{3.1}$$

$$\text{If } 0\le k\le k+1\le j \text{ then} m_{k,t,j}(\theta)\le m_{k+1,t,j}(\theta)\le 1. \tag{3.2}$$

Therefore, it is easily follows that $\nu_{j,t,j}(\theta)\ge\nu_{j-1,t,j}(\theta)\ge\cdots\ge\nu_{0,t,j}(\theta)$. From [7] lemma 3.4, it follows that

$$U_t^{(M)}(\nu)=\bigvee_{k=j}^{t}\nu_{k,t,j}(\theta).$$

Since $U_t^{(M)}(\nu)$ is the maximum of nondecreasing functions, it is nondecreasing on the interval $\left[\frac{j}{t-1},\frac{j+1}{t-1}\right]$. $U_t^{(M)}(\nu)$ is continuous, therefore ν is nondecreasing on the interval $\left[0,\frac{j_c}{t-1}\right]$. Let take arbitrary $j\in\{j_c+1,\cdots,t-2\}$ and $\theta\in\left[\frac{j}{t-1},\frac{j+1}{t-1}\right]$. By using the monotonicity of ν, it follows that $\nu\left(\frac{i}{t}\right)\ge\nu\left(\frac{i+1}{t}\right)\ge\cdots\ge\nu(1)$. It means that $U_t^{(M)}(\nu)(\theta)=\bigvee_{k=0}^{j}\nu_{k,t,j}(\theta)$ from the assertion (3.1). Thus, it is nonincreasing on $\left[\frac{j}{t-1},\frac{j+1}{t-1}\right]$, because $U_t^{(M)}(\nu)$ is the maximum of nonincreasing functions. Considering the continuity of $U_t^{(M)}(\nu)$, it is immediate that ν is nonincreasing on $\left[\frac{j_c+1}{t-1},1\right]$. Finally let investigate the case when $j=j_c$. If $\frac{j}{t-1}\le c$, then by the monotoncity of ν it follows that $\nu\left(\frac{j_c}{t}\right)\ge\nu\left(\frac{j_c-1}{t}\right)\ge\cdots\ge\nu(0)$. Hence , in the situation we get ν is nondecreasing on $\left[\frac{j_c}{t-1}, \frac{j_c+1}{t-1}\right]$. As a result, ν is nondecreasing on $\left[0, \frac{j_c+1}{t-1}\right]$ and nonincreasing on $\left[\frac{j_c+1}{t-1}, 1\right]$. Also, $c' = \frac{j_c+1}{t-1}$ is the maximum point of $U_t^{(M)}(\nu)$ and It is simple to verify that $\mid c - c' \mid \leq \frac{1}{t-1}$. If $j_c/t \geq c$ then by the monotonicity of ν it follows that$\nu\left(\frac{j_c}{t}\right) \geq \nu\left(\frac{j_c+1}{t}\right) \geq \cdots \geq \nu(1)$. Therefore, we obtain that ν is nonincreasing on $\left[\frac{j_c}{t-1}, \frac{j_c+1}{t-1}\right]$. It follows that ν is nondecreasing on $[0, \frac{j_c}{t-1}]$ and nonincreasing on $[\frac{j_c}{t-1}, 1]$. In addition,$c' = \frac{j_c}{t-1}$ is the maximum point of $U_t^{(M)}(\nu)$ and it is easy to check that $\mid c - c' \mid \leq \frac{1}{t-1}$.

The last part of the theorem is now proved. Let start by noting that for all $\theta \in [0,1]$, $U_t^{(M)}(\nu) \leq \nu(c)$. In fact, the description of $U_t^{(M)}(\nu)$ and the fact that c is global maximum point of ν imply this obvious. It indicates that

$$\begin{aligned} \mid U_t^{(M)}(\nu)(c) - \nu(c) \mid &= \nu(c) - U_t^{(M)}(\nu)(c) = \nu(c) - \bigvee_{k=0}^{t} \nu_{k,t,j_c}(c) \\ &\leq \nu_{j_c,t,j_c}(c) = \nu(c) - \nu\left(\frac{j_c}{t}\right). \end{aligned}$$

Since $c, \frac{j_c}{t} \in \left[\frac{j}{t-1}, \frac{j+1}{t-1}\right]$, we easily get $\nu(c) - \nu\left(\frac{j_c}{t}\right) \leq \omega_1(\nu; 1/t+1)$ and the theorem is proved completely. □

Definition 3.1. *([8]) Let $\nu : [\eta, \zeta] \to \mathbb{R}$ be continuous on $[\eta, \zeta]$. The function ν is called:*

i. *quasi-convex if $\nu(\lambda\theta + (1-\lambda)\vartheta) \leq \max\{\nu(\theta), \nu(\vartheta)\}$, for all $\theta, \vartheta \in [\eta, \zeta]$, $\lambda \in [0,1]$,*
ii. *quasi-concave, if $-\nu$ is quasi-convex.*

Remark. *The continuous function ν is quasi-convex on $[a, b]$ according to [9], which is similar to implying there is a point $c \in [\eta, b]$ where ν is nonincreasing on $[\eta, c]$ and nondecreasing on $[c, b]$. It is clear from the definition above that the function ν is quasi-concave on $[\eta, \zeta]$, similarly means that there exists a point $c \in [\eta, \zeta]$ such that ν is nondecreasing on $[\eta, c]$ and nonincreasing on $[c, b]$.*

Now, the main results of this section can be presented.

Theorem 3.3. i. *If $\eta, \zeta \in \mathbb{R}$, $\eta < \zeta$ and $\nu : [\eta, \zeta] \to \mathbb{R}_+$ is continuous then we get*

$$\mid U_t^{(M)}(\nu; [\eta, \zeta])(\theta) - \nu(\theta) \mid \leq 24\left([\zeta - \eta] + 1\right)\omega_1\left(\nu; \frac{1}{\sqrt{n+1}}\right)_{[\eta,\zeta]}, \ t \in \mathbb{N}, \ t \geq 2, \ \theta \in [\eta, \zeta].$$

ii. *If $\nu : [\eta, \zeta] \to \mathbb{R}_+$ is concave then we get*

$$\mid U_t^{(M)}(\nu; [\eta, \zeta])(\theta) - \nu(\theta) \mid \leq 2\left([\zeta - \eta] + 1\right)\omega_1\left(\nu; \frac{1}{t}\right)_{[\eta,\zeta]}, \ t \in \mathbb{N}, \ \theta \in [\eta, \zeta].$$

Proof. Consider the continuous function $h(\vartheta)$ on $[0,1]$ as $h(\vartheta) = \nu(\eta + (\zeta - \eta)\vartheta)$. It is simple to verify that $h(\frac{k}{t}) = \nu\left(\eta + k.\frac{(\zeta-\eta)}{t}\right)$ for all $k \in \{0, 1, \cdots, t\}$. Let any $\theta \in [\eta, \zeta]$ and $\vartheta \in [0,1]$ such that $\theta = \eta + (\zeta - \eta)\vartheta$. So we have $\vartheta = (\theta - \eta)/(\zeta - \eta)$ and $1 + \vartheta = \frac{\zeta+\theta-2\eta}{\zeta-\eta}$. From these equalities and noting the expressions for $h\left(\frac{k}{t}\right)$, we get $U_t^{(M)}(\nu; [\eta, \zeta])(\theta) = U_t^{(M)}(h; [0,1])(\vartheta)$. From Theorem 3.1

$$\mid U_t^{(M)}(\nu; [\eta, \zeta])(\theta) - \nu(\theta) \mid = \mid U_t^{(M)}(h; [0,1])(\vartheta) - h(\vartheta) \mid \leq 24\omega_1\left(h; \frac{1}{\sqrt{t+1}}\right)_{[0,1]}. \quad (3.3)$$

Since $\omega_1\left(h; \frac{1}{\sqrt{t+1}}\right)_{[0,1]} \leq \omega_1\left(\nu; \frac{\zeta-\eta}{\sqrt{t+1}}\right)_{[\eta,\zeta]}$ and the property $\omega_1(\nu; \lambda\delta)_{[\eta,\zeta]} \leq ([\lambda]+1)\,\omega_1(\nu;\delta)_{[\eta,\zeta]}$, we obtain

$$\omega_1\left(h; \frac{1}{\sqrt{t+1}}\right)_{[0,1]} \leq ([\zeta-\eta]+1)\,\omega_1\left(\nu; \frac{1}{\sqrt{t+1}}\right)_{[\eta,\zeta]}$$

which proves (i).

Using the notation from the above point (i), we get $U_n^{(M)}(\nu;[\eta,\zeta])(\theta) = U_t^{(M)}(h;[0,1])(\vartheta)$, where $h(\vartheta) = \nu(\eta+(\zeta-\eta)\vartheta) = \nu(\theta)$ for all $\vartheta \in [0,1]$. The last equality is equivalent to $\nu(u) = h\left(\frac{u-\eta}{\zeta-\eta}\right)$ for all $u \in [\eta,\zeta]$. By the property of concavity for ν, $\nu(\lambda u_1 + (1-\lambda)u_2) \geq \lambda\nu(u_1) + (1-\lambda)\nu(u_2)$, for all $\lambda \in [0,1]$, $u_1, u_2 \in [\eta,\zeta]$, in terms of h can be written as

$$h\left(\lambda\frac{u_1-\eta}{\zeta-\eta} + (1-\lambda)\frac{u_2-\eta}{\zeta-\eta}\right) \geq \lambda h\left(\frac{u_1-\eta}{\zeta-\eta}\right) + (1-\lambda)h\left(\frac{u_2-\eta}{\zeta-\eta}\right).$$

Denoting $\vartheta_1 = \frac{u_1-\eta}{\zeta-\eta} \in [0,1]$ and $\vartheta_2 = \frac{u_2-\eta}{\zeta-\eta} \in [0,1]$ this immediately implies the concavity of h on $[0,1]$. Then, by Theorem 3.1 (ii), we get

$$|\ U_t^{(M)}(\nu;[\eta,\zeta])(\theta) - \nu(\theta)\ |=|\ U_t^{(M)}(h;[0,1])(\vartheta) - h(\vartheta)\ | \leq 2\omega_1\left(h; \frac{1}{t}\right)_{[0,1]}.$$

□

Theorem 3.4. *Let the function $\nu : [\eta,\zeta] \to \mathbb{R}_+$ and let fix $t \in \mathbb{N}$, $t \geq 1$. Assume that there exists $c \in [\eta,\zeta]$ such that ν is nondecreasing on $[\eta,c]$ and nonincreasing on $[c,\zeta]$. Then, there exists $c' \in [\eta,\zeta]$ such that $U_t^{(M)}(\nu;[\eta,\zeta])$ is nondecreasing on $[\eta,c']$ and nonincreasing on $[c',\zeta]$. Also, we have $|\ c-c'\ | \leq (\zeta-\eta)/(t+1)$ and $|\ U_t^{(M)}(\nu;[\eta,\zeta])(c) - \nu(c)\ | \leq ([\zeta-\eta]+1)\,\omega_1\left(\nu; \frac{1}{\sqrt{t+1}}\right)_{[\eta,\zeta]}$.*

Proof. As in the prior theorem, we establish the function h. Let $c_1 \in [0,1]$ be such that $h(c_1) = c$, we observe that h is nondecreasing on $[0,c_1]$ and nonincreasing on $[c_1,1]$, because it is the composition of ν and the linear nondecreasing function $t \to \eta + (\zeta-\eta)t$. By Theorem 3.2 it results that there exists $c' \in [0,1]$ such that $U_t^{(M)}(h;[0,1])$ is nondecreasing on $[0,c_1']$ and nonincreasing on $[c_1',1]$ and also, we have $|\ U_t^{(M)}(h;[0,1])(c_1) - h(c_1)\ | \leq \omega_1(h; 1/t+1)$ and $|\ c_1 - c_1'\ | \leq 1/(t+1)$. Let $c' = \eta + (\zeta-\eta)c_1'$. If $\theta_1, \theta_2 \in [\eta,c']$ with $\theta_1 \leq \theta_2$ then let $\vartheta_1, \vartheta_2 \in [0,c_1']$ be such that $\theta_1 = \eta+(\zeta-\eta)\vartheta_1$ and $\theta_2 = \eta+(\zeta-\eta)\vartheta_2$. Than it follows that $U_t^{(M)}(\nu;[\eta,\zeta])(\theta_1) = U_t^{(M)}(h;[0,1])(\vartheta_1)$ and $U_t^{(M)}(\nu;[\eta,\zeta])(\theta_2) = U_t^{(M)}(h;[0,1])(\vartheta_2)$. the monotonicity of $U_t^{(M)}(h;[0,1])$ means $U_t^{(M)}(h;[0,1])(\vartheta_1) \leq U_t^{(M)}(h;[0,1])(\vartheta_2)$ that is
$U_t^{(M)}(\nu;[\eta,\zeta])(\theta_1) \leq U_t^{(M)}(\nu;[\eta,\zeta])(\theta_2)$. We obtain that $U_t^{(M)}(\nu;[\eta,\zeta])$ is nondecreasing on $[\eta,c']$. By the same way, we get that $U_t^{(M)}(\nu;[\eta,\zeta])$ is nonincreasing on $[c',\zeta]$. For the remainder of the proof, noting that $|\ c_1-c_1'\ | \leq 1/(t+1)$. we get $|\ c-c'\ |=|\ (\zeta-\eta)(c_1-c_1')\ | \leq (\zeta-\eta)/(t+1)$. In addition, mentioning

$$|\ U_t^{(M)}(h;[0,1])(c_1) - h(c_1)\ | \leq \omega_1\left(h; \frac{1}{t+1}\right)_{[0,1]}$$

and taking into account that $\omega_1\left(h; \frac{1}{t+1}\right)_{[0,1]} \leq ([b-\eta]+1)\,\omega_1\left(\nu; \frac{1}{t+1}\right)_{[\eta,\zeta]}$, we obtain

$$\begin{aligned}|\ U_t^{(M)}(\nu;[\eta,\zeta])(c) - \nu(c)\ | &= |\ U_t^{(M)}(h;[0,1])(c_1) - h(c_1)\ | \\ &\leq \omega_1\left(h; \frac{1}{\sqrt{t+1}}\right)_{[0,1]} \leq ([\zeta-\eta]+1)\,\omega_1\left(\nu; \frac{1}{\sqrt{t+1}}\right)_{[\eta,\zeta]}\end{aligned}$$

and the proof is complete. □

Remark. *The preceding theorem and Remark 3 contribute to the conclusion that $U_t^{(M)}(\nu)$ is also quasi-concave if $\nu : [\eta, \zeta] \to \mathbb{R}_+$ is continuous and quasi-concave. As we indicated in the Introduction, $U_t^{(M)}$ maintains monotonicity and quasi-convexity for functions in the space $C_+([0, 1])$. It can be demonstrated that these conservation features hold in the general case of the space $C_+([\eta, \zeta])$ using reasoning similar to that used in the demonstration of Theorem 3.4.*

4. Applications to fuzzy number approximation

Lemma 4.1. *Let $\eta, \zeta \in \mathbb{R}$, $\eta < \zeta$. For $t \in \mathbb{N}$, $t \geq 2$ and $k \in \{0, 1, \cdots, t\}$, $j \in \{0, 1, \cdots, t-2\}$ and $\theta \in (\eta + j.(\zeta - \eta)/(t-1), \eta + (j+1).(\zeta - \eta)/(t-1))$.*
Let $m_{k,t,j}(\theta) = \frac{b_{t,k}(\theta)}{b_{t,j}(\theta)}$, where $b_{t,k}(\theta) = \binom{t+k-1}{k} \left(\frac{\theta-\eta}{\zeta-\eta}\right)^k \left(\frac{\zeta-2\eta+\theta}{\zeta-\eta}\right)^{-t-k}$. Then
$m_{k,t,j}(\theta) \leq 1$.

Proof. We can assume that $\eta = 0$ and $\zeta = 1$, by using the same reasoning as in the proof of Theorems 3.3 and 3.4 we may simply get the conclusion of the lemma in the general case. For fix $\theta \in (j/(t-1), (j+1)/(t-1))$ and from Lemma 3.2 in [7], we obtain

$$m_{0,t,j}(\theta) \leq m_{1,t,j}(\theta) \leq \cdots \leq m_{j,t,j}(\theta),$$

$$m_{j,t,j}(\theta) \geq m_{j+1,t,j}(\theta) \geq \cdots \geq m_{t,t,j}(\theta).$$

From $m_{j,t,j}(\theta) = 1$, it is enough to show that $m_{j+1,t,j}(\theta) < 1$ and $m_{j-1,t,j}(\theta) < 1$. Then, we have

$$\frac{m_{j,t,j}(\theta)}{m_{j+1,t,j}(\theta)} = \frac{j+1}{t+j} \frac{1+\theta}{\theta}.$$

Because the function $g(\vartheta) = (1+\theta)/\theta$ is strictly decreasing on the interval $\left[\frac{j}{t-1}, \frac{(j+1)}{t-1}\right]$, it results that $\frac{1+\theta}{\theta} > \frac{t+j}{j+1}$. Obviously, this suggests $m_{j,t,j}(\theta)/m_{j+1,t,j}(\theta) > 1$ that is $m_{j,t,j}(\theta) < 1$. Similar conclusions lead us to the result that $m_{j-1,t,j}(\theta) < 1$ and we get the proof. □

Now consider a function $\nu \in C_+([\eta, \zeta])$. We may simplify the method to compute $U_t^{(M)}(\nu; [\eta, \zeta])(\theta)$ for some $\theta \in [\eta, \zeta]$ by combining formula(3) with the conclusion of Lemma 4.1. Let take $j \in \{0, 1, ..., t-2\}$ and $\theta \in [\eta + (\zeta - \eta)j/(t-1), \eta + (\zeta - \eta)(j+1)/(t-1)]$. As a result of the features of continuous functions, an immediate result of Lemma 4.1 is that $m_{k,t,j}(\theta) \leq 1$ for all $k \in \{0, 1, ..., t\}$. This means that

$$\bigvee_{k=0}^{t} b_{t,k}(\theta) = b_{t,j}(\theta), \theta \in [\eta + (\zeta - \eta)j/(t-1), \eta + (\zeta - \eta)(j+1)/(t-1)]. \tag{4.1}$$

As a result, for each $k \in \{0, 1, ..., t\}$ and
$\theta \in [\eta + (\zeta - \eta)j/(t-1), \eta + (\zeta - \eta)(j+1)/(t-1)]$

$$\nu_{k,t,j}(\theta) = m_{k,t,j}(\theta).\nu(\eta + (\zeta - \eta)k/t), \tag{4.2}$$

by (3) and (4.1) we obtain

$$U_t^{(M)}(\nu; [\eta, \zeta])(\theta) = \bigvee_{k=0}^{t} \nu_{k,t,j}(\theta),\ \theta \in [\eta + (\zeta - \eta)j/(t-1), \eta + (\zeta - \eta)(j+1)/(t-1)]. \tag{4.3}$$

A similar method from paper [7] that takes into account the particular situation $a = 0, b = 1$ is generalized in the formula above. From Lemma 4.1, for any $k \in \{0, 1, ..., t\}$ and $\theta \in [\eta + (\zeta - \eta)j/(t-1), \eta + (\zeta - \eta)(j+1)/(t-1)]$, we get $\nu_{k,t,j}(\theta) \leq \nu(\eta + (\zeta - \eta)k/t)$.

Lemma 4.2. *Let $\eta, b \in \mathbb{R}, \eta < \zeta$. If $\nu : [\eta, \zeta] \to \mathbb{R}_+$ is bounded then we get $U_t^{(M)}(\nu; [\eta, \zeta])(\eta + j(\zeta - \eta)/(t-1)) \geq \nu(\eta + j(\zeta - \eta)/(t-1))$ for all $j \in \{0, 1, \cdots t-2\}$.*

Proof. From Lemma 4.1, since
$\eta + j(\zeta - \eta)/(t-1) \in (\eta + (\zeta - \eta)j/(t-1), \eta + (\zeta - \eta)(j+1)/(t-1))$ and $m_{k,t,j}(\eta + j(\zeta - \eta)/(t-1)) = \frac{b_{t,k}(\eta + j(\zeta-\eta)/(t-1))}{b_{t,j}(\eta + j(\zeta-\eta)/(t-1))}$ for all $k \in \{0, 1, \cdots, t\}$ it means that

$$\bigvee_{k=0}^{t} b_{t,k}(\eta + j(\zeta - \eta)/t) = b_{t,j}(\eta + j(\zeta - \eta)/t).$$

Then we obtain

$$\begin{aligned} U_t^{(M)}(\nu; [\eta, \zeta])(\eta + j(\zeta - \eta)/t) &= \frac{\bigvee_{k=0}^{t} b_{t,k}(\eta + j(\zeta - \eta)/t)\, \nu(\eta + j(\zeta - \eta)/t)}{b_{t,j}(\eta + j(\zeta - \eta)/t)} \\ &\geq \frac{b_{t,j}(\eta + j(\zeta - \eta)/t)\, \nu(\eta + j(\zeta - \eta)/t)}{b_{t,j}(\eta + j(\zeta - \eta)/t)} \\ &= \nu(\eta + j(\zeta - \eta)/t). \end{aligned}$$

and lemma is proved. □

Theorem 4.3. *Let u be a fuzzy number with $supp(u) = [\eta, b]$ and $core(u) = [c, d]$ such that $\eta \leq c < d \leq \zeta$. Then for sufficiently large t, it result that $\widetilde{U}_t^{(M)}(u; [\eta, \zeta])$ is a fuzzy number such that :*

i. *$supp(u) = supp(\widetilde{U}_t^{(M)}(u; [\eta, \zeta]))$;*
ii. *if $core(\widetilde{U}_t^{(M)}(u; [\eta, \zeta])) = [c_t, d_t]$, then c_t and d_t can be identified precisely and also we get $|c - c_t| \leq (\zeta - \eta)/t$ and $|d - d_t| \leq (\zeta - \eta)/t$;*
iii. *if u is continuous on $[a, b]$, then*

$$\left|\widetilde{U}_t^{(M)}(u; [\eta, \zeta]) - u(\theta)\right| \leq 6\left([\zeta - \eta] + 1\right)\omega_1\left(u, \frac{1}{\sqrt{t}}\right)_{[\eta, \zeta]},$$

for all $\theta \in \mathbb{R}$.

Proof. Let $t \in \mathbb{N}$, with the inequality $(\zeta - \eta)/t < d - c$. From Theorem 3.4, there is $c' \in [\eta, \zeta]$ such that $\widetilde{U}_t^{(M)}(u; [\eta, \zeta])$ is nondecreasing on $[\eta, c']$ and nonincreasing on $[c', \zeta]$. Beside, from the description of $\widetilde{U}_t^{(M)}(u; [\eta, \zeta])$, it gives us that
By indicating $\| \, . \, \|$ the uniform norm on $B([\eta, \zeta])$ the space of bounded functions on $[\eta, \zeta]$, $\| \widetilde{U}_t^{(M)}(u; [\eta, \zeta]) \| \leq \| u \|$ and since $\| u \| = 1$, it means that $\| \widetilde{U}_t^{(M)}(u) \| \leq 1$. Consequently, it is sufficient to demonstrate that $\widetilde{U}_t^{(M)}(u)$ is a fuzzy number in order to obtain existence of $\alpha \in [\eta, \zeta]$ such that $\widetilde{U}_t^{(M)}(u)(\alpha) = 1$. Let $\alpha = \eta + j(\zeta - \eta)/n$ where j is choosen with the property that $c < \alpha < d$. Such j exists as $(\zeta - \eta)/t < d - c$. Since $\alpha \in core(u)$, it results $u(\alpha) = 1$. Also,from Lemma 4.2, we can write that $\widetilde{U}_t^{(M)}(u; [\eta, \zeta])(\alpha) \geq u(\alpha)$ and obviously this means that $\widetilde{U}_t^{(M)}(u; [\eta, \zeta])$ is a fuzzy number. Since we have $F_t^{(M)}(u; [\eta, \zeta])(\eta) = u(\eta)$, $U_t^{(M)}(u; [\eta, \zeta])(b) = u(b)$ and the description of u and $\widetilde{U}_t^{(M)}(u; [\eta, \zeta])$, it follows that $\widetilde{U}_t^{(M)}(u; [\eta, \zeta])(\theta) = 0$ outside $[\eta, \zeta]$. Now, by $u(\theta) > 0$ and $\widetilde{U}_t^{(M)}(u; [\eta, \zeta])(\theta) = U_t^{(M)}(u; [\eta, \zeta])(\theta)$ for all $\theta \in (\eta, \zeta)$, we can obtain that $\widetilde{U}_t^{(M)}(u; [\eta, \zeta])(\theta) > 0$ for all $\theta \in (\eta, \zeta)$ which proves (i).

Now, let $t \in \mathbb{N}$ with $(\zeta - \eta)/t \leq d - c$. Then take $k(t, c), k(t, d) \in \{1, \cdots, t-1\}$ be with the property that $\eta + (\zeta - \eta)(k(t, d) - 1)/t < c \leq \eta + (\zeta - \eta)k(t, c)/n$ and $\eta + (\zeta - \eta)k(t, c)/t \leq d < \eta + (\zeta - \eta)(k(t, d) + 1)/n$. Since $(\zeta - \eta)/t \leq d - c$ it is obvious that $k(t, c) \leq k(t, d)$. Also, $k(t, c)$ and $k(t, d)$ were chosen, we get that $u(\eta + (\zeta - \eta)k/t) = 1$ for any

$k \in \{k(t,c), \cdots, k(t,d)\}$ and $u(\eta+(\zeta-\eta)k/t) < 1$ for any $k \in \{0, \cdots, t\} \setminus \{k(t,c), \cdots, k(t,d)\}$. For some $\theta \in [\eta + k(t,c)(\zeta-\eta)/t, \eta + (k(t,c)+1)(\zeta-\eta)/t]$, we have

$$\widetilde{U}_t^{(M)}(u;[\eta,\zeta])(\theta) = \bigvee_{k=0}^{n} u_{k,t,k(t,c)}(\theta).$$

We have

$$u_{k(t,c),t,k(t,c)}(\theta) = m_{k(t,c),t,k(t,c)}(\zeta)u(\eta+(\zeta-\eta)k(t,c)/n) = u(\eta+(\zeta-\eta)k(t,c)/t) = 1$$

and from the description of $k(t,c)$ and by Lemma 4.2,so for any $k \in \{0, \cdots, t\}$, we get

$$\widetilde{U}_t^{(M)}(u;[\eta,\zeta])(\theta) = u(\eta+(\zeta-\eta)k(t,c)/t) = 1,$$

$\forall\, \theta \in [\eta + k(t,c)(\zeta-\eta)/t, \eta + (k(t,c)+1)(\zeta-\eta)/t]$. Similarly we obtain that

$$\widetilde{U}_t^{(M)}(u;[\eta,\zeta])(\theta) = u(\eta+(\zeta-\eta)k(t,d)/t) = 1,$$

$\forall\, \theta \in [\eta + k(t,d)(\zeta-\eta)/t, \eta + (k(t,d)+1)(\zeta-\eta)/t]$. Let take arbitrarily

$$\theta \in (\eta + (k(t,c)-1)(\zeta-\eta)/t, \eta + k(t,c)(\zeta-\eta)/t),$$

we have

$$\widetilde{U}_t^{(M)}(u;[\eta,\zeta])(\theta) = \bigvee_{k=0}^{n} u_{k,t,k(t,c)-1}(\theta).$$

If $k \in \{k(t,c), \cdots, k(t,d)\}$, then we get

$$\begin{aligned} u_{k,t,k(t,c)-1}(\theta) =& m_{k,t,k(t,c)-1}(\theta)u(\eta+(\zeta-\eta)k/t) < u(\eta+(\zeta-\eta)k/t) \\ &= u(\eta+(\zeta-\eta)k(t,c)/n) = \widetilde{U}_t^{(M)}(u;[\eta,\zeta])(\eta+(\zeta-\eta)k(t,c)/t). \end{aligned}$$

Let $k \notin \{k(t,c), \cdots, k(t,d)\}$, then we get

$$\begin{aligned} u_{k,t,k(t,c)-1}(\theta) =& m_{k,t,k(t,c)-1}(\theta)u(\eta+(\zeta-\eta)k/t) \leq u(\eta+(\zeta-\eta)k/t) \\ &< u(\eta+(\zeta-\eta)k(t,c)/t) = \widetilde{U}_t^{(M)}(u;[\eta,\zeta])(\eta+(\zeta-\eta)k(t,c)/t). \end{aligned}$$

From the propertiy of quasi-concavity of $\widetilde{U}_t^{(M)}(u;[\eta,\zeta])$ on $[\eta,\zeta]$ it easily results that

$$\widetilde{U}_t^{(M)}(u;[\eta,\zeta])(\theta) < \widetilde{U}_t^{(M)}(u;[\eta,\zeta])(\eta+(\zeta-\eta)k(t,c)/t),\ \forall\theta \in [\eta, \eta+k(t,c)(\zeta-\eta)/t)].$$

Similarly, we get

$$\widetilde{U}_t^{(M)}(u;[\eta,\zeta])(\theta) < \widetilde{U}_t^{(M)}(u;[\eta,\zeta])(\eta+(\zeta-\eta)(k(t,d)+1)/t),$$

$\forall\, \theta \in [\eta+(k(t,d)+1)(\zeta-\eta)/t), \zeta]$. From the previous inequalities

$$\begin{aligned} \widetilde{U}_t^{(M)}(u;[\eta,\zeta])(\eta+(\zeta-\eta)k(t,c)/t) =& u(\eta+(\zeta-\eta)k(t,c)/t) = u(\eta+(\zeta-\eta)k(t,d)/t) \\ =& \widetilde{U}_t^{(M)}(u;[\eta,\zeta])(\eta+(\zeta-\eta)(k(t,d)+1)/t) = 1, \end{aligned}$$

we get that $\widetilde{U}_t^{(M)}(u;[\eta,\zeta])$ reaches its maximum value only in the range $[\eta+(\zeta-\eta)k(t,c)/t, \eta+(\zeta-\eta)(k(t,d)+1)/t]$ which by the description of $\widetilde{U}_t^{(M)}(u;[\eta,\zeta])$ implies that $core\left(\widetilde{U}_t^{(M)}(u;[\eta,\zeta])\right) = [\eta+(\zeta-\eta)k(t,c)/t, \eta+(\zeta-\eta)(k(t,d)+1)/t]$. Then, indicating $c_t = \eta+(b-\eta)k(t,c)/n$ one can see that both c_t and c belong to the interval $[\eta+(\zeta-\eta)(k(t,c)-1)/t, \eta+(\zeta-\eta)k(t,c)/t]$ of length $(\zeta-\eta)/n$ and so $|c-c_t| \leq (\zeta-\eta)/t$. Correlatively, indicating $d_t = \eta+(\zeta-\eta)(k(t,d)+1)/(t+1)$ we get that $|d-dn| \leq (\zeta-\eta)/t$ and the statement (ii) has been proven.

(iii) From Theorem 3.3, the proof is immediate.

□

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