INTERNATIONAL ELECTRONIC JOURNAL OF GEOMETRY
VOLUME 16 NO. 2 PAGE 715–726 (2023)
DOI: HTTPS://DOI.ORG/10.36890/IEJG.1362590

Approximations of Parallel Surfaces Along Curves

Büşra Köse* and Yusuf Yaylı

(Communicated by Cihan Özgür)

ABSTRACT

In this paper, we study developable surfaces which are flat and normal approximation of parallel surfaces along curves associated with three special vector fields. It is known that a surface whose points are at a constant distance along the normal of the surface is called a parallel surface. We investigate singularities of such developable surfaces. We show that under what conditions the approach surfaces are parallel. Also, we show that the approach surfaces are constant angle ruled surfaces if the curves selected on the surfaces are isophote, relatively normal-slant helix and helix.

Keywords: *Parallel surfaces, developable surfaces, curves on surfaces, flat approximations, normal approximations.*
AMS Subject Classification (2020): *Primary: 53Axx ; Secondary: 57R45.*

1. Introduction

Izumiya and Otani [5] introduced the osculating developable surface tangent to the surface along the curve for a regular curve on a surface in Euclidean 3-space. Such a surface gives flat approximation of the surface along the curve. They gave existence and the uniquesness of the flat approximation surface. On the other hand; Hananoi, Ito and Izumiya [3] introduced 3 type of Darboux vector fields called normal, rectifying and osculating Darboux vector fields. They showed that [5] an flat approximation surface is directed osculating Darboux vector field. Also, they introduced two invariants which interested in the singularities of flat approximation surfaces by using three invariants related to the Darboux frame of a curve on a surface. In another work, Izumiya and Hananoi [4] introduced the normal developable surface normal to the surface along the curve for a regular curve on a surface in Euclidean 3-space. Such a surface gives normal approximation of the surface along the curve. They showed that a normal approximation surface is directed rectifying Darboux vector field. Also, they introduced two invariants which interested in the singularities of normal approximation surfaces by using three invariants related to the Darboux frame of a curve on a surface.

In this paper, we investigate a flat approximation of a parallel surface along a curve. In §3, we introduce a developable surface tangent to the parallel surface along the curve. We show that, flat approximation of the surface along the curve and flat approximation of its parallel surface along curve are parallel surfaces. Similar to given in their study [5] by Izumiya and Otani, we introduce an invariant which interested in the singularities of flat approximation surfaces of the parallel surfaces along the curves by using three invariants related to the Darboux frame of a curve on a surface. We show that if the curve on the surface is isophote curve then flat approximation surface of the parallel surface is a constant angle ruled surface. Then we give an example of helicoid. Also, we investigate a normal approximation of a parallel surface along a curve, too. In §4, we investigate a developable surface normal to the parallel surface along the curve and we saw that we can't define developable surface associated with rectifying Darboux vector field, directly. Therefore, we can't find normal approximation of the parallel surface along the curve, directly. Then, we consider 2 type of parallel surfaces which are ruled surfaces and normal approximation of those surfaces by using rectifying and normal Darboux vector fields. In §4.1, we introduce one of those parallel surfaces which is Y parallel surface and normal approximation of Y parallel surface associated with rectifying Darboux vector field. Then, we

Received : *19-09-2023,* ***Accepted :*** *29-10-2023*
** Corresponding author*

show that normal approximation of Y parallel surface and normal approximation of the surface are parallel surfaces. Also, if the curve on the surface is relatively normal-slant helix then normal approximation surface of Y parallel surface is a constant angle ruled surface. We give an example of helicoid. Then, in §4.2, we introduce other parallel surface which is T parallel surface and normal approximation of T parallel surface associated with normal Darboux vector field. Finally, we show that if the curve on the surface is helix then normal approximation of T parallel surface is a constant angle ruled surface.

2. Basic Consept

Let M be a surface in $\mathbb{R}^3$ with its unit normal vector $\boldsymbol{n}$ and $\boldsymbol{\alpha} : I \subset \mathbb{R} \to M$ be a curve. $\boldsymbol{T} = \frac{\boldsymbol{\alpha}'}{\|\boldsymbol{\alpha}'\|}$ is tangent vector field and $\boldsymbol{Y} = \boldsymbol{n} \wedge \boldsymbol{T}$. Due to properties of the cross product, moving frame $\{\boldsymbol{T}, \boldsymbol{Y}, \boldsymbol{n}\}$ is orthonormal frame of $T_{\alpha(s)}\mathbb{R}^3$. $\{\boldsymbol{T}, \boldsymbol{Y}, \boldsymbol{n}\}$ frame is called Darboux frame or Surface frame of curved surface pair $(\boldsymbol{\alpha}, M)$ [8]. Equations of this surface frame:

$$\begin{pmatrix} \boldsymbol{T}'(s) \\ \boldsymbol{Y}'(s) \\ \boldsymbol{n}'(s) \end{pmatrix} = \begin{pmatrix} 0 & k_g(s) & k_n(s) \\ -k_g(s) & 0 & t_r(s) \\ -k_n(s) & -t_r(s) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{T}(s) \\ \boldsymbol{Y}(s) \\ \boldsymbol{n}(s) \end{pmatrix}$$

Here, k_n is normal curvature of surface, k_g is geodesic curvature of surface and t_r is geodesic torsion of surface [8].

In addition to Darboux frame, we can give the following frame associated with Darboux frame:
M is a surface on $\mathbb{E}^3$ with $\boldsymbol{n}$ unit normal vector and $\boldsymbol{D}_o$ is osculating Darboux vector field[3]. $\boldsymbol{\alpha} : I \subset \mathbb{R} \to M$ is a curve on surface M.

$$\begin{aligned} \boldsymbol{C}_o(s) &= \frac{-k_n(s)\boldsymbol{T}(s) - t_r(s)\boldsymbol{Y}(s)}{\sqrt{t_r^2(s) + k_n^2(s)}} \\ \boldsymbol{D}_o(s) &= \frac{t_r(s)\boldsymbol{T}(s) - k_n(s)\boldsymbol{Y}(s)}{\sqrt{t_r^2(s) + k_n^2(s)}} \end{aligned}$$

Equations of $\{\boldsymbol{n}, \boldsymbol{C}_o, \boldsymbol{D}_o\}$ frame:

$$\begin{pmatrix} \boldsymbol{n}'(s) \\ \boldsymbol{C}_o'(s) \\ \boldsymbol{D}_o'(s) \end{pmatrix} = \begin{pmatrix} 0 & \bar{f}(s) & 0 \\ -\bar{f}(s) & 0 & \bar{g}(s) \\ 0 & -\bar{g}(s) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{n}(s) \\ \boldsymbol{C}_o(s) \\ \boldsymbol{D}_o(s) \end{pmatrix}$$

Here, $\bar{f}(s) = \sqrt{t_r^2(s) + k_n^2(s)}$, $\bar{g}(s) = k_g(s) + \frac{k_n(s)t_r'(s) - k_n'(s)t_r(s)}{t_r^2(s) + k_n^2(s)}$.
The formula between $\{\boldsymbol{n}, \boldsymbol{C}_o, \boldsymbol{D}_o\}$ and $\{\boldsymbol{T}, \boldsymbol{Y}, \boldsymbol{n}\}$ Darboux frame:

$$\begin{pmatrix} \boldsymbol{T}(s) \\ \boldsymbol{Y}(s) \\ \boldsymbol{n}(s) \end{pmatrix} = \begin{pmatrix} \frac{-k_n(s)}{\sqrt{t_r^2(s)+k_n^2(s)}} & \frac{t_r(s)}{\sqrt{t_r^2(s)+k_n^2(s)}} & 0 \\ \frac{-t_r(s)}{\sqrt{t_r^2(s)+k_n^2(s)}} & \frac{-k_n(s)}{\sqrt{t_r^2(s)+k_n^2(s)}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{C}_o(s) \\ \boldsymbol{D}_o(s) \\ \boldsymbol{n}(s) \end{pmatrix}$$

Definition 2.1. Let M and M_r be an orientable surfaces and $\boldsymbol{n}$ an orthogonal positive normal vector field of M. If there is exist smooth function $f : \mathbb{R}^3 \to \mathbb{R}^3$, impending a fixed number r and its restriction to surface M is bijection defined by

$$f : M \to M_r, \ f(p) = p + r\boldsymbol{n}(p)$$

then M_r is called parallel to M [8].

Let M and M_r be an oriented surface in Euclidean 3-space. $\boldsymbol{\alpha}(s)$ is curve on surface M and $\beta(s_\beta)$ is curve on parallel surface M_r. Then, the formula between Darboux frame of $\boldsymbol{\alpha}(s)$ and $\beta(s_\beta)$ is

$$\begin{pmatrix} \boldsymbol{T}^*(s) \\ \boldsymbol{Y}^*(s) \\ \boldsymbol{n}(s) \end{pmatrix} = \begin{pmatrix} \frac{1-rk_n(s)}{\sqrt{(1-rk_n(s))^2+(rt_r(s))^2}} & \frac{-rt_r(s)}{\sqrt{(1-rk_n(s))^2+(rt_r(s))^2}} & 0 \\ \frac{rt_r(s)}{\sqrt{(1-rk_n(s))^2+(rt_r(s))^2}} & \frac{1-rk_n(s)}{\sqrt{(1-rk_n(s))^2+(rt_r(s))^2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{T}(s) \\ \boldsymbol{Y}(s) \\ \boldsymbol{n}(s) \end{pmatrix}$$

Then we have the following Frenet-Serret type formula:

$$\frac{d}{ds_\beta}\begin{pmatrix}\boldsymbol{T}^*(s)\\ \boldsymbol{Y}^*(s)\\ \boldsymbol{n}(s)\end{pmatrix}=\begin{pmatrix}0 & k_g^*(s) & k_n^*(s)\\ -k_g^*(s) & 0 & t_r^*(s)\\ -k_n^*(s) & -t_r^*(s) & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{T}^*(s)\\ \boldsymbol{Y}^*(s)\\ \boldsymbol{n}(s)\end{pmatrix}$$

Here,

$$\begin{aligned}
k_g^* &= \frac{k_g}{\sqrt{(1-rk_n)^2+(rt_r)^2}}-\frac{rt_r(rk_n^{'})+(1-rk_n)(rt_r^{'})}{((1-rk_n)^2+(rt_r)^2)^{3/2}}\\
k_n^* &= \frac{k_n(1-rk_n)-t_r(rt_r)}{(1-rk_n)^2+(rt_r)^2}\\
t_r^* &= \frac{t_r}{(1-rk_n)^2+(rt_r)^2}
\end{aligned}$$

where k_g^* is geodesic curvature, k_n^* is normal curvature and t_r^* is geodesic torsion [6].
A constant angle surface in Euclidean 3-space $\mathbb{E}^3$ is a surface whose normal $\boldsymbol{n}$ of the curve makes a constant angle with the fixed direction $\boldsymbol{k}$ [7],

$$\widehat{(\boldsymbol{n},\boldsymbol{k})}=\theta.$$

In addition, we review special curves. Let $\boldsymbol{\alpha}$ be a unit speed curve on an oriented surface M and $\{\boldsymbol{T},\boldsymbol{Y},\boldsymbol{n}\}$ be the Darboux frame along $\boldsymbol{\alpha}$. The curve $\boldsymbol{\alpha}$ is called a relatively normal-slant helix if the vector field $\boldsymbol{Y}$ of $\boldsymbol{\alpha}$ makes a constant angle with a fixed direction, i.e. there exists a fixed unit vector $\boldsymbol{d}$ and a constant angle θ such that $\langle \boldsymbol{Y},\boldsymbol{d}\rangle = cos\theta$ [2].
Let M be a regular surface and let $\boldsymbol{\alpha}: I\in\mathbb{R}\to M$ be a unit-speed isophote curve. Then from the definiton of the isophote curve

$$\langle \boldsymbol{n}(u,v),\boldsymbol{d}\rangle = cos\theta = constant,$$

where $\boldsymbol{n}(u,v)$ is the unit normal vector of the surface $S(u,v)$ (a parameterization of M) and $\boldsymbol{d}$ is the unit fixed vector on the axis of isophote curve[1].
Finally, we review three special vector fields and approximations of surfaces along curves. Hananoi, Ito and Izumiya in [3] defined three vector fields $\bar{\boldsymbol{D}}_n(s)$, $\bar{\boldsymbol{D}}_r(s)$, $\bar{\boldsymbol{D}}_o(s)$ along $\boldsymbol{\alpha}$ by

$$\begin{aligned}
\bar{\boldsymbol{D}}_n(s) &= -k_n(s)\boldsymbol{Y}(s)+k_g(s)\boldsymbol{n}(s),\\
\bar{\boldsymbol{D}}_r(s) &= t_r(s)\boldsymbol{T}(s)+k_g(s)\boldsymbol{n}(s),\\
\bar{\boldsymbol{D}}_o(s) &= t_r(s)\boldsymbol{T}(s)-k_n(s)\boldsymbol{Y}(s)
\end{aligned}$$

which are called the normal Darboux vector field, the rectifying Darboux vector field and the osculating Darboux vector field along $\boldsymbol{\alpha}$, respectively. They also defined the spherical image of each Darboux vector field as follows:

$$\begin{aligned}
\boldsymbol{D}_n(s) &= \frac{-k_n(s)\boldsymbol{Y}(s)+k_g(s)\boldsymbol{n}(s)}{\sqrt{k_n^2(s)+k_g^2(s)}} \qquad if\ (k_n(s),k_g(s))\neq(0,0),\\
\boldsymbol{D}_r(s) &= \frac{t_r(s)\boldsymbol{T}(s)+k_g(s)\boldsymbol{n}(s)}{\sqrt{k_g^2(s)+t_r^2(s)}} \qquad if\ (t_r(s),k_g(s))\neq(0,0),\\
\boldsymbol{D}_o(s) &= \frac{t_r(s)\boldsymbol{T}(s)-k_n(s)\boldsymbol{Y}(s)}{\sqrt{t_r^2(s)+k_n^2(s)}} \qquad if\ (k_n(s),t_r(s))\neq(0,0).
\end{aligned}$$

Izumiya and Otani [5] defined osculating developable surface of M along $\boldsymbol{\alpha}$. $\boldsymbol{\alpha}$ is a regular curve on a surface M with $t_r^2(s)+k_n^2(s)\neq 0$, they define a map $\boldsymbol{OD}_\alpha: I\times\mathbb{R}\to\mathbb{R}^3$ by

$$\begin{aligned}
\boldsymbol{OD}_\alpha(s,u) &= \boldsymbol{\alpha}(s)+u\boldsymbol{D}_o(s)\\
&= \boldsymbol{\alpha}(s)+u\left(\frac{t_r(s)\boldsymbol{T}(s)-k_n(s)\boldsymbol{Y}(s)}{\sqrt{t_r^2(s)+k_n^2(s)}}\right).
\end{aligned}$$

$\boldsymbol{OD}_\alpha$ is called an osculating developable surface of M along $\boldsymbol{\alpha}$. If $(s_0,0)\in I\times\mathbb{R}$ is a regular point (i.e., $k_n(s_0)\neq 0$), the normal vector of $\boldsymbol{OD}_\alpha$ at $\boldsymbol{OD}_\alpha(s_0,0)=\boldsymbol{\alpha}(s_0)$ has the same direction of the normal vector of

M at $\boldsymbol{\alpha}(s_0)$. Therefore, $\boldsymbol{OD}_\alpha$ is called osculating developable surface of M along $\boldsymbol{\alpha}$. Moreover, Hananoi and Izumiya [4] defined normal developable surface of M along $\boldsymbol{\alpha}$. $\boldsymbol{\alpha}$ is a regular curve on a surface M with $k_g^2(s) + t_r^2(s) \neq 0$, they define a map $\boldsymbol{ND}_\alpha : I \times \mathbb{R} \to \mathbb{R}^3$ by

$$\begin{aligned}\boldsymbol{ND}_\alpha(s,u) &= \boldsymbol{\alpha}(s) + u\boldsymbol{D}_r(s)\\ &= \boldsymbol{\alpha}(s) + u\left(\frac{t_r(s)\boldsymbol{T}(s) + k_g(s)\boldsymbol{n}(s)}{\sqrt{t_r^2(s) + k_g^2(s)}}\right).\end{aligned}$$

$\boldsymbol{ND}_\alpha$ is called a normal developable surface of M along $\boldsymbol{\alpha}$. If $(s_0, 0) \in I \times \mathbb{R}$ is a regular point (i.e., $k_g(s_0) \neq 0$), the normal vector of $\boldsymbol{ND}_\alpha$ at $\boldsymbol{ND}_\alpha(s_0, 0) = \boldsymbol{\alpha}(s_0)$ is orthogonal to the normal vector of M along $\boldsymbol{\alpha}$. Therefore, $\boldsymbol{ND}_\alpha$ is called normal developable surface of M along $\boldsymbol{\alpha}$.

3. Flat Approximations of Parallel Surfaces Along Curves and Constant Angle Surfaces

In this section we introduce flat approximation of M_r which is parallel of M. Let $\boldsymbol{\alpha} : I \subset \mathbb{R} \to M \subset \mathbb{R}^3$ be a regular curve on surface M with $k_n^2(s) + t_r^2(s) \neq 0$ and ruled surface given with $\boldsymbol{\Phi}_\alpha(s,u) = \boldsymbol{\alpha}(s) + u\boldsymbol{D}_o(s)$ is flat approximation surface of M along $\boldsymbol{\alpha}$ [5]. $\boldsymbol{\beta} = \boldsymbol{\alpha} + r\boldsymbol{n}$ and $\boldsymbol{\beta} : U \subset \mathbb{R} \to M_r \subset \mathbb{R}^3$ is a regular curve on surface M_r with $k_n^2(s) + t_r^2(s) \neq 0$. Flat approximation of M_r along $\boldsymbol{\beta}$ can be given a map $\boldsymbol{\Phi}_\beta : U \times \mathbb{R} \to \mathbb{R}^3$ by

$$\boldsymbol{\Phi}_\beta(s,u) = \boldsymbol{\beta}(s) + u\left(\frac{t_r(s)\boldsymbol{T}(s) - k_n(s)\boldsymbol{Y}(s)}{\sqrt{t_r^2(s) + k_n^2(s)}}\right) = \boldsymbol{\beta}(s) + u\boldsymbol{D}_o(s).$$

This is a developable surface being that

$$det(\boldsymbol{\beta}', \boldsymbol{D}_o, \boldsymbol{D}_o') = \left((1 - rk_n)\boldsymbol{T} - rt_r\boldsymbol{Y}, \frac{t_r\boldsymbol{T} - k_n\boldsymbol{Y}}{\sqrt{t_r^2 + k_n^2}}, \left(k_g + \frac{k_n t_r' - k_n' t_r}{t_r^2 + k_n^2}\right)\frac{k_n\boldsymbol{T} + t_r\boldsymbol{Y}}{\sqrt{t_r^2 + k_n^2}}\right) = 0.$$

We can also give two invariants of $(\boldsymbol{\beta}, M_r)$. One of these invariants is $\delta(s)$ given in [5] by Izumiya and Otani. Also, we introduce the other invariant $\bar{\sigma}(s)$ as follows:

$$\begin{aligned}\delta(s) &= \bar{g}(s) = k_g(s) + \frac{k_n(s)t_r'(s) - k_n'(s)t_r(s)}{t_r^2(s) + k_n^2(s)},\\ \bar{\sigma}(s) &= \frac{t_r(s)}{\sqrt{t_r^2(s) + k_n^2(s)}} - \left(\frac{k_n(s)(1 - rk_n(s)) - rt_r^2(s)}{\delta(s)\sqrt{t_r^2(s) + k_n^2(s)}}\right)' ;\ (\text{when } \delta(s) \neq 0).\end{aligned}$$

Besides,

$$\frac{\partial \boldsymbol{\Phi}_\beta}{\partial s} \times \frac{\partial \boldsymbol{\Phi}_\beta}{\partial u} = \left(\frac{-k_n + r(k_n^2 + t_r^2)}{\sqrt{t_r^2 + k_n^2}} - u\delta\right)\boldsymbol{n}.$$

Hence, $(s_0, u_0) \in U \times \mathbb{R}$ is a singular point of $\boldsymbol{\Phi}_\beta$ if and only if $\delta(s_0) \neq 0$ and

$$u_0 = \frac{-k_n(s_0) + r(k_n^2(s_0) + t_r^2(s_0))}{\delta(s_0)\sqrt{k_n^2(s_0) + t_r^2(s_0)}}.$$

If $(s_0, 0)$ is a regular point, the normal vector of $\boldsymbol{\Phi}_\beta$ at $\boldsymbol{\Phi}_\beta(s_0, 0) = \boldsymbol{\beta}(s_0)$ has the same direction of the normal vector of M_r at $\boldsymbol{\beta}(s_0)$. Therefore, $\boldsymbol{\Phi}_\beta$ is osculating developable surface of M_r along $\boldsymbol{\beta}$.
Also, if $r = 0$, flat approximation of M coincide with flat approximation of M_r. In other words, osculating developable surface $\boldsymbol{\Phi}_\beta$ and invariant $\bar{\sigma}(s)$ coincide with, respectively, osculating developable surface $\boldsymbol{OD}_\alpha$ and invariant $\sigma(s)$ given in [5] by Izumiya and Otani.

Theorem 3.1. *Let M_r be parallel surface of M. $\boldsymbol{\alpha} : I \subset \mathbb{R} \to M \subset \mathbb{R}^3$ is a regular curve on M with $k_n^2(s) + t_r^2(s) \neq 0$ and $\boldsymbol{\beta} = \boldsymbol{\alpha} + r\boldsymbol{n}$, $\boldsymbol{\beta} : U \subset \mathbb{R} \to M_r \subset \mathbb{R}^3$ is a regular curve on M_r with $k_n^2(s) + t_r^2(s) \neq 0$. $\boldsymbol{\Phi}_\alpha(s,u) = \boldsymbol{\alpha}(s) + u\boldsymbol{D}_o(s)$ and $\boldsymbol{\Phi}_\beta(s,u) = \boldsymbol{\beta}(s) + u\boldsymbol{D}_o(s)$ are flat approximation surfaces of M and M_r, respectively. Then, $\boldsymbol{\Phi}_\beta$ is parallel surface of $\boldsymbol{\Phi}_\alpha$.*

Proof.

$$\begin{aligned}\boldsymbol{\Phi}_\beta(s,u) &= \boldsymbol{\beta}(s) + u\left(\frac{t_r(s)\boldsymbol{T}(s) - k_n(s)\boldsymbol{Y}(s)}{\sqrt{t_r^2(s)+k_n^2(s)}}\right)\\ &= \boldsymbol{\alpha}(s) + r\boldsymbol{n}(s) + u\boldsymbol{D}_o(s)\\ &= \boldsymbol{\alpha}(s) + u\boldsymbol{D}_o(s) + r\boldsymbol{n}(s)\\ &= \boldsymbol{\Phi}_\alpha(s,u) + r\boldsymbol{n}(s).\end{aligned}$$

□

In the study of Izumiya and Otani in [5], they gave the Theorem 3.1. for flat approximation surface of the surface along the curve. Now, we give the following theorem which is similar to Theorem 3.1.[5], for flat approximation surface of the parallel surface along the curve:

Theorem 3.2. *Let M_r be parallel surface of M and $\boldsymbol{\beta} = \boldsymbol{\alpha} + r\boldsymbol{n}$, $\boldsymbol{\beta} : U \subset \mathbb{R} \to M_r \subset \mathbb{R}^3$ is a regular curve on M_r with $k_n^2(s) + t_r^2(s) \neq 0$. $\boldsymbol{\Phi}_\beta(s,u) = \boldsymbol{\beta}(s) + u\boldsymbol{D}_o(s)$ is flat approximation surfaces of M_r. Then we have:*

(A) The following are equivalent:
  - *(1) $\boldsymbol{\Phi}_\beta$ is a cylinder,*
  - *(2) $\delta(s) = \bar{g}(s) = 0$,*
  - *(3) $\boldsymbol{\beta}$ is a contour generator with respect to an orthogonal projection.*

(B) If $\delta(s) \neq 0$, then the following are equivalent:
  - *(1) $\boldsymbol{\Phi}_\beta$ is a conical surface,*
  - *(2) $\bar{\sigma}(s) = 0$,*
  - *(3) $\boldsymbol{\beta}$ is a contour generator wiht respect to a central projection.*

In the study of Izumiya and Otani in [5] gave singularities of flat approximation surface of surface along the curve through Theorem 3.3. and now we give the following theorem which is similar to Theorem 3.3.[5], for singularities of flat approximation surface of the parallel surface along the curve:

Theorem 3.3. *Let M_r be parallel surface of M and $\boldsymbol{\beta} = \boldsymbol{\alpha} + r\boldsymbol{n}$, $\boldsymbol{\beta} : U \subset \mathbb{R} \to M_r \subset \mathbb{R}^3$ is a regular curve on M_r with $k_n^2(s) + t_r^2(s) \neq 0$. $\boldsymbol{\Phi}_\beta(s,u) = \boldsymbol{\beta}(s) + u\boldsymbol{D}_o(s)$ is flat approximation surfaces of M_r. Then we have the following:*

(1) The image of flat approximation surface $\boldsymbol{\Phi}_\beta$ of M_r along $\boldsymbol{\beta}$ is nonsingular at (s_0, u_0) if and only if

$$\frac{-k_n(s_0) + r(k_n^2(s_0) + t_r^2(s_0))}{\sqrt{t_r^2(s_0) + k_n^2(s_0)}} - u_0\delta(s_0) \neq 0,$$

(2) The image of flat approximation surface $\boldsymbol{\Phi}_\beta$ of M_r along $\boldsymbol{\beta}$ is locally diffeomorphic to the cuspidaledge $C \times \mathbb{R}$ at (s_0, u_0) if $\delta(s) \neq 0$, $\bar{\sigma}(s) \neq 0$ and

$$u_0 = \frac{-k_n(s_0) + r(k_n^2(s_0) + t_r^2(s_0))}{\delta(s_0)\sqrt{t_r^2(s_0) + k_n^2(s_0)}},$$

(3) The image of flat approximation surface $\boldsymbol{\Phi}_\beta$ of M_r along $\boldsymbol{\beta}$ is locally diffeomorphic to the swallowtail at (s_0, u_0) if $\delta(s) \neq 0$, $\bar{\sigma}(s) \neq 0$, $\bar{\sigma}'(s) \neq 0$ and

$$u_0 = \frac{-k_n(s_0) + r(k_n^2(s_0) + t_r^2(s_0))}{\delta(s_0)\sqrt{t_r^2(s_0) + k_n^2(s_0)}}.$$

Here, $C \times \mathbb{R} = \{(x_1, x_2, x_3) | x_1^2 = x_2^3\}$ is the cuspidal edge and $SW = \{(x_1, x_2, x_3) | x_1 = 3u^4 + u^2v, x_2 = 4u^3 + 2uv, x_3 = v\}$ is the swallowtail.

Theorem 3.4. *Let M_r be parallel surface of M. $\boldsymbol{\alpha} : I \subset \mathbb{R} \to M \subset \mathbb{R}^3$ is a regular curve on M with $k_n^2(s) + t_r^2(s) \neq 0$ and $\boldsymbol{\beta} = \boldsymbol{\alpha} + r\boldsymbol{n}$, $\boldsymbol{\beta} : U \subset \mathbb{R} \to M_r \subset \mathbb{R}^3$ is a regular curve on M_r with $k_n^2(s) + t_r^2(s) \neq 0$. $\boldsymbol{\Phi}_\alpha(s,u) = \boldsymbol{\alpha}(s) + u\boldsymbol{D}_o(s)$ and $\boldsymbol{\Phi}_\beta(s,u) = \boldsymbol{\beta}(s) + u\boldsymbol{D}_o(s)$ are flat approximation surfaces of M and M_r, respectively. Then, the following are equivalent:*

(1) Curve $\boldsymbol{\alpha}$ on M is isophote curve,

(2) *Curve $\boldsymbol{\beta}$ on M_r is isophote curve,*
(3) *$\boldsymbol{\Phi}_\alpha$ is constant angle ruled surface,*
(4) *$\boldsymbol{\Phi}_\beta$ is constant angle ruled surface,*
(5) *$\frac{\bar{g}}{f} = constant$,*
(6) *$\boldsymbol{n}$ is circle.*

Proof. Let $\boldsymbol{\alpha}$ be isophote curve on M. Then we have $\langle \boldsymbol{n}, \boldsymbol{d} \rangle = cos\theta = constant$ where $\boldsymbol{d}$ is a unit fixed vector. Due to surface M_r is parallel of surface M, normal vector of M_r is $\boldsymbol{n}$, too. Therefore, curve $\boldsymbol{\beta}$ is isophote curve on M_r. Namely, the condition (1) is equivalent to the condition (2).
Let $\boldsymbol{\beta}$ be isophote curve on M_r. Then we have $\langle \boldsymbol{n}, \boldsymbol{d} \rangle = cos\theta = constant$ where $\boldsymbol{d}$ is a unit fixed vector. On the other hand, normal vector of flat approximation surface $\boldsymbol{\Phi}_\alpha$ is $\boldsymbol{n}$, too. Therefore, surface $\boldsymbol{\Phi}_\alpha$ is a constant angle ruled surface. Namely, the condition (2) is equivalent to the condition (3).
Let $\boldsymbol{\Phi}_\alpha$ is a constant angle ruled surface. Then we have $\langle \boldsymbol{n}, \boldsymbol{d} \rangle = cos\theta = constant$ where $\boldsymbol{d}$ is a unit fixed vector. Besides, normal vector of $\boldsymbol{\Phi}_\beta$ is $\boldsymbol{n}$, too, because of $\boldsymbol{\Phi}_\beta$ is flat approximation surface of M_r. Therefore, flat approximation surface $\boldsymbol{\Phi}_\beta$ is a constant angle ruled surface, too. Namely, the condition (3) is equivalent to the condition (4).
Let $\boldsymbol{\Phi}_\beta$ is a constant angle ruled surface. Then we have $\langle \boldsymbol{n}, \boldsymbol{d} \rangle = cos\theta = constant$ where $\boldsymbol{d}$ is a unit fixed vector. Also, $\boldsymbol{\alpha}$ is isophote curve on M because of normal vector of M is $\boldsymbol{n}$. We know that $\frac{\bar{g}}{f}$ is geodesic curvature of $\boldsymbol{\alpha}$. Then $\frac{\bar{g}}{f}$ is constant. Namely, the condition (4) is equivalent to the contition (5).
Suppose that, geodesic curvature $\frac{\bar{g}}{f}$ is constant. Then, $\boldsymbol{n}$ is circle. Namely, the condition (5) is equivalent to the contition (6).
Let $\boldsymbol{n}$ is circle. Then, curve $\boldsymbol{\alpha}$ on M is isophote curve. Namely, the condition (6) is equivalent to the contition (1). □

Example 3.1. Let M_r is parallel surface of M. $M = \varphi(u, v) = (ucosv, usinv, v)$ is a helicoid (Figure 1a) and $\boldsymbol{\alpha}(t) = \varphi\left(1, \frac{t}{\sqrt{2}}\right) = \left(cos\left(\frac{t}{\sqrt{2}}\right), sin\left(\frac{t}{\sqrt{2}}\right), \frac{t}{\sqrt{2}}\right)$ is a helix on M. Darboux frame is $\{\boldsymbol{T}, \boldsymbol{Y}, \boldsymbol{n}\}$ along $\boldsymbol{\alpha}$. Let's attain flat approximations of M and M_r. By directlyforward calculations, we have

$$\begin{aligned}
\boldsymbol{n}|_{\alpha(t)} &= \frac{1}{\sqrt{2}}\left(sin\left(\frac{t}{\sqrt{2}}\right), -cos\left(\frac{t}{\sqrt{2}}\right), 1\right), \\
\boldsymbol{T} &= \frac{1}{\sqrt{2}}\left(-sin\left(\frac{t}{\sqrt{2}}\right), cos\left(\frac{t}{\sqrt{2}}\right), 1\right), \\
\boldsymbol{Y}|_{\alpha(t)} &= \left(-cos\left(\frac{t}{\sqrt{2}}\right), -sin\left(\frac{t}{\sqrt{2}}\right), 0\right)
\end{aligned}$$

and

$$\begin{aligned}
k_n(t) &= 0, \\
k_g(t) &= \frac{1}{2}, \\
t_r(t) &= -\frac{1}{2}.
\end{aligned}$$

Then, flat approximation surface of M as below (Figure 1b):

$$\boldsymbol{\Phi}_\alpha(v, w_1) = (cosv, sinv, v) + w_1(-sinv, cosv, 1).$$

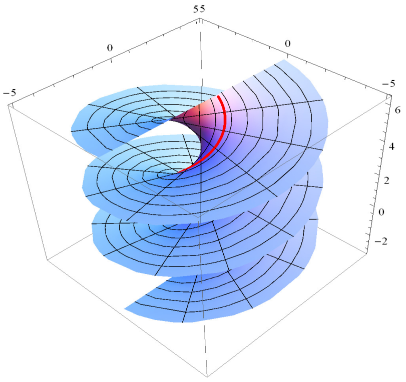
Indeed, we have

$$det\left((-sinv, cosv, 1), (-sinv, cosv, 1), (-cosv, -sinv, 0)\right) = 0.$$

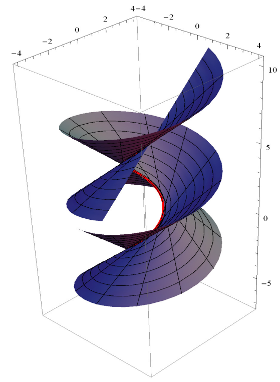
Namely, $\boldsymbol{\Phi}_\alpha$ is a developable surface. Also, we have

$$\begin{aligned}
\frac{\partial \boldsymbol{\Phi}_\alpha}{\partial v} \times \frac{\partial \boldsymbol{\Phi}_\alpha}{\partial w_1} &= w_1(-sinv, cosv, -1) \\
&= -\sqrt{2} w_1 \boldsymbol{n}
\end{aligned}$$

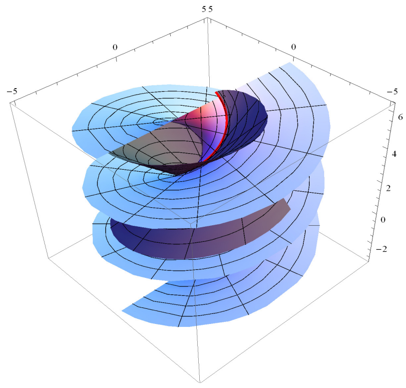
That is, the normal vector of $\boldsymbol{\Phi}_\alpha$ has the same direction normal vector of M.



(a) Surface M



(b) Surface $\mathbf{\Phi}_\alpha$



(c) M and $\mathbf{\Phi}_\alpha$

Figure 1. Flat approximation of surface M with curve α in red color

Now, parallel surface of M as below (Figure 2a):

$$M_r = \varphi_r(u,v) = \varphi(u,v) + \sqrt{2}\boldsymbol{n}$$
$$= (ucosv, usinv, v) + \frac{\sqrt{2}}{\sqrt{1+u^2}}(sinv, -cosv, u)$$

and $\boldsymbol{\beta}$ on M_r with $r = \sqrt{2}$ as below:

$$\boldsymbol{\beta}(t) = \varphi_r\left(1, \frac{t}{\sqrt{2}}\right) = \left(cos\left(\frac{t}{\sqrt{2}}\right) + sin\left(\frac{t}{\sqrt{2}}\right), sin\left(\frac{t}{\sqrt{2}}\right) - cos\left(\frac{t}{\sqrt{2}}\right), \frac{t}{\sqrt{2}} + 1\right).$$

Then, flat approximation surface of M_r (Figure 2b) as below :

$$\mathbf{\Phi}_\beta(v, w_2) = (cosv + sinv, sinv - cosv, v + 1) + w_2(-sinv, cosv, 1).$$

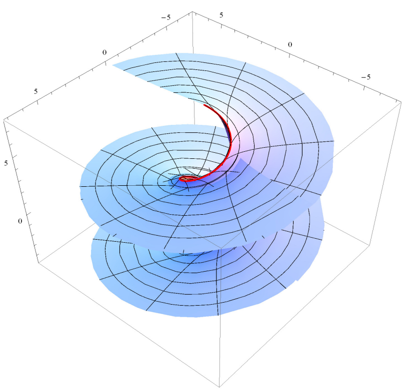
Indeed, we have

$$det((-sinv + cosv, cosv + sinv, 1), (-sinv, cosv, 1), (-cosv, -sinv, 0)) = 0.$$

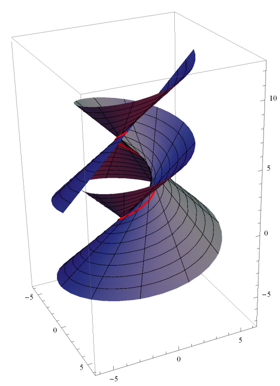
Namely, $\mathbf{\Phi}_\beta$ is a developable surface. Also, we have

$$\frac{\partial \mathbf{\Phi}_\beta}{\partial v} \times \frac{\partial \mathbf{\Phi}_\beta}{\partial w_2} = (1 - w_2)(sinv, -cosv, 1)$$
$$= \sqrt{2}(1 - w_2)\boldsymbol{n}.$$

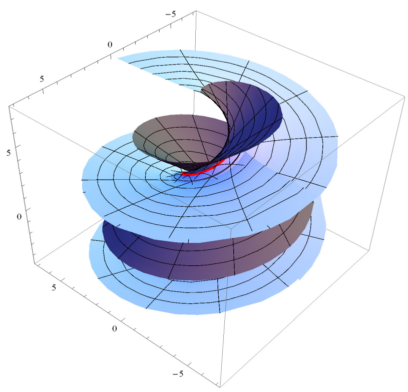
That is, the normal vector of $\mathbf{\Phi}_\beta$ has the same direction normal vector of M_r.



(a) Surface M_r



(b) Surface $\mathbf{\Phi}_\beta$



(c) M_r and $\mathbf{\Phi}_\beta$

Figure 2. Flat approximation of surface M_r with curve β in red color

4. Normal Approximations of Parallel Surfaces Along Curves and Constant Angle Surfaces

In this section we investigate normal approximations of $\boldsymbol{M_y}$, $\boldsymbol{M_t}$ which are parallel of M, associated with $\boldsymbol{D}_r$ and $\boldsymbol{D}_n$ Darboux vector fields. We know that, $\boldsymbol{\alpha}: I \subset \mathbb{R} \to M \subset \mathbb{R}^3$ is a regular curve on surface M with $k_g^2(s) + t_r^2(s) \neq 0$ and $\bar{\boldsymbol{\Phi}}_\alpha(s,u) = \boldsymbol{\alpha}(s) + u\boldsymbol{D}_r(s)$ is normal approximation surface of M [4]. $M_r = M + r\boldsymbol{n}$ is parallel surface of M and $\boldsymbol{\beta} = \boldsymbol{\alpha} + r\boldsymbol{n}$, $\boldsymbol{\beta}: U \subset \mathbb{R} \to M_r \subset \mathbb{R}^3$ is a regular curve on surface M_r with $k_g^2(s) + t_r^2(s) \neq 0$. With the thought in §3, ruled surface $\bar{\boldsymbol{\Phi}}_\beta(s,u) = \boldsymbol{\beta}(s) + u\boldsymbol{D}_r(s)$ is not normal approximation surface of M_r, directly. Because, this ruled surface can't be developable directly. Accordingly, we introduce $\boldsymbol{M_y} = \boldsymbol{\alpha} + w\boldsymbol{Y}$, $\boldsymbol{M_t} = \boldsymbol{\alpha} + w\boldsymbol{T}$ which are ruled surfaces and their normal approximations.

4.1. Y Parellel Surface

Let $\boldsymbol{\alpha}: I \subset \mathbb{R} \to M \subset \mathbb{R}^3$ be a regular curve on surface M. $\boldsymbol{M_y} = \boldsymbol{\alpha} + w\boldsymbol{Y}$ is Y parallel surface of M and $\boldsymbol{\beta} = \boldsymbol{\alpha} + r\boldsymbol{Y}$, $\boldsymbol{\beta}: U \subset \mathbb{R} \to \boldsymbol{M_y} \subset \mathbb{R}^3$ is a regular curve on surface $\boldsymbol{M_y}$ with $k_g^2(s) + t_r^2(s) \neq 0$. Normal approximation of $\boldsymbol{M_y}$ along $\boldsymbol{\beta}$ can be given a map $\bar{\boldsymbol{\Phi}}_\beta : U \times \mathbb{R} \to \mathbb{R}^3$ by

$$\bar{\boldsymbol{\Phi}}_\beta(s,v) = \boldsymbol{\beta}(s) + v\boldsymbol{D}_r(s) = \boldsymbol{\alpha}(s) + r\boldsymbol{Y}(s) + v\left(\frac{t_r(s)\boldsymbol{T}(s) + k_g(s)\boldsymbol{n}(s)}{\sqrt{k_g^2(s)+t_r^2(s)}}\right).$$

This is a developable surface being that

$$det(\boldsymbol{\beta}', \boldsymbol{D}_r, \boldsymbol{D}_r') = \left((1-rk_g)\boldsymbol{T} + rt_r\boldsymbol{n}, \frac{t_r\boldsymbol{T} + k_g\boldsymbol{n}}{\sqrt{k_g^2+t_r^2}}, \left(k_n + \frac{t_rk_g' - k_gt_r'}{k_g^2+t_r^2}\right)\frac{t_r\boldsymbol{n} - k_g\boldsymbol{T}}{\sqrt{k_g^2+t_r^2}}\right) = 0.$$

We can also give an invariant $\delta_r(s)$ of $(\boldsymbol{\beta}, \boldsymbol{M_y})$ given in the study of [4] by Hananoi and Izumiya as follows:

$$\delta_r(s) = k_n(s) + \frac{t_r(s)k_g'(s) - k_g(s)t_r'(s)}{k_g^2(s)+t_r^2(s)}.$$

Besides,

$$\frac{\partial\bar{\boldsymbol{\Phi}}_\beta}{\partial s} \times \frac{\partial\bar{\boldsymbol{\Phi}}_\beta}{\partial v} = \left(\frac{-k_g + r(k_g^2+t_r^2)}{\sqrt{k_g^2+t_r^2}} + v\delta_r\right)\boldsymbol{Y}.$$

Hence, $(s_0, v_0) \in U \times \mathbb{R}$ is a singular point of $\bar{\boldsymbol{\Phi}}_\beta$ if and only if $\delta_r(s_0) \neq 0$ and

$$v_0 = \frac{k_g(s_0) - r(k_g^2(s_0) + t_r^2(s_0))}{\delta_r(s_0)\sqrt{k_g^2(s_0)+t_r^2(s_0)}}.$$

Also, normal vector of $\boldsymbol{M_y}$ is

$$\frac{\partial\boldsymbol{M_y}}{\partial s} \times \frac{\partial\boldsymbol{M_y}}{\partial w} = (1-wk_g)\boldsymbol{n} - wt_r\boldsymbol{T}.$$

Thus, we can say that if $(s_0, 0)$ is a regular point the normal vector of $\bar{\boldsymbol{\Phi}}_\beta$ at $\boldsymbol{\beta}(s_0) = \bar{\boldsymbol{\Phi}}_\beta(s_0, 0)$ is orthogonal to the normal vector of $\boldsymbol{M_y}$ at $\boldsymbol{\beta}(s_0)$. $\bar{\boldsymbol{\Phi}}_\beta$ is the normal approximation surface of $\boldsymbol{M_y}$ along $\boldsymbol{\beta}$.

Theorem 4.1. *Let $\boldsymbol{\alpha}: I \subset \mathbb{R} \to M \subset \mathbb{R}^3$ be a regular curve on M with $k_g^2(s) + t_r^2(s) \neq 0$ and $\bar{\boldsymbol{\Phi}}_\alpha(s,v) = \boldsymbol{\alpha}(s) + v\boldsymbol{D}_r(s)$ is normal approximation of M along $\boldsymbol{\alpha}$. $\boldsymbol{M_y} = \boldsymbol{\alpha} + w\boldsymbol{Y}$ is Y parallel surface of M. $\boldsymbol{\beta} = \boldsymbol{\alpha} + r\boldsymbol{Y}$, $\boldsymbol{\beta}: U \subset \mathbb{R} \to \boldsymbol{M_y} \subset \mathbb{R}^3$ is a regular curve on $\boldsymbol{M_y}$ with $k_g^2(s) + t_r^2(s) \neq 0$ and $\bar{\boldsymbol{\Phi}}_\beta(s,v) = \boldsymbol{\beta}(s) + v\boldsymbol{D}_r(s)$ is normal approximation of $\boldsymbol{M_y}$ along $\boldsymbol{\beta}$. Then, $\bar{\boldsymbol{\Phi}}_\beta$ is parallel surface of $\bar{\boldsymbol{\Phi}}_\alpha$.*

Proof. Unit normal vector of $\bar{\boldsymbol{\Phi}}_\alpha$ is $\boldsymbol{Y}$.

$$\begin{aligned}\bar{\boldsymbol{\Phi}}_\beta(s,v) &= \boldsymbol{\beta}(s) + v\boldsymbol{D}_r(s)\\ &= \boldsymbol{\alpha}(s) + r\boldsymbol{Y}(s) + v\boldsymbol{D}_r(s)\\ &= \boldsymbol{\alpha}(s) + v\boldsymbol{D}_r(s) + r\boldsymbol{Y}(s)\\ &= \bar{\boldsymbol{\Phi}}_\alpha(s,v) + r\boldsymbol{Y}(s).\end{aligned}$$

□

Theorem 4.2. *Let $\boldsymbol{\alpha} : I \subset \mathbb{R} \to M \subset \mathbb{R}^3$ be a regular curve on M with $k_g^2(s) + t_r^2(s) \neq 0$. $\boldsymbol{M_y} = \boldsymbol{\alpha} + w\boldsymbol{Y}$ is Y parallel surface of M. $\boldsymbol{\beta} = \boldsymbol{\alpha} + r\boldsymbol{Y}$, $\boldsymbol{\beta} : U \subset \mathbb{R} \to \boldsymbol{M_y} \subset \mathbb{R}^3$ is a regular curve on surface $\boldsymbol{M_y}$ with $k_g^2(s) + t_r^2(s) \neq 0$ and $\bar{\boldsymbol{\Phi}}_\beta(s, v) = \boldsymbol{\beta}(s) + v\boldsymbol{D}_r(s)$ is normal approximation of $\boldsymbol{M_y}$ along $\boldsymbol{\beta}$. Then, $\boldsymbol{\alpha}$ is relatively normal-slant helix on M if and only if $\bar{\boldsymbol{\Phi}}_\beta$ is a constant angle ruled surface.*

Proof. Let $\boldsymbol{\alpha}$ is relativly normal-slant helix on M and Darboux frame is $\{\boldsymbol{T}, \boldsymbol{Y}, \boldsymbol{n}\}$ along $\boldsymbol{\alpha}$. Then, there exists a fixed unit vector $\boldsymbol{d}$ which provide equation $\langle \boldsymbol{Y}, \boldsymbol{d} \rangle = cos\theta = constant$. On the other hand, unit normal vector of $\bar{\boldsymbol{\Phi}}_\beta$ is $\boldsymbol{Y}$. Therefore, ruled surface $\bar{\boldsymbol{\Phi}}_\beta$ is a constant angle ruled surface. Backwards, let $\bar{\boldsymbol{\Phi}}_\beta$ is a constant angle ruled surface. Then, we have $\langle \boldsymbol{Y}, \boldsymbol{d} \rangle = cos\theta = constant$ with $\boldsymbol{d}$ is a fixed unit vector. It follows that, $\boldsymbol{\alpha}$ is relatively normal-slant helix on M. □

Example 4.1. Let $\boldsymbol{M_y}$ is Y parallel surface of M. $M = \varphi(u, v) = (ucosv, usinv, v)$ is a helicoid (Figure 3a) and $\alpha(v) = \varphi(1, v) = (cosv, sinv, v)$ is a helix on M. Darboux frame is $\{\boldsymbol{T}, \boldsymbol{Y}, \boldsymbol{n}\}$ along α and $r \neq 0$. Let's attain normal approximations of M and $\boldsymbol{M_y}$. By straightforward calculations, we have

$$\begin{aligned} \boldsymbol{n}|_{\alpha(v)} &= \frac{1}{\sqrt{2}}(sinv, -cosv, 1), \\ \boldsymbol{T} &= \frac{1}{\sqrt{2}}(-sinv, cosv, 1), \\ \boldsymbol{Y}|_{\alpha(v)} &= (-cosv, -sinv, 0) \end{aligned}$$

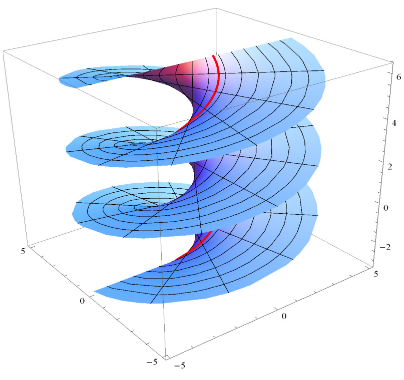
and

$$\begin{aligned} k_n(v) &= 0, \\ k_g(v) &= \frac{1}{\sqrt{2}}, \\ t_r(v) &= \frac{1}{\sqrt{2}}. \end{aligned}$$

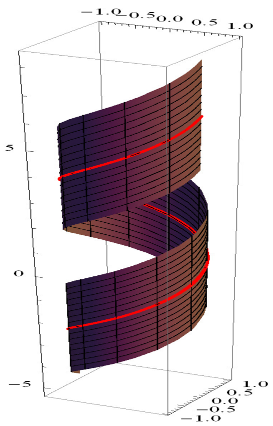
Then, normal approximation surface of M (Figure 3b) as below:

$$\begin{aligned} \boldsymbol{\Phi}_\alpha(v, w_1) &= \alpha(v) + w_1\boldsymbol{D}_r(v) \\ &= (cosv, sinv, v) + w_1(0, 0, 1) \end{aligned}$$

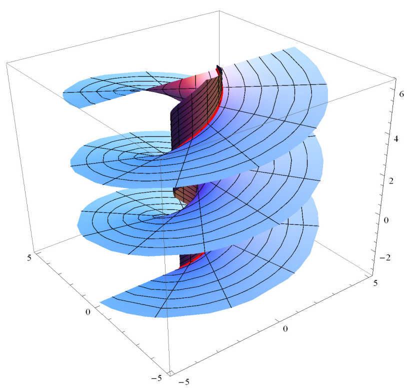
Namely, normal vectors of these two surfaces are orthogonal. Also, normal approximation surface of M is cylinder.



(a) Surface M



(b) Surface $\boldsymbol{\Phi}_\alpha$



(c) M and $\boldsymbol{\Phi}_\alpha$

Figure 3. Normal approximation of surface M with curve α in red color

Now, Y parallel surface of M (Figure 4a) as below:

$$\begin{aligned}\boldsymbol{M_y}(v, w_1) &= \boldsymbol{\alpha}(v) + w_1\boldsymbol{Y}(s)\\ &= (cosv, sinv, v) + w_1(-cosv, -sinv, 0)\\ &= ((1-w_1)cosv, (1-w_1)sinv, v)\end{aligned}$$

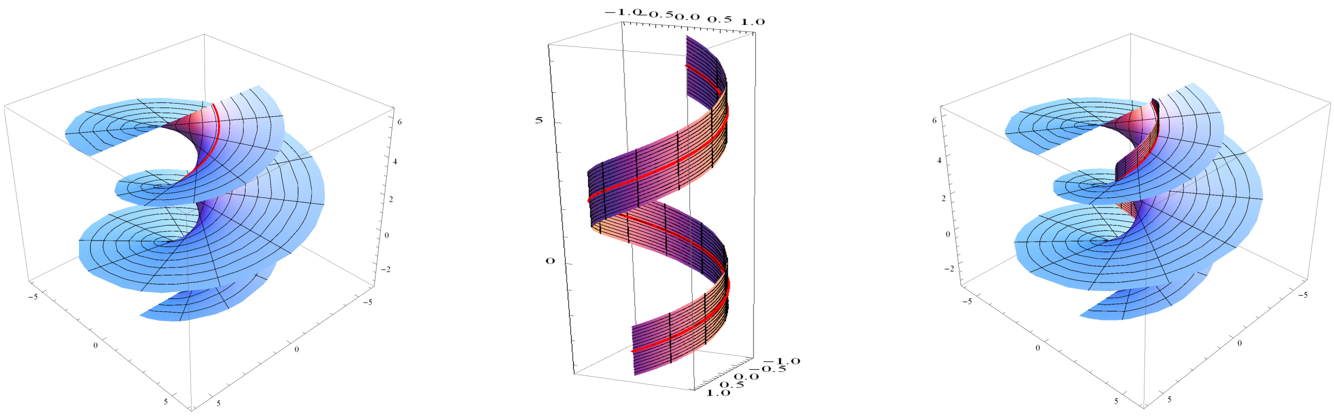
and $\boldsymbol{\beta}$ on $\boldsymbol{M_y}$ with $r = 2$ as below:

$$\begin{aligned}\boldsymbol{\beta}(v) &= \boldsymbol{\alpha}(v) + r\boldsymbol{Y}(v)\\ &= (cosv, sinv, v) + r(-cosv, -sinv, 0)\\ &= (-cosv, -sinv, v)\end{aligned}$$

Then, normal approximation surface of $\boldsymbol{M_y}$ (Figure 4b):

$$\begin{aligned}\bar{\boldsymbol{\Phi}}_\beta(v, w_2) &= \boldsymbol{\beta}(v) + w_2\boldsymbol{D}_r(v)\\ &= (-cosv, -sinv, v) + w_2(0, 0, 1)\end{aligned}$$

This means that, normal approximation surface of $\boldsymbol{M_y}$ along $\boldsymbol{\beta}$ is a cylinder, too.



(a) Surface $\boldsymbol{M_y}$ **(b)** Surface $\bar{\boldsymbol{\Phi}}_\beta$ **(c)** $\boldsymbol{M_y}$ and $\bar{\boldsymbol{\Phi}}_\beta$

Figure 4. Normal approximation of surface $\boldsymbol{M_y}$ with curve β in red color

4.2. *T Parallel Surface*

Let $\boldsymbol{\alpha} : I \subset \mathbb{R} \to M \subset \mathbb{R}^3$ be a regular curve on surface M. $\boldsymbol{M_t} = \boldsymbol{\alpha} + w\boldsymbol{T}$ is T parallel surface of M and $\boldsymbol{\beta}(s) = \boldsymbol{\alpha}(s) + (c-s)\boldsymbol{T}(s)$, $\boldsymbol{\beta} : U \subset \mathbb{R} \to \boldsymbol{M_t} \subset \mathbb{R}^3$ is a involute of $\boldsymbol{\alpha}$ on surface $\boldsymbol{M_t}$ with $k_n^2(s) + k_g^2(s) \neq 0$. Normal approximation of $\boldsymbol{M_t}$ along $\boldsymbol{\beta}$ can be given a map $\bar{\boldsymbol{\Psi}}_\beta : U \times \mathbb{R} \to \mathbb{R}^3$ by

$$\bar{\boldsymbol{\Psi}}_\beta(s, v) = \boldsymbol{\beta}(s) + v\boldsymbol{D}_n(s) = \boldsymbol{\alpha}(s) + (c-s)\boldsymbol{T}(s) + v\left(\frac{-k_n(s)\boldsymbol{Y}(s)+k_g(s)\boldsymbol{n}(s)}{\sqrt{k_n^2(s)+k_g^2(s)}}\right)$$

This is a developable surface being that

$$det(\boldsymbol{\beta}', \boldsymbol{D}_n, \boldsymbol{D}_n') = det\left((c-s)(k_g\boldsymbol{Y} + k_n\boldsymbol{n}), \frac{-k_n\boldsymbol{Y} + k_g\boldsymbol{n}}{\sqrt{k_n^2 + k_g^2}}, \left(-t_r + \frac{k_nk_g' - k_gk_n'}{k_n^2 + k_g^2}\right)\frac{k_g\boldsymbol{Y} + k_n\boldsymbol{n}}{\sqrt{k_n^2 + k_g^2}}\right) = 0.$$

We can also give an invariant $\bar{\delta}_n(s)$ of $(\boldsymbol{\beta}, \boldsymbol{M_t})$ as follows:

$$\bar{\delta}_n(s) = -t_r(s) + \frac{k_n(s)k_g'(s) - k_g(s)k_n'(s)}{k_n^2(s) + k_g^2(s)}.$$

Besides,

$$\frac{\partial\bar{\boldsymbol{\Psi}}_\beta}{\partial s} \times \frac{\partial\bar{\boldsymbol{\Psi}}_\beta}{\partial v} = \left((c-s)\sqrt{k_g^2 + k_n^2} + v\bar{\delta}_n\right)\boldsymbol{T}.$$

Hence, $(s_0, u_0) \in U \times \mathbb{R}$ is a singular point of $\bar{\Psi}_\beta$ if and only if $\bar{\delta}_n(s_0) \neq 0$ and

$$v_0 = \frac{(s_0 - c)\sqrt{k_g^2(s_0) + k_n^2(s_0)}}{\bar{\delta}_n(s_0)}.$$

Also, normal vector of $\boldsymbol{M_t}$ is

$$\frac{\partial \boldsymbol{M_t}}{\partial s} \times \frac{\partial \boldsymbol{M_t}}{\partial w} = -wk_g\boldsymbol{n} + wk_n\boldsymbol{Y}.$$

Thus, we can say that if $(s_0, 0)$ is a regular point the normal vector of $\bar{\boldsymbol{\Psi}}_\beta$ at $\boldsymbol{\beta}(s_0) = \bar{\boldsymbol{\Psi}}_\beta(s_0, 0)$ is orthogonal to the normal vector of $\boldsymbol{M_t}$ at $\boldsymbol{\beta}(s_0)$. $\bar{\boldsymbol{\Psi}}_\beta$ is the normal approximation surface of $\boldsymbol{M_t}$ along $\boldsymbol{\beta}$.

Theorem 4.3. *Let $\boldsymbol{\alpha} : I \subset \mathbb{R} \to M \subset \mathbb{R}^3$ be a regular curve on M with $k_n^2(s) + k_g^2(s) \neq 0$. $\boldsymbol{M_t} = \boldsymbol{\alpha} + w\boldsymbol{T}$ is T parallel surface of M and $\boldsymbol{\beta}(s) = \boldsymbol{\alpha}(s) + (c - s)\boldsymbol{T}(s)$, $\boldsymbol{\beta} : U \subset \mathbb{R} \to \boldsymbol{M_t} \subset \mathbb{R}^3$ is a involute of $\boldsymbol{\alpha}$ on surface $\boldsymbol{M_t}$ with $k_n^2(s) + k_g^2(s) \neq 0$. $\bar{\boldsymbol{\Psi}}_\beta(s, v) = \boldsymbol{\beta}(s) + v\boldsymbol{D}_n(s)$ is normal approximation of $\boldsymbol{M_t}$ along $\boldsymbol{\beta}$. Then, $\boldsymbol{\alpha}$ is helix on M if and only if $\bar{\boldsymbol{\Psi}}_\beta$ is a constant angle ruled surface.*

Proof. Let $\boldsymbol{\alpha}$ is helix on M and Darboux frame is $\{\boldsymbol{T}, \boldsymbol{Y}, \boldsymbol{n}\}$ along $\boldsymbol{\alpha}$. Then, there exists a fixed unit vector $\boldsymbol{d}$ which provide equation $\langle \boldsymbol{T}, \boldsymbol{d} \rangle = cos\theta = constant$. On the other hand, unit normal vector of $\bar{\boldsymbol{\Psi}}_\beta$ is $\boldsymbol{T}$. Therefore, $\bar{\boldsymbol{\Psi}}_\beta$ is a constant angle ruled surface. Backwards, let $\bar{\boldsymbol{\Psi}}_\beta$ is a constant angle ruled surface. Then, we have $\langle \boldsymbol{T}, \boldsymbol{d} \rangle = cos\theta = constant$ with *d* is a fixed unit vector. It follows that, $\boldsymbol{\alpha}$ is helix on M. □

5. Conclusions

In this paper, we gave flat approximation of parallel surface and showed that flat approximation surface of the surface and flat approximation surface of parallel surface are parallel, too. Also, if curve on the surface is isophote curve then flat approximation surface is constant angle ruled surface. Then, we saw that we can't define normal approximation of parallel surface associated with rectifying and normal Darboux vector fields. After, we gave special ruled surfaces called Y parallel surface and T parallel surface. We gave normal approximations of these parallel surfaces. We showed that normal approximation surface of the surface and normal approximation surface of Y parallel surface are parallel, too. Finally, we gave that if curve on the surface be chosen special then normal approximation surface of these parallel surfaces are constant angle ruled surfaces.

Acknowledgements

We would like to thank to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

There is no funding.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] Doğan, F., Yaylı, Y.: *On isophote curves and their characterizations*. Turkish Journal of Mathematics, **39**, 650–664 (2015). https://doi.org/10.3906/mat-1410-4

[2] Düldül, M., Macit, N.: *Relatively normal-slant helices lying on a surface and their characterizations*. Hacettepe Journal of Mathematics and Statistics, **46**, 397–408 (2017).

[3] Hananoi, S., Ito, N., Izumiya, S.: *Spherical Darboux images of curves on surfaces*. Beitr. Algebra Geom., **56**, 575–585 (2015). https://doi.org/10.1007/s13366-015-0240-z
[4] Hananoi, S., Izumiya, S.: *Normal developable surfaces of surfaces along curves*. Proceedings of the Royal Society of Edinburgh Section A: Mathematics, **147**, 177–203 (2017). https://doi.org/10.1017/S030821051600007X
[5] Izumiya, S., Otani, S.: *Flat Approximations of Surfaces Along Curves*. Demonstratio Mathematica, **48**, 217–241 (2015). https://doi.org/10.1515/dema-2015-0018
[6] Kızıltuğ, S., Tarakcı, Ö., Yaylı, Y.: *On the curves lying on parallel surfaces in the Euclidean 3-space* $\mathbb{E}^3$. Journal of Advanced Research in Dynamical and Control Systems, **5**, 26–35 (2013).
[7] Özkaldı, S., Yaylı, Y.: *Constant Angle Surfaces and Curves in* $\mathbb{E}^3$. International Electronic Journal of Geometry, **4**, 70–78 (2011).
[8] Sabuncuoğlu, A.: Diferensiyel Geometri. Nobel Akademik Yayıncılık. Ankara (2016).

Affiliations

BÜŞRA KÖSE
ADDRESS: Department of Mathematics, Faculty of Science, Ankara University, Ankara-Turkey.
E-MAIL: bsrkose@ankara.edu.tr
ORCID ID: 0009-0005-1739-4207

YUSUF YAYLI
ADDRESS: Department of Mathematics, Faculty of Science, Ankara University, Ankara-Turkey.
E-MAIL: yayli@science.ankara.edu.tr
ORCID ID: 0000-0003-4398-3855