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On Lacunary $\mathscr{I}_2^*$-Convergence and Lacunary $\mathscr{I}_2^*$-Cauchy Sequence

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Abstract
In the study conducted here, we have given some new concepts in summability. In this sense, firstly, we have given the concept of lacunary $\mathscr{I}_2^*$-convergence and we have investigated the relations between lacunary $\mathscr{I}_2$-convergence and lacunary $\mathscr{I}_2^*$-convergence. Also, we have given the concept of lacunary $\mathscr{I}_2^*$-Cauchy sequence and investigated the relations between lacunary $\mathscr{I}_2$-Cauchy sequence and lacunary $\mathscr{I}_2^*$-Cauchy sequence.

Keywords: Double sequence, Ideal convergence, Ideal Cauchy sequence, Lacunary sequence
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1. Introduction and Definitions

During this study, we take $\mathbb{N}$ as the set of all positive integers and $\mathbb{R}$ as the set of all real numbers. The convergence in sequences of real numbers is generalized to the concept of statistical convergence by Fast [1] and Schoenberg [2], independently. The concept of ideal convergence, which is a generalization of statistical convergence that would later inspire many researchers, was first defined by Kostyrko et al. [3]. Nabiev [4] studied on $\mathscr{I}$-Cauchy sequence and $\mathscr{I}^*$-Cauchy sequence with some characteristics. Using the ideal notion, ideal-statistical convergence and ideal lacunary statistical convergence were introduced by Das et al.[5] as new notions. In the topology induced by random n-normed spaces, the lacunary ideal convergence and lacunary ideal Cauchy with some important characteristics investigated by Yamanci and Gürdal [6]. The lacunary ideal convergence was studied by Debnath [7] in intuitionistic fuzzy normed linear spaces. The ideal lacunary convergence was introduced by Tripathy et al.[8]. In recent times, the concepts of the lacunary $\mathscr{I}^*$-convergence, strongly lacunary $\mathscr{I}^*$-convergence, lacunary $\mathscr{I}^*$-Cauchy sequence and strongly lacunary $\mathscr{I}^*$-Cauchy sequence were introduced by Akın and Dündar [9, 10]. Das et al. [11] studied $\mathscr{I}$ and $\mathscr{I}^*$-convergence for double sequences. Dündar and Altay [12, 13] introduced $\mathscr{I}_2$-ideal convergence and ideal Cauchy double sequences in the linear metric space and they investigated some characteristics and between relations. Dündar et al. [14] studied strongly $\mathscr{I}_2$-lacunary convergence and $\mathscr{I}_2$lacunary Cauchy double sequences of sets. Hazarika [15] studied the lacunary ideal convergence for double sequences.

In recently, the notions of convergence, statistical convergence and ideal convergence in some metric spaces and normed spaces were studied in summability theory by a lot of mathematicians. In the study conducted here, we defined the lacunary $\mathscr{I}_2^*$-convergence. We investigate the connections between lacunary $\mathscr{I}_2$-convergence and lacunary $\mathscr{I}_2^*$-convergence. Also, we defined the concept of lacunary $\mathscr{I}_2^*$-Cauchy sequence and investigate the relations between lacunary $\mathscr{I}_2$-Cauchy sequence and lacunary $\mathscr{I}_2^*$-Cauchy sequence.

Some basic definitions, concepts and characteristics that will be used throughout the study and are available in the literature will now be noted (see [3, 4], [6]-[10], [12, 13], [16]-[20])

For $\mathscr{I} \subseteq 2^{\mathbb{N}}$, if the following propositions
(i) $\emptyset \in \mathscr{I}$, (ii) If $G, H \in \mathscr{I}$, then $G \cup H \in \mathscr{I}$, (iii) If $G \in \mathscr{I}$ and $H \subseteq G$, then $H \in \mathscr{I}$
hold, then $\mathscr{I} \subseteq 2^{\mathbb{N}}$ is named an ideal.

If $\mathbb{N} \notin \mathscr{I}$, then $\mathscr{I}$ is named a non-trivial ideal. Also, if $\{k\} \in \mathscr{I}$ for each $k \in \mathbb{N}$, then a non-trivial ideal is named an admissible ideal.

For $\mathscr{F} \subseteq 2^{\mathbb{N}}$, if the following propositions
(i) $\emptyset \notin \mathscr{F}$, (ii) If $G, H \in F$, then $G \cap H \in \mathscr{F}$, (iii) If $G \in \mathscr{F}$ and $H \supseteq G$, then $H \in \mathscr{F}$
hold, then $\mathscr{F} \subseteq 2^{\mathbb{N}}$ is named a filter.

For a non-trivial ideal $\mathscr{I} \subseteq 2^{\mathbb{N}}$

$$\mathscr{F}(\mathscr{I}) = \{G \subset \mathbb{N} : (\exists H \in \mathscr{I})(G = \mathbb{N} \backslash H)\}$$

is named the filter associated with $\mathscr{I}$.

By a lacunary sequence $\theta = \{k_r\}$, we mean an increasing integer sequence such that

$$k_0 = 0 \text{ and } h_r = k_r - k_{r-1} \to \infty, \text{ as } r \to \infty.$$

During this study, the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r.

Then after this, we take $\theta = \{k_r\}$ be a lacunary sequence and $\mathscr{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal.

A sequence $(x_k) \subset \mathbb{R}$ is lacunary convergent to $\ell \in \mathbb{R}$, if

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} x_k = \ell.$$

A sequence $(x_k) \subset \mathbb{R}$ is lacunary Cauchy sequence if

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k,p \in I_r} (x_k - x_p) = 0.$$

If for each $\varepsilon > 0$

$$\left\{ r \in \mathbb{N} : \left| \frac{1}{h_r} \sum_{k \in I_r} x_k - \ell \right| \geq \varepsilon \right\} \in \mathscr{I}$$

holds, then the sequence $(x_k) \subset \mathbb{R}$ is lacunary $\mathscr{I}$-convergent to $\ell \in \mathbb{R}$ and we write $x_k \to \ell[\mathscr{I}_\theta]$.

A sequence $(x_k) \subset \mathbb{R}$ is lacunary $\mathscr{I}$-Cauchy if for every $\varepsilon > 0$ there exists $N_0 = N_0(\varepsilon)$ such that

$$\left\{ r \in \mathbb{N} : \left| \frac{1}{h_r} \sum_{k \in I_r} (x_k - x_{N_0}) \right| \geq \varepsilon \right\} \in \mathscr{I}.$$

A sequence $(x_k) \subset \mathbb{R}$ is lacunary $\mathscr{I}^*$-convergent to $\ell \in \mathbb{R}$ iff there exists any set $G = \{g_1 < g_2 < \cdots < g_k < \cdots\} \subset \mathbb{N}$ such that for the set $G' = \{r \in \mathbb{N} : g_k \in I_r\} \in \mathscr{F}(\mathscr{I})$, we have

$$\lim_{\substack{r \to \infty \\ (r \in G')}} \frac{1}{h_r} \sum_{k \in I_r} x_{g_k} = \ell$$

and we write $x_k \to \ell(\mathscr{I}_\theta^*)$.

A sequence $(x_k) \subset \mathbb{R}$ is lacunary $\mathscr{I}^*$-Cauchy sequence iff there exists any set $G = \{g_1 < g_2 < \cdots < g_k < \cdots\} \subset \mathbb{N}$ such that for the set $G' = \{r \in \mathbb{N} : g_k \in I_r\} \in \mathscr{F}(\mathscr{I})$, we have

$$\lim_{\substack{r \to \infty \\ (r \in G')}} \frac{1}{h_r} \sum_{k,p \in I_r} (x_{g_k} - x_{g_p}) = 0.$$

For a double sequence $\theta = \{(k_r, j_u)\}$, if there exist two increasing sequence of integers such that

$$k_0 = 0,\ h_r = k_r - k_{r-1} \to \infty \text{ and } j_0 = 0,\ \bar{h}_u = j_u - j_{u-1} \to \infty \text{ as } r,u \to \infty,$$

then $\theta = \{(k_r, j_u)\}$ is named a double lacunary sequence. We take the following screenings for double lacunary sequence:

$$k_{ru} = k_r j_u,\ h_{ru} = h_r \bar{h}_u,\ I_{ru} = \{(k,j) : k_{r-1} < k \leq k_r \text{ and } j_{u-1} < j \leq j_u\},\ \ q_r = \frac{k_r}{k_{r-1}} \text{ and } q_u = \frac{j_u}{j_{u-1}}.$$

Then after this, we think $\mathscr{I}_2 \subset 2^{\mathbb{N}^2}$ as a non-trivial admissible ideal.

For each $k \in \mathbb{N}$ and a non-trivial ideal $\mathscr{I}_2 \subset 2^{\mathbb{N}^2}$, if $\{k\} \times \mathbb{N} \in \mathscr{I}_2$ and $\mathbb{N} \times \{k\} \in \mathscr{I}_2$, then we say that $\mathscr{I}_2$ is named strongly admissible ideal.

If $\mathscr{I}_2 \subset 2^{\mathbb{N}^2}$ is a strongly admissible ideal, then clearly $\mathscr{I}_2 \subset 2^{\mathbb{N}^2}$ is an admissible ideal.

Let $\mathscr{I}_2^0 = \{A \subset \mathbb{N}^2 : \big(\exists m(A) \in \mathbb{N}\big)\big(i, j \geq m(A) \Rightarrow (i, j) \notin A\big)\}$. Then, $\mathscr{I}_2^0$ is a non-trivial strongly admissible ideal and clearly $\mathscr{I}_2$ is a strongly admissible if and only if $\mathscr{I}_2^0 \subset \mathscr{I}_2$.

There is a filter $\mathscr{F}(\mathscr{I}_2)$ corresponding with $\mathscr{I}_2$ such that

$$\mathscr{F}(\mathscr{I}_2) = \big\{G \subset \mathbb{N}^2 : (\exists H \in \mathscr{I}_2)(G = \mathbb{N}^2 \backslash H)\big\}.$$

An admissible ideal $\mathscr{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ satisfies the property (AP2) if for every countable family of mutually disjoint sets $\{G_1, G_2, ...\} \in \mathscr{I}_2$, there exists a countable family of sets $\{H_1, H_2, ...\}$ such that $G_k \Delta H_k \in \mathscr{I}_2^0$, i.e., $G_k \Delta H_k$ is included in the finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$ for each $k \in \mathbb{N}$ and $H = \bigcup_{k=1}^{\infty} H_k \in \mathscr{I}_2$ (hence $H_k \in \mathscr{I}_2$ for each $k \in \mathbb{N}$).

If for each $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $|x_{kj} - \ell| < \varepsilon$ whenever $k, j > n_\varepsilon$, then the double sequence $x = (x_{kj}) \subset \mathbb{R}$ is convergent to $\ell \in \mathbb{R}$ and denoted with

$$\lim_{k,j\to\infty} x_{kj} = \ell \ \text{ or } \ \lim_{k,j\to\infty} x_{kj} = \ell.$$

Then after this, we take $\theta = \{(k_r, j_u)\}$ as a double lacunary sequence and $\mathscr{I}_2 \subset 2^{\mathbb{N}^2}$ as a strongly admissible ideal.

For a double sequence $(x_{kj}) \subset \mathbb{R}$, if

$$\lim_{r,u\to\infty} \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} x_{kj} = \ell$$

hold, then (x_{kj}) is lacunary convergent to $\ell \in \mathbb{R}$.

For a double sequence $(x_{kj}) \subset \mathbb{R}$, if

$$\lim_{r,u\to\infty} \frac{1}{h_{ru}} \sum_{(k,j),(s,t)\in I_{ru}} (x_{kj} - x_{st}) = 0$$

hold, then (x_{kj}) is lacunary Cauchy double sequence.

For a double sequence $(x_{kj}) \subset \mathbb{R}$, if for every $\varepsilon > 0$

$$\left\{(r,u) \in \mathbb{N}^2 : \left|\frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} x_{kj} - \ell\right| \geq \varepsilon\right\} \in \mathscr{I}_2$$

hold, then (x_{kj}) is lacunary $\mathscr{I}_2$-convergent to $\ell \in \mathbb{R}$ and denoted with $x_{kj} \to \ell(\mathscr{I}_{\theta_2})$.

If for every $\varepsilon > 0$ there exist $N = N(\varepsilon)$ and $S = S(\varepsilon)$

$$\left\{(r,u) \in \mathbb{N}^2 : \left|\frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} (x_{kj} - x_{NS})\right| \geq \varepsilon\right\} \in \mathscr{I}_2$$

hold, then (x_{kj}) is lacunary $\mathscr{I}_2$-Cauchy double sequence.

Lemma 1.1. *[12] Let $\{P_k\}_{k=1}^{\infty}$ be a countable collection of subsets of $\mathbb{N}^2$ such that $P_k \in F(\mathscr{I}_2)$ for each k, where $\mathscr{F}(\mathscr{I}_2)$ is a filter associate with a strongly admissible ideal $\mathscr{I}_2$ by (AP2). Therefore, there exists a set $P \subset \mathbb{N}^2$ such that $P \in \mathscr{F}(\mathscr{I}_2)$ and the set $P \backslash P_k$ is finite for all k.*

2. Main Results

For double sequences, we first defined lacunary $\mathscr{I}_2^*$-convergence and gave theorems examining its relationship with lacunary $\mathscr{I}_2$-convergence.

Definition 2.1. *A double sequence $(x_{kj}) \subset \mathbb{R}$ is lacunary $\mathscr{I}_2^*$-convergent to $\ell \in \mathbb{R}$ iff there exists a set $G = \{(k,j) \in \mathbb{N}^2\}$ such that for the set $G' = \{(r,u) \in \mathbb{N}^2 : (k,j) \in I_{ru}\} \in \mathscr{F}(\mathscr{I}_2)$, we have*

$$\lim_{\substack{r,u\to\infty \\ ((r,u)\in G')}} \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} x_{kj} = \ell$$

and so we can write $x_{kj} \to \ell(\mathscr{I}_{\theta_2}^*)$.

Theorem 2.2. *If the double sequence $(x_{kj}) \subset \mathbb{R}$ is lacunary $\mathscr{I}_2^*$-convergent to $\ell \in \mathbb{R}$, then it is lacunary $\mathscr{I}_2$-convergent to $\ell \in \mathbb{R}$.*

Proof. Let $x_{kj} \to L(\mathscr{I}_{\theta_2}^*)$. Then, there exists a set $G = \{(k,j) \in \mathbb{N}^2\}$ such that for the set $G' = \{(r,u) \in \mathbb{N}^2 : (k,j) \in I_{ru}\} \in \mathscr{F}(\mathscr{I}_2)$ $(i.e., H = \mathbb{N}^2 \backslash G' \in \mathscr{I}_2)$ and for every $\varepsilon > 0$ there exists $r_0 = r_0(\varepsilon) \in \mathbb{N}$ such that for all $r,u > r_0$ we have

$$\left| \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} x_{kj} - \ell \right| < \varepsilon,\ ((r,u) \in G').$$

Then,

$$A(\varepsilon) = \left\{ (r,u) \in \mathbb{N}^2 : \left| \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} x_{kj} - \ell \right| \geq \varepsilon \right\} \subset H \cup \left[G' \cap \left((\{1,2,\cdots,r_0\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,\cdots,r_0\}) \right) \right].$$

Since $\mathscr{I}_2$ is a strongly admissible ideal, we have

$$H \cup \left[G' \cap \left((\{1,2,\cdots,r_0\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,\cdots,r_0\}) \right) \right] \in \mathscr{I}_2$$

and so $A(\varepsilon) \in \mathscr{I}_2$. Hence, $x_{kj} \to \ell(\mathscr{I}_{\theta_2})$. □

Theorem 2.3. *Let $\mathscr{I}_2$ be a strongly admissible ideal by $(AP2)$. If the double sequence $(x_{kj}) \subset \mathbb{R}$ is lacunary $\mathscr{I}_2$-convergent to $\ell \in \mathbb{R}$, then it is lacunary $\mathscr{I}_2^*$-convergent to $\ell \in \mathbb{R}$.*

Proof. Assume that $x_{kj} \to \ell(\mathscr{I}_{\theta_2})$. Then for each $\varepsilon > 0$,

$$T(\varepsilon) = \left\{ (r,u) \in \mathbb{N}^2 : \left| \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} x_{kj} - \ell \right| \geq \varepsilon \right\} \in \mathscr{I}_2.$$

Put

$$T_1 = \left\{ (r,u) \in \mathbb{N}^2 : \left| \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} x_{kj} - \ell \right| \geq 1 \right\}$$

and

$$T_p = \left\{ (r,u) \in \mathbb{N}^2 : \frac{1}{p} \leq \left| \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} x_{kj} - \ell \right| < \frac{1}{p-1} \right\},$$

for $p \geq 2$ and $p \in \mathbb{N}$. It is clear that $T_i \cap T_j = \emptyset$ for $i \neq j$ and $T_i \in \mathscr{I}_2$ for each $i \in \mathbb{N}$. By property $(AP2)$, there is a sequence $\{V_p\}_{p\in\mathbb{N}}$ such that $T_j \Delta V_j$ is included in finite union of rows and columns in $\mathbb{N}^2$ for each $j \in \mathbb{N}$ and

$$V = \bigcup_{j=1}^{\infty} V_j \in \mathscr{I}_2.$$

We prove that,

$$\lim_{\substack{r,u\to\infty \\ ((r,u)\in G')}} \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} x_{kj} = \ell,$$

for $G' = \mathbb{N}^2 \backslash V \in \mathscr{F}(\mathscr{I}_2)$. Take $\delta > 0$. Select $q \in \mathbb{N}$ such that $\frac{1}{q} < \delta$. Therefore,

$$\left\{ (r,u) \in \mathbb{N}^2 : \left| \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} x_{kj} - \ell \right| \geq \delta \right\} \subset \bigcup_{j=1}^{q} T_j.$$

Since $T_j \Delta V_j$ is a finite set for $j \in \{1,2,\cdots,q\}$, there exists $r_0 \in \mathbb{N}$ such that

$$\left(\bigcup_{j=1}^{q} T_j\right) \cap \left\{(r,u) \in \mathbb{N}^2 : r \geq r_0 \wedge u \geq r_0\right\} = \left(\bigcup_{j=1}^{q} V_j\right) \cap \left\{(r,u) \in \mathbb{N}^2 : r \geq r_0 \wedge u \geq r_0\right\}.$$

If $r,u \geq r_0$ and $(r,u) \notin V$, then

$$(r,u) \notin \bigcup_{j=1}^{q} V_j \ \text{ and so } \ (r,u) \notin \bigcup_{j=1}^{q} T_j.$$

We have

$$\left| \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} x_{kj} - \ell \right| < \frac{1}{q} < \delta.$$

This implies that

$$\lim_{\substack{r,u\to\infty \\ ((r,u)\in G')}} \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} x_{kj} = \ell,$$

Hence, we have $x_{kj} \to \ell(\mathscr{I}_{\theta_2}^*)$. This completes the proof. □

Now, for double sequences, we have defined lacunary $\mathscr{I}_2^*$-Cauchy sequence and given theorems examining its relationship with lacunary $\mathscr{I}_2$-Cauchy sequence.

Definition 2.4. *The double sequence $(x_{kj}) \subset \mathbb{R}$ is lacunary $\mathscr{I}_2^*$-Cauchy sequence iff there exists a set $G = \{(k,j) \in \mathbb{N}^2\}$ such that for the set $G' = \{(r,u) \in \mathbb{N}^2 : (k,j) \in I_{ru}\} \in \mathscr{F}(\mathscr{I}_2)$, we have*

$$\lim_{\substack{r,u\to\infty \\ ((r,u)\in G')}} \frac{1}{h_{ru}} \sum_{(k,j),(s,t)\in I_{ru}} (x_{kj} - x_{st}) = 0.$$

Theorem 2.5. *If the double sequence $(x_{kj}) \subset \mathbb{R}$ is lacunary $\mathscr{I}_2^*$-Cauchy sequence, then (x_{kj}) is lacunary $\mathscr{I}_2$-Cauchy sequence.*

Proof. Let $(x_{kj}) \subset \mathbb{R}$ is a lacunary $\mathscr{I}_2^*$-Cauchy double sequence. Then, there exists a set $G = \{(k,j) \in \mathbb{N}^2\}$ such that for the set $G' = \{(r,u) \in \mathbb{N}^2 : (k,j) \in I_{ru}\} \in \mathscr{F}(\mathscr{I}_2)$ and for every $\varepsilon > 0$ there exists $r_0 = r_0(\varepsilon)$ such that

$$\left| \frac{1}{h_{ru}} \sum_{(k,j),(s,t)\in I_{ru}} (x_{kj} - x_{st}) \right| < \varepsilon, \ \ ((r,u) \in G')$$

for all $r,u > r_0$. Now, let $H = \mathbb{N} \setminus G'$. It is clear that $H \in \mathscr{I}_2$. Then, for $(r,u) \in G'$

$$A(\varepsilon) = \left\{ (r,u) \in \mathbb{N}^2 : \left| \frac{1}{h_{ru}} \sum_{(k,j),(s,t)\in I_{ru}} (x_{kj} - x_{st}) \right| \geq \varepsilon \right\} \subset H \cup \left[G' \cap \left((\{1,2,\cdots,r_0\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,\cdots,r_0\})\right)\right].$$

Since $\mathscr{I}_2$ is an admissible ideal, we have

$$H \cup \left[G' \cap \left((\{1,2,\cdots,r_0\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,\cdots,r_0\})\right)\right] \in \mathscr{I}_2$$

and so $A(\varepsilon) \in \mathscr{I}_2$. Hence, (x_{kj}) is lacunary $\mathscr{I}_2$-Cauchy double sequence. □

Theorem 2.6. *Let $\mathscr{I}_2$ be a strongly admissible ideal by $(AP2)$. If the double sequence $(x_{kj}) \subset \mathbb{R}$ is lacunary $\mathscr{I}_2$-Cauchy double sequence, then (x_{kj}) is lacunary $\mathscr{I}_2^*$-Cauchy double sequence.*

Proof. Assume that (x_{kj}) is lacunary $\mathscr{I}_2$-Cauchy sequence. Then, for each $\varepsilon > 0$ there exist $N = N(\varepsilon)$ and $S = S(\varepsilon)$ such that

$$A(\varepsilon) = \left\{ (r,u) \in \mathbb{N}^2 : \left| \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} (x_{kj} - x_{NS}) \right| \geq \varepsilon \right\} \in \mathscr{I}_2.$$

Let

$$P_i = \left\{ (r,u) \in \mathbb{N}^2 : \left| \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} (x_{kj} - x_{s_i t_i}) \right| \geq \frac{1}{i} \right\},\ i = 1,2,\ldots,$$

where $s_i = N\left(\frac{1}{i}\right)$ and $t_i = S\left(\frac{1}{i}\right)$. It is clear that $P_i \in \mathscr{F}(\mathscr{I}_2)$ for $i = 1,2,\cdots$. Using the Lemma 1.1, since $\mathscr{I}_2$ has the $(AP2)$ so there exists a set $P \subset \mathbb{N}^2$ such that $P \in \mathscr{F}(\mathscr{I}_2)$ and $P \setminus P_i$ is finite for all i. At the moment, we demonstrate that

$$\lim_{\substack{r,u\to\infty \\ ((r,u)\in P)}} \frac{1}{h_{ru}} \sum_{(k,j),(s,t)\in I_{ru}} (x_{kj} - x_{st}) = 0.$$

For prove this let $\varepsilon > 0$, $m \in \mathbb{N}$ such that $m > \frac{2}{\varepsilon}$. If $(r,u) \in P$ then $P \setminus P_m$ is a finite set, so there exists $r_0 = r_0(m)$ such that $(r,u) \in P_m$ for all $r,u > r_0(m)$. Therefore, for all $r,u > r_0(m)$

$$\left| \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} (x_{kj} - x_{s_m t_m}) \right| < \frac{1}{m}$$

and

$$\left| \frac{1}{h_{ru}} \sum_{(s,t)\in I_{ru}} (x_{st} - x_{s_m t_m}) \right| < \frac{1}{m}.$$

Hence, for all $r,u > r_0(m)$ it follows that

$$\begin{aligned} \left| \frac{1}{h_{ru}} \sum_{(k,j),(s,t)\in I_{ru}} (x_{kj} - x_{st}) \right| &\leq \left| \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} (x_{kj} - x_{s_m t_m}) \right| + \left| \frac{1}{h_{ru}} \sum_{(s,t)\in I_{ru}} (x_{st} - x_{s_m t_m}) \right| \\ &< \frac{1}{m} + \frac{1}{m} < \varepsilon. \end{aligned}$$

Therefore, for any $\varepsilon > 0$ there exists $r_0 = r_0(\varepsilon)$ such that for $r,u > r_0(\varepsilon)$ and $(r,u) \in P \in \mathscr{F}(\mathscr{I}_2)$

$$\left| \frac{1}{h_{ru}} \sum_{(k,j),(s,t)\in I_{ru}} (x_{kj} - x_{st}) \right| < \varepsilon.$$

This demonstrates that (x_{kj}) is lacunary $\mathscr{I}_2^*$-Cauchy double sequence. □

Theorem 2.7. *If the double sequence $(x_{kj}) \subset \mathbb{R}$ is lacunary $\mathscr{I}_2^*$-convergent to $\ell \in \mathbb{R}$, so (x_{kj}) is lacunary $\mathscr{I}_2$-Cauchy double sequence.*

Proof. Let $x_{kj} \to \ell(\mathscr{I}_{\theta_2}^*)$. So, there exists a set $G = \{(k,j) \in \mathbb{N}^2\}$ such that for the set $G' = \{(r,u) \in \mathbb{N}^2 : (k,j) \in I_{ru}\} \in \mathscr{F}(\mathscr{I}_2)$, we have

$$\lim_{\substack{r,u\to\infty \\ ((r,u)\in G')}} \left| \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} x_{kj} - \ell \right| = 0.$$

It shows that there exist $r_0 = r_0(\varepsilon)$ such that

$$\left| \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} x_{kj} - \ell \right| < \frac{\varepsilon}{2},\ ((r,u) \in G')$$

for every $\varepsilon > 0$ and all $r, u > r_0$. Since

$$\begin{aligned}\left|\frac{1}{h_{ru}}\sum_{(k,j),(s,t)\in I_{ru}}(x_{kj}-x_{st})\right| &\leq \left|\frac{1}{h_{ru}}\sum_{(k,j)\in I_{ru}}x_{kj}-\ell\right|+\left|\frac{1}{h_{ru}}\sum_{(s,t)\in I_{ru}}x_{st}-\ell\right| \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,\ ((r,u)\in G')\end{aligned}$$

for all $r, u > r_0$. Hence, we have

$$\lim_{\substack{r,u\to\infty \\ ((r,u)\in G')}}\frac{1}{h_{ru}}\sum_{(k,j),(s,t)\in I_{ru}}(x_{kj}-x_{st})=0.$$

That is, (x_{kj}) is a lacunary $\mathscr{I}_2^*$-Cauchy double sequence. Therefore, by Theorem 2.5, (x_{kj}) is a lacunary $\mathscr{I}_2$-Cauchy double sequence. □

3. Conclusion

In summability theory, the notions of classical convergence, statistical and ideal convergence in some metric spaces and normed spaces were studied by a lot of mathematicians in recently. For double sequences, we investigated the lacunary $\mathscr{I}^*$-convergence and lacunary $\mathscr{I}^*$-Cauchy sequence in $\mathbb{R}$. In the future, for double sequences, the notions of strongly lacunary $\mathscr{I}^*$-convergence and strongly lacunary $\mathscr{I}^*$-Cauchy sequence in $\mathbb{R}$ are defined.

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