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Inference for the Jones and Faddy's Skewed t-Distribution Based on Progressively Type-II Censored Samples

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Abstract

In this paper, the location and the scale parameters of Jones and Faddy's skewed t (JFST) distribution are estimated based on progressively Type-II censored samples. We obtain maximum likelihood (ML) and modified maximum likelihood (MML) estimators of unknown parameters. Then, confidence intervals for the estimators of μ and σ are obtained. The performances of proposed methodologies are compared via Monte-Carlo simulation study. It is concluded that the ML and MML estimators are close, especially for moderate and large sample sizes. At the end of the study, real life data is analyzed for illustrative proposes.

1. INTRODUCTION

In many life testing and reliability studies, a number of the experimental units are lost or removed from experimentation before failure. Data obtained from these experiments is known as censored data. In the literature, the most commonly-used censoring schemes are Type-I and Type-II censoring. In Type-I censoring, the experimenter continues the experiment until a pre-fixed time T is reached. Observations which fail after time T are censored. In Type-II censoring, a predetermined number of failures $m < n$ are censored. Progressive censoring is different from Type-I and Type-II censoring, as it is a more general censoring scheme, where the experimenter is allowed to remove experimental units between experiments as well, see Cohen (1963) and Rastogi and Tripathi (2014). This censoring scheme provides time-saving and cost-effective life testing plans to the experimenter, see Balakrishnan and Aggarwala (2000). In the literature, there are considerable number of studies dealing with progressive censoring, see Balakrishnan et al. (2003), Raqab et al. (2010), Basak and Balakrishnan (2012), Krishna and Kumar (2013) and Musleh and Helu (2014).

Progressively Type-II censoring scheme is described briefly as follows: Consider life-testing experiment starting with n identical units and a progressive censoring scheme with $\boldsymbol{R} = (R_1, R_2, \dots, R_m)$. When the first failure occurs, R_1 surviving units are removed randomly from the remaining $n-1$ units. Similarly, when the second failure occurs, R_2 surviving units are removed from the remaining $n-2-R_1$ units. This process continues to the mth failure and R_m surviving units are removed from the remaining $n-m-\sum_{i=1}^{m-1} R_i$ units. Here, R_i's $i = 1, \dots, m$ are fixed prior to study. It is clear that the entire sample is complete when $R_1 = R_2 = \cdots = R_m = 0$. If $R_1 = R_2 = \cdots = R_{m-1} = 0$ and $R_m = n-m$, then the entire sample is a traditional Type-II right censored sample.

Our aim in this paper is to derive estimators of the location and scale parameters of Jones and Faddy's skewed-t (JFST) distribution based on progressively type-II censored samples. The reason for using JFST

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distribution is that it is flexible for modeling data sets having heavy tailed symmetric, positively skewed or negatively skewed distributions.

We use the well-known and widely used maximum likelihood (ML) methodology to obtain estimators of unknown parameters. However, it can be seen that solutions of likelihood equations cannot be obtained explicitly because of the nonlinear functions of the parameters. We, therefore, resort to different approaches, namely iterative and non-iterative, in order to solve them.

The rest of the paper is organized as follows. A brief description of JFST distribution is given in Section 2. In Section 3, ML and MML estimators of the location and the scale parameters of the JSFT distribution are obtained based on the progressively Type-II censored samples. Observed Fisher information matrices are computed for both the ML and the MML estimators. In addition, the confidence intervals (CIs) for unknown parameters and the coverage probabilities (CPs) for pivotal quantities based on the asymptotic normality are obtained. In section 4, an extensive Monte-Carlo simulation study is performed for different sample sizes, censoring schemes and parameter settings. A real data set is analyzed in Section 6. Final comments and conclusions are given in Section 7.

2. JFST'S SKEWED t-DISTRIBUTION

t-distribution is one of the most commonly used distributions as an alternative to normal distribution. Although this distribution is symmetric and bell-shaped, it has heavier tails. This property provides flexibility for modeling real life data sets. In practice, heavy tailed distribution may not be enough, because the data sets may be skew. Therefore, the skew alternatives of the t-distribution are more suitable for use with such data sets. A wide range of study in this field; see for example, Nadarajah and Kotz (2003), Jones and Faddy (2003), Sartori (2006), Azzalini and Genton (2008), Carota (2010) and Massacci (2014). In this study, we use JFST distribution as a skewed alternative to t-distribution.

The JFST distribution with the location parameter μ and the scale parameter σ has the following probability density function (pdf) and cumulative distribution function (cdf)

$$f(z) = \frac{1}{\sigma} C_{a,b}^{-1} \left\{1 + \frac{z}{\sqrt{v+z^2}}\right\}^{a+1/2} \left\{1 - \frac{z}{\sqrt{v+z^2}}\right\}^{b+1/2}, \quad z \in \mathbb{R}, a, b \in \mathbb{R}^+ \tag{1}$$

and

$$F(z) = I_{\{1+z/\sqrt{v+z^2}\}/2}(a,b), \quad z \in \mathbb{R}, a, b \in \mathbb{R}^+ \tag{2}$$

respectively. Here, $z = (x-\mu)/\sigma$, a and b are the shape parameters, $v = a + b$ and $C_{a,b} = 2^{v-1}B(a,b)\sqrt{v}$. In addition, $B(.,.)$ and $I_z(.,.)$ represent the Beta function and the incomplete Beta function, respectively. If X has a JFST distribution with parameters μ, σ, a and b then it is denoted by $X \sim JFST(\mu, \sigma, a, b)$.

Now, we discuss some important properties of JFST distribution.

i. JFST distribution is positively and negatively skewed for $a > b$ and $a < b$, respectively.
ii. It reduces to t-distribution with $2a$ degrees of freedom for $a = b$. It should be noted that, when $a, b \to \infty$ the JFST distribution is convergence to normal distribution.
iii. If $a > k/2$ and $b > k/2$, then $E(Z^k)$ is computed as

$$E(Z^k) = \frac{(a+b)^{k/2}}{2^k B(a,b)} \sum_{i=0}^{k} \binom{k}{i} (-1)^i B\left(a + \frac{k}{2} - i, b - \frac{k}{2} + i\right).$$

iv. For the effects of shape parameters a and b for the JFST distribution see Figure 1.

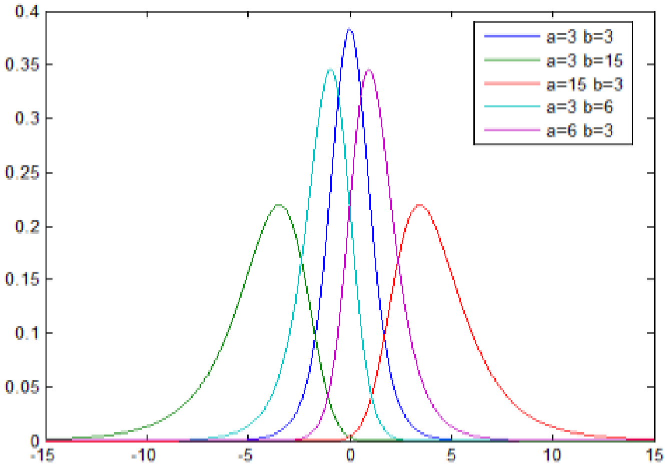

Figure 1. *JFST distributions for different values of shape parameters*

For more detailed information, see Jones and Faddy (2003), Jones (2008), Acıtaş et al. (2013), Arslan (2015) and Arslan and Şenoğlu (2017).

3. PARAMETER ESTIMATION

In this section, we consider ML and MML estimation of the location and scale parameters of JFST distribution based on progressively Type-II censored data. In this context, in subsection 3.1 and 3.2, ML estimators are obtained by using iterative methods and MML estimators of the unknown parameters are derived in explicit form. In subsection 3.3, observed Fisher information matrices are computed for both the ML and the MML estimators of μ and σ. Using observed variances, the pivotal quantities are calculated. After this, CPs and CIs are obtained.

3.1. Maximum Likelihood Estimators

Let $\boldsymbol{X} = (X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n})$ with $X_{1:m:n} < X_{2:m:n} < \cdots < X_{m:m:n}$ be a progressively Type-II censored sample of size m under the censoring schema $\boldsymbol{R} = (R_1, R_2, \ldots, R_m)$. We use the notation X_i instead of $X_{i:m:n}$, for simplicity. The likelihood function, based on progressively Type-II censored sample is shown below

$$L(\mu, \sigma, a, b; X_1, X_2, \ldots, X_m) = c \prod_{i=1}^{m} f(x_i)[1 - F(x_i)]^{R_i},$$

where $c = n(n - R_1 - 1)(n - R_1 - R_2 - 2) \ldots (n - R_1 - R_2 - \cdots - R_{m-1} - m + 1)$. The log-likelihood function is given by

$$\ln L \propto m \ln \sigma + (a + 0.5) \sum_{i=1}^{m} \ln \left\{ 1 + \frac{z_i}{\sqrt{v + z_i^2}} \right\} + (b + 0.5) \sum_{i=1}^{m} \ln \left\{ 1 - \frac{z_i}{\sqrt{v + z_i^2}} \right\}$$
$$+ \sum_{i=1}^{m} R_i \ln[1 - F(z_i)], \quad (3)$$

where $z_i = (x_i - \mu)/\sigma$. By taking derivation with respect to μ and σ, likelihood equations are obtained as given below

$$\frac{\partial \ln L}{\partial \mu} = -\frac{a+0.5}{\sigma}\sum_{i=1}^{m} g_1(z_i) + \frac{b+0.5}{\sigma}\sum_{i=1}^{m} g_2(z_i) + \frac{1}{\sigma}\sum_{i=1}^{m} R_i g_3(z_i) = 0, \quad (4)$$

$$\frac{\partial \ln L}{\partial \sigma} = -\frac{m}{\sigma} - \frac{a+0.5}{\sigma}\sum_{i=1}^{m} z_i g_1(z_i) + \frac{b+0.5}{\sigma}\sum_{i=1}^{m} z_i g_2(z_i) + \frac{1}{\sigma}\sum_{i=1}^{m} R_i z_i g_3(z_i) = 0. \quad (5)$$

Here,

$$g_1(z_i) = \frac{v}{(v+z_i^2)^{3/2} + z_i(v+z_i^2)}, \quad g_2(z_i) = \frac{v}{(v+z_i^2)^{3/2} - z_i(v+z_i^2)} \quad \text{and}$$
$$g_3(z_i) = \frac{f(z_i)}{1-F(z_i)}, \quad i = 1, \dots, m. \quad (6)$$

The ML estimators of the unknown parameters μ and σ are the solutions of the likelihood equations (4)-(5). However, because of the non-linear functions given in equation (6), the likelihood equations cannot be solved explicitly. Therefore, we resort to iterative methods.

3.2. Modified Maximum Likelihood Estimators

We use MML methodology proposed by Tiku (1967) to obtain close form estimators of the location and scale parameters in this subsection. To obtain MML estimators of the parameters, we first linearize the non-linear functions $g_1(z_i)$, $g_2(z_i)$ and $g_3(z_i)$ given in equation (6) using Taylor series expansion around $E(Z_{i:m:n}) = t_{i:m:n}$. From Balakrishnan and Sandhu (1995)

$$F(Z_{i:m:n}) = U_{i:m:n},$$

where $U_{i:m:n}$ is the ith order statistics from a progressively Type-II censored sample whose distribution is uniform $U(0,1)$. Then

$$Z_{i:m:n} = F^{-1}(U_{i:m:n})$$

and hence

$$t_{i:m:n} = E(Z_{i:m:n}) \approx F^{-1}(\eta_{i:m:n}),$$

where $\eta_{i:m:n} = E(U_{i:m:n})$. From Balakrishnan and Aggarwala (2000), it is known that

$$\eta_{i:m:n} = 1 - \prod_{j=m-i+1}^{m} \frac{j + R_{m-j+1} + \cdots + R_m}{j + 1 + R_{m-j+1} + \cdots + R_m}, \quad i = 1, \dots, m.$$

It should be noted that, for simplicity notation, we use t_i instead of $t_{i:m:n}$.

The linearized functions are obtained as shown below

$$g_1(z_i) = \alpha_{1i} - \beta_{1i} z_i, \quad g_2(z_i) = \alpha_{2i} + \beta_{1i} z_i \quad \text{and} \quad g_3(z_i) = \alpha_{3i} + \beta_{3i} z_i, \quad i = 1, \dots, m,$$

where

$$\beta_{1i} = \frac{v\left[3t_i\sqrt{v+t_i^2} + v + 3t_i^2\right]}{\left[(v+t_i^2)^{3/2} + t_i(v+t_i^2)\right]^2}, \quad \beta_{2i} = \frac{v\left[3t_i\sqrt{v+t_i^2} - v - 3t_i^2\right]}{\left[(v+t_i^2)^{3/2} - t_i(v+t_i^2)\right]^2}, \quad \beta_{3i} = \frac{f'(t_i)}{1-F(t_i)} + \frac{f(t_i)^2}{[1-F(t_i)]^2},$$

$$\alpha_{1i} = g_1(t_i) + t_i\beta_{1i}, \quad \alpha_{2i} = g_2(t_i) - t_i\beta_{2i}, \quad \alpha_{3i} = g_3(t_i) - t_i\beta_{3i}. \tag{7}$$

By incorporating the linearized functions given in equation (7) into the likelihood equations (4)-(5), modified likelihood equations are obtained as follows

$$\frac{\partial \ln L^*}{\partial \mu} = -\frac{a+0.5}{\sigma}\sum_{i=1}^{m}(\alpha_{1i} - \beta_{1i}z_i) + \frac{b+0.5}{\sigma}\sum_{i=1}^{m}(\alpha_{2i} + \beta_{2i}z_i) + \frac{1}{\sigma}\sum_{i=1}^{m} R_i(\alpha_{3i} + \beta_{3i}z_i) = 0, \tag{8}$$

$$\frac{\partial \ln L^*}{\partial \sigma} = -\frac{m}{\sigma} - \frac{a+0.5}{\sigma}\sum_{i=1}^{m}(\alpha_{1i} - \beta_{1i}z_i)z_i + \frac{b+0.5}{\sigma}\sum_{i=1}^{m}(\alpha_{2i} + \beta_{2i}z_i)z_i + \frac{1}{\sigma}\sum_{i=1}^{m} R_i(\alpha_{3i} + \beta_{3i}z_i)z_i = 0. \tag{9}$$

The solutions of these equations are the following closed form MML estimators

$$\hat{\mu}_{MML} = K + \frac{\Delta}{w}\hat{\sigma}_{MML} \quad \text{and} \quad \hat{\sigma}_{MML} = \frac{B + \sqrt{B^2 + 4AC}}{2A},$$

where

$$\delta_i = (a+0.5)\beta_{1i} + (b+0.5)\beta_{2i} + R_i\beta_{3i}, \quad w = \sum_{i=1}^{m}\delta_i, \quad K = \frac{\sum_{i=1}^{m}\delta_i x_i}{w},$$

$$\Delta_i = -(a+0.5)\alpha_{1i} + (b+0.5)\alpha_{2i} + R_i\alpha_{3i}, \quad \Delta = \sum_{i=1}^{m}\Delta_i,$$

$$A = m = n - \sum_{i=1}^{m} R_i, \quad B = \sum_{i=1}^{m}\Delta_i(x_i - K), \quad C = \sum_{i=1}^{m}\delta_i(x_i - K)^2. \tag{10}$$

Remark. On occasion, the values of the β_{1i} coefficients may be negative. This situation makes $\hat{\sigma}_{MML}$ unreal or negative. To overcome this problem, the coefficients β_{1i} and α_{1i} are replaced by β_{1i}^* and α_{1i}^*

$$\beta_{1i}^* = \frac{v\left[3\sqrt{v+t_i^2} + v + 3t_i^2\right]}{\left[(v+t_i^2)^{3/2} + t_i(v+t_i^2)\right]^2}, \quad \alpha_{2i}^* = g(t_i) + t_i\beta_{1i}^*, \quad i = 1, \dots, m,$$

respectively. It should be noted that this alternative representation does not change the asymptotic properties of the estimators because $z_i - t_i \cong 0$ and, consequently, $\alpha_{1i} + \beta_{1i}z_i \cong \alpha_{1i}^* + \beta_{1i}^* z_i$ $(i = 1, \dots, m)$, see Islam and Tiku (2004) and Acıtaş et al. (2013).

The MML estimators have the following properties:

i. They have closed form expressions.
ii. They are the functions of sample observations and are easy to compute.
iii. Asymptotically, they are fully efficient, i.e., their variances are equivalent to Rao-Cramer lower bound when regularity conditions hold, see Vaughan (2002). They also have high efficiencies, even for small n values.
iv. They are asymptotically equivalent to ML estimators, see Bhattacharyya (1985) and Vaughan and Tiku (2000).

In addition to being non-iterative, MML methodology provides initial values having the fastest convergence rate among the others for the iterative methods, see Acıtaş et al. (2011). Therefore, in this study, we use MML estimators of parameters as initial values for solving the likelihood equations in ML methodology.

3.3 Variances and Covariances

The asymptotic variance-covariance matrix of the ML estimators of the parameters μ and σ is derived from the inverse on the following Fisher information matrix

$$I = \begin{bmatrix} -E\left(\frac{\partial^2 \ln L}{\partial \mu^2}\right) & -E\left(\frac{\partial^2 \ln L}{\partial \mu \partial \sigma}\right) \\ -E\left(\frac{\partial^2 \ln L}{\partial \sigma \partial \mu}\right) & -E\left(\frac{\partial^2 \ln L}{\partial \sigma^2}\right) \end{bmatrix}. \quad (11)$$

Elements of the Fisher information matrix for ML estimators: The elements of the Fisher information matrix for the likelihood equations (4)-(5) are obtained from the following equations

$$\frac{\partial^2 \ln L}{\partial \mu^2} = -\frac{a+0.5}{\sigma^2}\sum_{i=1}^{m} g_1'(z_i) - \frac{b+0.5}{\sigma^2}\sum_{i=1}^{m} g_2'(z_i) - \frac{1}{\sigma^2}\sum_{i=1}^{m} g_3'(z_i), \quad (12)$$

$$\frac{\partial^2 \ln L}{\partial \sigma^2} = \frac{m}{\sigma^2} + \frac{2(a+0.5)}{\sigma^2}\sum_{i=1}^{m} z_i g_1(z_i) - \frac{a+0.5}{\sigma^2}\sum_{i=1}^{m} z_i^2 g_1'(z_i) - \frac{2(b+0.5)}{\sigma^2}\sum_{i=1}^{m} z_i g_2(z_i)$$
$$- \frac{b+0.5}{\sigma^2}\sum_{i=1}^{m} z_i^2 g_2'(z_i) - \frac{2}{\sigma^2}\sum_{i=1}^{m} R_i z_i g_3(z_i) - \frac{1}{\sigma^2}\sum_{i=1}^{m} R_i z_i^2 g_3'(z_i), \quad (13)$$

$$\frac{\partial^2 \ln L}{\partial \mu \partial \sigma} = \frac{a+0.5}{\sigma^2}\sum_{i=1}^{m} g_1(z_i) - \frac{a+0.5}{\sigma^2}\sum_{i=1}^{m} z_i g_1'(z_i) - \frac{b+0.5}{\sigma^2}\sum_{i=1}^{m} g_2(z_i)$$
$$- \frac{b+0.5}{\sigma^2}\sum_{i=1}^{m} z_i g_2'(z_i) - \frac{1}{\sigma^2}\sum_{i=1}^{m} R_i g_3(z_i) - \frac{1}{\sigma^2}\sum_{i=1}^{m} R_i z_i g_3'(z_i) \quad (14)$$

where $g_1'(.)$, $g_2'(.)$ and $g_3'(.)$ are the derivative of $g_1(.)$, $g_2(.)$ and $g_3(.)$, respectively. These are

$$g_1'(z_i) = \frac{3\left(v+z_i^2\right)^{1/2} v z_i + v^2 + 3 v z_i^2}{\left[\left(v+z_i^2\right)^{3/2} + z_i\left(v+z_i^2\right)\right]^2}, \quad g_2'(z_i) = \frac{3\left(v+z_i^2\right)^{1/2} v z_i - v^2 - 3 v z_i^2}{\left[\left(v+z_i^2\right)^{3/2} - z_i\left(v+z_i^2\right)\right]^2} \text{ and}$$
$$g_3'(z_i) = \frac{f'(z_i)\left(1-F(z_i)\right) + f^2(z_i)}{\left(1-F(z_i)\right)^2}. \quad (15)$$

However, because the exact mathematical expressions for the expectation of the equation (11) is extremely difficult, we use the observed variance-covariance matrix. Following this, the inverse of the observed Fisher information matrix is derived from following equation

$$I_{obs}^{-1} = \begin{bmatrix} -\frac{\partial^2 \ln L}{\partial \mu^2} & -\frac{\partial^2 \ln L}{\partial \mu \partial \sigma} \\ -\frac{\partial^2 \ln L}{\partial \sigma \partial \mu} & -\frac{\partial^2 \ln L}{\partial \sigma^2} \end{bmatrix}^{-1}_{\mu=\hat{\mu}, \sigma=\hat{\sigma}} = \sigma^2 \begin{bmatrix} I^{11} & I^{12} \\ I^{21} & I^{22} \end{bmatrix} \quad (16)$$

where $I^{12} = I^{21}$.

Elements of the Fisher information matrix for MML estimators: The elements of the Fisher information matrix for the modified likelihood equations (8)-(9) are obtained from the following equations

$$\frac{\partial^2 \ln L^*}{\partial \mu^2} = -\frac{1}{\sigma^2}\sum_{i=1}^{m} \delta_i, \quad (17)$$

$$\frac{\partial^2 \ln L^*}{\partial \sigma^2} = \frac{m}{\sigma^2} - \frac{2}{\sigma^2}\sum_{i=1}^{m} \Delta_i z_i - \frac{3}{\sigma^2}\sum_{i=1}^{m} \delta_i z_i^2, \tag{18}$$

$$\frac{\partial^2 \ln L^*}{\partial \mu \partial \sigma} = -\frac{1}{\sigma^2}\sum_{i=1}^{m} \Delta_i - \frac{2}{\sigma^2}\sum_{i=1}^{m} \delta_i z_i. \tag{19}$$

Similar to equation (16), the inverse of the observed Fisher information matrix is derived by replacing $\frac{\partial^2 \ln L}{\partial}$ to $\frac{\partial^2 \ln L^*}{\partial}$. As a result, we get

$$I_{obs}^{-1\,*} = \sigma^2 \begin{bmatrix} I_*^{11} & I_*^{12} \\ I_*^{21} & I_*^{22} \end{bmatrix}. \tag{20}$$

The observed variances and covariance of the ML estimators of $\hat{\mu}$ and $\hat{\sigma}$ are computed as shown below

$$Var(\hat{\mu}) = \hat{\sigma}^2\sqrt{I^{11}}, \quad Var(\hat{\sigma}) = \hat{\sigma}^2\sqrt{I^{22}} \quad \text{and} \quad Cov(\hat{\mu},\hat{\sigma}) = \hat{\sigma}^2\sqrt{I^{12}}. \tag{21}$$

By incorporating the MML estimators of $\hat{\mu}$, $\hat{\sigma}$ and I_*^{11}, I_*^{12} and I_*^{22} into equation (21), the observed variances and covariance of the MML $\hat{\mu}$ and $\hat{\sigma}$ are obtained in similar manner.

Then, we calculate pivotal quantities for both the ML and the MML estimators as given below

$$P_1 = \frac{\hat{\mu}-\mu}{\sqrt{Var(\hat{\mu})}} \quad \text{and} \quad P_2 = \frac{\hat{\sigma}-\sigma}{\sqrt{Var(\hat{\sigma})}}. \tag{22}$$

The asymptotic distribution of the pivotal quantities is to becomes standard normal, based on the asymptotic normality of $\hat{\mu}$ and $\hat{\sigma}$. Using Monte-Carlo simulation, CPs are computed as

$$CP_1 = P\left(|P_1| \le z_{\alpha/2}\right) \quad \text{and} \quad CP_2 = P\left(|P_2| \le z_{\alpha/2}\right), \tag{23}$$

where $z_{\alpha/2}$ is the $(\alpha/2)th$ percentile of the standard normal distribution.

The two-sided normal approximate CIs for the ML and the MML estimators of μ and σ are obtained as $\hat{\mu} \pm z_{\alpha/2}\sqrt{Var(\hat{\mu})}$ and $\hat{\sigma} \pm z_{\alpha/2}\sqrt{Var(\hat{\sigma})}$, respectively.

4. SIMULATION STUDY

In this section, an extensive Monte-Carlo simulation study was conducted to compare the performances of the proposed estimators of the parameters of the JFST distribution examined in Section 3 for various different parameter settings, sample sizes, number of failures and censoring schemes. Comparisons are made for both the point and the interval estimators of the parameters. Bias and mean square error (MSE) are used for comparisons of point estimators and the CP criterion is used for comparisons of the interval estimators.

The progressive type-II censored samples are generated using the algorithm originated by Balakrishnan and Sandhu (1995), and without loss of generality, μ and σ are taken to be 0 and 1, respectively. Four different sets of shape parameters are used, i.e., $a = b = 3$ and $a = b = 15$ (symmetric cases); $a = 6$, $b = 3$ and $a = 9$, $b = 3$ (positively skewed cases). It should be noted that the simulation results for $a = 3$, $b = 6$ and $a = 3$, $b = 9$ (negatively skewed cases) have also been obtained. However, the simulation results are very similar with those obtained from the positively skewed cases. Therefore, for the sake of brevity, we therefore did not reproduce them.

The following sample sizes and the number of failures were considered in the simulations

n =15 ⇒ m = 5 and 10,
n =30 ⇒ m = 10 and 20 and
n =50 ⇒ m = 25 and 40.

We use different censoring schemes, depending on the shape of the distribution to make comparisons meaningful. For simplicity, we use '*' notation to illustrate censoring schemes. For example, $(1^{*3},0^{*5},1^{*2})$ denotes the censoring scheme (1,1,1,0,0,0,0,0,1,1).

All the simulations were conducted using Matlab R2013a for $[\![100{,}000/m]\!]$ Monte-Carlo runs. Here, $[\![.]\!]$ represents the greatest integer value. It should be noted that, as mentioned in subsection 3.1, the likelihood equations (4) and (5) cannot be solved explicitly. Therefore, ML estimators of the parameters μ and σ are obtained using the iterative method. Because of the reasons given in subsection 3.2, we use MML estimates of the parameters in (10) as initial values for the iterations.

In Table 1, means and MSEs of the ML and the MML estimators of the location and scale parameters of the JFST distribution, i.e. μ and σ, are reported. Discussions about the simulation results for the parameters μ and σ are presented separately for the sake of simplicity.

For μ: In terms of the bias criterion, the ML estimator has smaller bias than the corresponding MML estimator when the shape of the JFST distribution is symmetric, i.e., $a = b$ =3 and $a = b$ =15. On the other hand, the ML estimator has larger bias than the MML estimator for positively skewed cases, i.e., a =6, b =3 and a =9, b =3. In terms of the MSE criterion, the ML estimator shows better performance than the corresponding MML estimator in all cases. It should be noted that the performances of the ML and the MML estimators are more or less the same when m ≥20 for all the parameter settings, sample sizes and the censoring schemes as expected, since it is known that the MML estimator is asymptotically equivalent to the ML estimator.

For σ: The MML estimator outperforms the corresponding ML estimator with respect to the bias criterion in all cases, because, the ML estimator underestimates the scale parameter σ. The ML estimator of σ is more efficient than the corresponding MML estimator with respect to the MSE criterion in most of the cases. For large values of n and m, the ML and the MML estimators are close to each other due to the reason given in subsection 3.2.

It should be noted that the performances of the all the estimators increase significantly as the sample proportion m/n increases (in other words, the proportion of censoring decreases) as expected.

The exact variances of $\hat{\mu}$ and $\hat{\sigma}$ cannot be obtained. We therefore compute the observed Fisher information matrix to obtain approximate variances and covariance of the parameter estimators. In Table 2, simulated variances and covariance of the ML and the MML estimators of μ and σ are compared with the corresponding variances and covariance obtained from the observed Fisher information matrix. It is easy to see that variances and covariance determined from the observed Fisher information matrix are close to simulated variances and covariance, even when m is moderate, i.e. m =20 for both the ML and the MML estimators.

Table 1. *Mean and MSEs of ML, MML and LS estimators of μ and σ.*

			ML		MML	
n	m	Scheme	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\mu}$	$\hat{\sigma}$
			$(a = 3, b = 3)$			
15	5	(3,0,4,0,3)	-0.1225(0.2015)	0.8773(0.1543)	-0.1286(0.2024)	0.9336(0.1710)
	10	$(2,0^{*8},3)$	-0.0139(0.1011)	0.9453(0.0739)	-0.0307(0.1024)	0.9883(0.0848)
	10	$(1^{*3},0^{*5},1^{*2})$	-0.0151(0.1056)	0.9481(0.0713)	-0.0253(0.1064)	0.9912(0.0802)
30	10	$(5^{*2},0^{*6},5^{*2})$	-0.0603(0.0913)	0.9375(0.0751)	-0.0613(0.0918)	0.9846(0.0859)
	20	$(5,0^{*18},5)$	-0.0081(0.0497)	0.9709(0.0354)	-0.0193(0.0501)	1.0027(0.0405)
	20	$(1^{*5},0^{*10},1^{*5})$	-0.0128(0.0521)	0.9705(0.0358)	-0.0162(0.0523)	0.9974(0.0381)
50	25	$(0^{*12},2^{*12},1)$	-0.0180(0.0350)	0.9691(0.0321)	-0.0150(0.0351)	0.9863(0.0330)
	40	$(5,0^{*38},5)$	-0.0005(0.0298)	0.9857(0.0176)	-0.0056(0.0299)	1.0027(0.0185)
	40	$(1^{*5},0^{*30},1^{*5})$	-0.0013(0.0282)	0.9856(0.0177)	-0.0036(0.0282)	1.0010(0.0184)
			$(a = 6, b = 3)$			
15	5	(2^{*5})	0.0281(0.1442)	0.8649(0.1340)	0.0127(0.1457)	0.8871(0.1352)
	10	$(0^{*9},5)$	0.0603(0.1283)	0.9301(0.0682)	0.0516(0.1286)	0.9406(0.0685)
	10	$(0^{*5},1^{*5})$	0.0515(0.1243)	0.9359(0.0672)	0.0440(0.1246)	0.9556(0.0685)
30	10	(2^{*10})	0.0153(0.0786)	0.9382(0.0619)	-0.0022(0.0799)	0.9574(0.0632)
	20	$(0^{*19},10)$	0.0328(0.0639)	0.9634(0.0340)	0.0250(0.0641)	0.9716(0.0342)
	20	$(1^{*5},0^{*10},1^{*5})$	0.0205(0.0665)	0.9704(0.0325)	0.0117(0.0670)	0.9861(0.0332)
50	25	(1^{*25})	0.0101(0.0405)	0.9793(0.0255)	0.0034(0.0406)	0.9944(0.0263)
	40	$(0^{*39},10)$	0.0143(0.0384)	0.9839(0.0166)	0.0090(0.0387)	0.9901(0.0167)
	40	$(1^{*5},0^{*30},1^{*5})$	0.0184(0.0378)	0.9884(0.0169)	0.0114(0.0379)	0.9989(0.0173)
			$(a = 9, b = 3)$			
15	5	(2^{*5})	0.1610(0.3008)	0.8601(0.1296)	0.1512(0.3006)	0.8741(0.1298)
	10	$(0^{*9},5)$	0.1379(0.2615)	0.9267(0.0665)	0.1426(0.2620)	0.9301(0.0664)
	10	$(0^{*5},1^{*5})$	0.1338(0.2541)	0.9284(0.0678)	0.1316(0.2548)	0.9424(0.0689)
30	10	(2^{*10})	0.0716(0.1514)	0.9352(0.0584)	0.0564(0.1524)	0.9462(0.0587)
	20	$(0^{*19},10)$	0.0653(0.1250)	0.9649(0.0324)	0.0647(0.1252)	0.9679(0.0324)
	20	$(1^{*5},0^{*10},1^{*5})$	0.0598(0.1218)	0.9693(0.0319)	0.0526(0.1221)	0.9804(0.0324)
50	25	(1^{*25})	0.0368(0.0775)	0.9766(0.0247)	0.0288(0.0778)	0.9872(0.0251)
	40	$(0^{*39},10)$	0.0380(0.0709)	0.9812(0.0159)	0.0362(0.0709)	0.9845(0.0159)
	40	$(1^{*5},0^{*30},1^{*5})$	0.0400(0.0685)	0.9837(0.0161)	0.0338(0.0687)	0.9911(0.0162)
			$(a = 15, b = 15)$			
15	5	(3,0,4,0,3)	-0.1310(0.1783)	0.8479(0.1246)	-0.1369(0.1790)	0.8630(0.1242)
	10	$(2,0^{*8},3)$	-0.0248(0.0884)	0.9259(0.0599)	-0.0306(0.0885)	0.9359(0.0599)
	10	$(1^{*3},0^{*5},1^{*2})$	-0.0267(0.0889)	0.9277(0.0567)	-0.0316(0.0891)	0.9368(0.0566)
30	10	$(5^{*2},0^{*6},5^{*2})$	-0.0620(0.0829)	0.9300(0.0598)	-0.0641(0.0830)	0.9437(0.0600)
	20	$(5,0^{*18},5)$	-0.0148(0.0439)	0.9687(0.0282)	-0.0193(0.0440)	0.9777(0.0285)
	20	$(1^{*5},0^{*10},1^{*5})$	-0.0130(0.0450)	0.9637(0.0282)	-0.0154(0.0450)	0.9698(0.0281)
50	25	$(0^{*12},2^{*12},1)$	-0.0243(0.0338)	0.9615(0.0248)	-0.0250(0.0339)	0.9651(0.0248)
	40	$(5,0^{*38},5)$	0.0003(0.0232)	0.9863(0.0139)	-0.0022(0.0232)	0.9913(0.0139)
	40	$(1^{*5},0^{*30},1^{*5})$	0.0003(0.0235)	0.9815(0.0134)	-0.0018(0.0235)	0.9854(0.0133)

Table 2. *Simulated and observed variance and covariances of the ML and the MML estimators.*

			ML						MML					
n	m	Scheme	$Var(\hat{\mu})$	$Var(\hat{\sigma})$	$Cov(\hat{\mu}.\hat{\sigma})$	$\hat{\sigma}^2 I^{11}$	$\hat{\sigma}^2 I^{22}$	$\hat{\sigma}^2 I^{12}$	$Var(\hat{\mu})$	$Var(\hat{\sigma})$	$Cov(\hat{\mu}.\hat{\sigma})$	$\hat{\sigma}^2 I_*^{11}$	$\hat{\sigma}^2 I_*^{22}$	$\hat{\sigma}^2 I_*^{12}$
							$(a = 3, b = 3)$							
15	5	(3,0,4,0,3)	0.1865	0.1392	0.0681	0.1583	0.1303	0.0598	0.1859	0.1666	0.0730	0.1738	0.1159	0.0666
	10	$(2,0^{*8},3)$	0.1010	0.0709	0.0055	0.0968	0.0690	0.0047	0.1014	0.0847	0.0026	0.0988	0.0639	0.0086
	10	$(1^{*3},0^{*5},1^{*2})$	0.1053	0.0686	0.0091	0.1019	0.0666	0.0074	0.1058	0.0801	0.0078	0.1020	0.0616	0.0098
30	10	$(5^{*2},0^{*6},5^{*2})$	0.0877	0.0712	0.0317	0.0801	0.0688	0.0282	0.0881	0.0856	0.0348	0.0873	0.0633	0.0308
	20	$(5,0^{*18},5)$	0.0496	0.0345	0.0018	0.0506	0.0346	0.0019	0.0497	0.0405	0.0009	0.0518	0.0329	0.0030
	20	$(1^{*5},0^{*10},1^{*5})$	0.0519	0.0350	0.0036	0.0502	0.0342	0.0039	0.0521	0.0381	0.0033	0.0506	0.0324	0.0045
50	25	$(0^{*12},2^{*12},1)$	0.0347	0.0312	0.0106	0.0335	0.0299	0.0099	0.0349	0.0328	0.0110	0.0339	0.0289	0.0099
	40	$(5,0^{*38},5)$	0.0298	0.0174	0.0003	0.0283	0.0176	0.0002	0.0298	0.0185	0.0001	0.0285	0.0171	0.0004
	40	$(1^{*5},0^{*30},1^{*5})$	0.0282	0.0175	0.0011	0.0282	0.0175	0.0006	0.0282	0.0184	0.0010	0.0282	0.0169	0.0007
							$(a = 6, b = 3)$							
15	5	(2^{*5})	0.1434	0.1157	-0.0343	0.1248	0.1010	-0.0308	0.1455	0.1225	-0.0378	0.1172	0.0956	-0.0259
	10	$(0^{*9},5)$	0.1247	0.0633	-0.0398	0.1156	0.0595	-0.0383	0.1260	0.0650	-0.0412	0.1114	0.0573	-0.0361
	10	$(0^{*5},1^{*5})$	0.1216	0.0631	-0.0364	0.1159	0.0611	-0.0369	0.1227	0.0665	-0.0380	0.1141	0.0581	-0.0352
30	10	(2^{*10})	0.0784	0.0581	-0.0215	0.0731	0.0547	-0.0206	0.0799	0.0614	-0.0238	0.0693	0.0517	-0.0167
	20	$(0^{*19},10)$	0.0628	0.0326	-0.0212	0.0599	0.0311	-0.0201	0.0635	0.0334	-0.0219	0.0585	0.0302	-0.0191
	20	$(1^{*5},0^{*10},1^{*5})$	0.0661	0.0316	-0.0188	0.0638	0.0305	-0.0182	0.0669	0.0330	-0.0197	0.0625	0.0292	-0.0171
50	25	(1^{*25})	0.0404	0.0250	-0.0117	0.0393	0.0246	-0.0114	0.0406	0.0262	-0.0122	0.0390	0.0237	-0.0108
	40	$(0^{*39},10)$	0.0382	0.0163	-0.0114	0.0363	0.0160	-0.0111	0.0386	0.0166	-0.0117	0.0357	0.0156	-0.0107
	40	$(1^{*5},0^{*30},1^{*5})$	0.0374	0.0167	-0.0114	0.0382	0.0161	-0.0107	0.0378	0.0173	-0.0117	0.0377	0.0156	-0.0102
							$(a = 9, b = 3)$							
15	5	(2^{*5})	0.2749	0.1101	-0.1120	0.2325	0.0965	-0.0968	0.2777	0.1139	-0.1152	0.2274	0.0925	-0.0940
	10	$(0^{*9},5)$	0.2425	0.0611	-0.0834	0.2179	0.0567	-0.0767	0.2416	0.0615	-0.0834	0.2188	0.0554	-0.0764
	10	$(0^{*5},1^{*5})$	0.2362	0.0627	-0.0832	0.2172	0.0587	-0.0771	0.2375	0.0656	-0.0855	0.2214	0.0566	-0.0765
30	10	(2^{*10})	0.1463	0.0542	-0.0579	0.1372	0.0514	-0.0551	0.1492	0.0558	-0.0599	0.1304	0.0495	-0.0518
	20	$(0^{*19},10)$	0.1207	0.0312	-0.0425	0.1148	0.0298	-0.0406	0.1210	0.0314	-0.0427	0.1143	0.0293	-0.0401
	20	$(1^{*5},0^{*10},1^{*5})$	0.1182	0.0310	-0.0395	0.1157	0.0299	-0.0386	0.1193	0.0321	-0.0405	0.1152	0.0290	-0.0377
50	25	(1^{*25})	0.0762	0.0242	-0.0287	0.0742	0.0238	-0.0279	0.0769	0.0249	-0.0294	0.0738	0.0232	-0.0273
	40	$(0^{*39},10)$	0.0695	0.0156	-0.0218	0.0665	0.0155	-0.0217	0.0696	0.0157	-0.0218	0.0662	0.0152	-0.0214
	40	$(1^{*5},0^{*30},1^{*5})$	0.0669	0.0158	-0.0211	0.0678	0.0157	-0.0213	0.0676	0.0162	-0.0215	0.0673	0.0154	-0.0209
							$(a = 15, b = 15)$							
15	5	(3,0,4,0,3)	0.1611	0.1014	0.0654	0.1320	0.0896	0.0564	0.1602	0.1054	0.0659	0.1367	0.0866	0.0598
	10	$(2,0^{*8},3)$	0.0878	0.0544	0.0126	0.0784	0.0510	0.0099	0.0876	0.0558	0.0124	0.0786	0.0495	0.0115
	10	$(1^{*3},0^{*5},1^{*2})$	0.0882	0.0515	0.0101	0.0804	0.0486	0.0104	0.0881	0.0526	0.0099	0.0799	0.0472	0.0115
30	10	$(5^{*2},0^{*6},5^{*2})$	0.0791	0.0549	0.0328	0.0728	0.0511	0.0300	0.0789	0.0568	0.0331	0.0750	0.0491	0.0312
	20	$(5,0^{*18},5)$	0.0437	0.0272	0.0044	0.0425	0.0268	0.0044	0.0437	0.0280	0.0043	0.0428	0.0260	0.0051
	20	$(1^{*5},0^{*10},1^{*5})$	0.0448	0.0269	0.0049	0.0417	0.0260	0.0056	0.0448	0.0272	0.0048	0.0416	0.0255	0.0059
50	25	$(0^{*12},2^{*12},1)$	0.0333	0.0233	0.0113	0.0298	0.0229	0.0103	0.0332	0.0235	0.0113	0.0299	0.0226	0.0104
	40	$(5,0^{*38},5)$	0.0232	0.0137	0.0010	0.0235	0.0137	0.0010	0.0232	0.0138	0.0009	0.0236	0.0134	0.0012
	40	$(1^{*5},0^{*30},1^{*5})$	0.0235	0.0130	0.0014	0.0231	0.0134	0.0013	0.0235	0.0131	0.0014	0.0231	0.0131	0.0014

Table 3. *Average confidence lengths and coverage probabilities for the ML and the MML estimators of* μ *and* σ.

			ML		MML	
n	m	Scheme	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\mu}$	$\hat{\sigma}$
				$(a = 3, b = 3)$		
15	5	(3,0,4,0,3)	1.5594(0.8230)	1.4148(0.7871)	1.6341(0.8318)	1.3348(0.7917)
	10	(2,0*8,3)	1.2198(0.9120)	1.0301(0.8678)	1.2322(0.9116)	0.9911(0.8692)
	10	(1*3,0*5,1*2)	1.2511(0.9161)	1.0114(0.8714)	1.2519(0.9127)	0.9733(0.8753)
30	10	(5*2,0*6,5*2)	1.1092(0.8817)	1.0284(0.8647)	1.1584(0.8905)	0.9865(0.8671)
	20	(5,0*18,5)	0.8821(0.9392)	0.7291(0.9108)	0.8924(0.9390)	0.7113(0.9086)
	20	(1*5,0*10,1*5)	0.8781(0.9324)	0.7244(0.9042)	0.8814(0.9330)	0.7061(0.9066)
50	25	(0*12,2*12,1)	0.7180(0.9260)	0.6780(0.9050)	0.7220(0.9275)	0.6661(0.9103)
	40	(5,0*38,5)	0.6597(0.9368)	0.5207(0.9304)	0.6616(0.9392)	0.5127(0.9296)
	40	(1*5,0*30,1*5)	0.6577(0.9388)	0.5188(0.9364)	0.6582(0.9400)	0.5103(0.9348)
				$(a = 6, b = 3)$		
15	5	(2*5)	1.3850(0.8584)	1.2460(0.7735)	1.3419(0.8500)	1.2118(0.7759)
	10	(0*9,5)	1.3327(0.8909)	0.9560(0.8574)	1.3085(0.8873)	0.9383(0.8578)
	10	(0*5,1*5)	1.3348(0.8968)	0.9688(0.8616)	1.3243(0.8953)	0.9447(0.8644)
30	10	(2*10)	1.0602(0.9088)	0.9166(0.8634)	1.0317(0.9036)	0.8915(0.8658)
	20	(0*19,10)	0.9596(0.9210)	0.6908(0.9002)	0.9479(0.9184)	0.6810(0.8998)
	20	(1*5,0*10,1*5)	0.9898(0.9326)	0.6846(0.9044)	0.9801(0.9308)	0.6701(0.9084)
50	25	(1*25)	0.7768(0.9343)	0.6148(0.9160)	0.7737(0.9333)	0.6037(0.9180)
	40	(0*39,10)	0.7464(0.9352)	0.4952(0.9268)	0.7410(0.9332)	0.4893(0.9248)
	40	(1*5,0*30,1*5)	0.7661(0.9388)	0.4971(0.9316)	0.7610(0.9384)	0.4895(0.9312)
				$(a = 9, b = 3)$		
15	5	(2*5)	1.8901(0.8161)	1.2175(0.7678)	1.8692(0.8160)	1.1922(0.7689)
	10	(0*9,5)	1.8298(0.8681)	0.9336(0.8503)	1.8335(0.8668)	0.9230(0.8491)
	10	(0*5,1*5)	1.8269(0.8664)	0.9499(0.8486)	1.8447(0.8690)	0.9329(0.8507)
30	10	(2*10)	1.4522(0.8935)	0.8883(0.8637)	1.4153(0.8892)	0.8726(0.8644)
	20	(0*19,10)	1.3282(0.9088)	0.6768(0.9010)	1.3251(0.9078)	0.6711(0.9002)
	20	(1*5,0*10,1*5)	1.3335(0.9162)	0.6783(0.9026)	1.3304(0.9170)	0.6678(0.9046)
50	25	(1*25)	1.0676(0.9223)	0.6049(0.9200)	1.0647(0.9230)	0.5968(0.9235)
	40	(0*39,10)	1.0112(0.9268)	0.4874(0.9236)	1.0084(0.9264)	0.4834(0.9240)
	40	(1*5,0*30,1*5)	1.0207(0.9312)	0.4917(0.9220)	1.0173(0.9324)	0.4862(0.9240)
				$(a = 15, b = 15)$		
15	5	(3,0,4,0,3)	1.4243(0.8143)	1.1734(0.7626)	1.4495(0.8172)	1.1533(0.7655)
	10	(2,0*8,3)	1.0979(0.9010)	0.8857(0.8518)	1.0992(0.9003)	0.8723(0.8530)
	10	(1*3,0*5,1*2)	1.1118(0.9035)	0.8638(0.8597)	1.1083(0.9025)	0.8520(0.8616)
30	10	(5*2,0*6,5*2)	1.0575(0.8823)	0.8864(0.8559)	1.0735(0.8848)	0.8684(0.8582)
	20	(5,0*18,5)	0.8085(0.9266)	0.6419(0.9064)	0.8105(0.9258)	0.6326(0.9058)
	20	(1*5,0*10,1*5)	0.8001(0.9226)	0.6321(0.8948)	0.7994(0.9220)	0.6255(0.8962)
50	25	(0*12,2*12,1)	0.6767(0.9180)	0.5938(0.9000)	0.6778(0.9188)	0.5898(0.9008)
	40	(5,0*38,5)	0.6015(0.9456)	0.4582(0.9332)	0.6016(0.9440)	0.4543(0.9356)
	40	(1*5,0*30,1*5)	0.5963(0.9424)	0.4530(0.9284)	0.5956(0.9420)	0.4493(0.9292)

In Table 3, the average lengths of the CIs and the CPs for the location parameter μ and the scale parameter σ are computed at 95% confidence level. Here, it should be noted that the observed Fisher information matrices, based on the ML and the MML estimators, are used for computing pivotal quantities.

The average lengths of the CIs based on the ML and the MML estimators for both parameters μ and σ are more or less the same. However, the average length of the CI based on the ML estimator of μ is slightly shorter than the corresponding MML estimator for symmetrical cases. On the other hand, the MML estimator is preferable to the ML estimator in computing the average length of the CI for positively skewed

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cases. The average length of the CI based on the MML estimator of σ is a little bit shorter than the ML estimator.

The CPs based on the ML and the MML estimators of μ and σ are extremely unsatisfactory, especially when m/n is small. On the other hand, the values of CPs based on the ML and the MML estimators of μ and σ are close to the expected value 95% while m/n is increasing.

5. DATA ANALYSIS

In this section, we analyze a real data set taken from the literature to demonstrate implementation of the proposed estimation methods. For this purpose, we use the ball bearing data set originally discussed by Lieblein and Zelen (1956). This concern the results of a test on endurance of deep groove ball bearings (each measurement in 10^6 revolutions). The complete data contains 23 observations and are given as follows:

17.88, 28.92, 33.0, 41.52, 42.12, 45.60, 48.40, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04 and 173.4.

In the literature, this data set was modeled using several statistical distributions, such as Weibull, logistic-exponential, generalized inverted exponential and log-normal, see Lawless (1982), Lieblein and Zelen (1956), Krishna and Kumar (2013) and Singh and Tripathi (2015). Different to earlier studies, here, we use JFST distribution for modeling the ball bearing data.

Before analyzing this data set, we first need to estimate the shape parameters a and b. We use the profile likelihood method to identify estimates of the shape parameters a and b. The steps of the profile likelihood method are given below, see for example, Islam and Tiku (2004) and Acıtaş and Şenoğlu (2016).

Step 1. Calculate $\hat{\mu}$ and $\hat{\sigma}$ for the given a and b values.
Step 2. Calculate the log-likelihood value by incorporating $\hat{\mu}$ and $\hat{\sigma}$ into (3).
Step 3. Repeat step 1 and step 2 for a serious values of a and b.
Step 4. Find a and b values maximizing the log-likelihood function among the others and choose them as conceivable values of the shape parameters.

Following these steps, a and b are obtained as 4.1 and 1.8, respectively. Then, we draw the Q-Q plots of the observations for the various values of a and b. It can be observed that the JFST distribution a =4.1 and b =1.8 beautifully models the ball bearing data; see Figure 2. It is clear that both profile likelihood methodology and the Q-Q plot technique are in agreement in identifying the shape parameters of the JFST distribution.

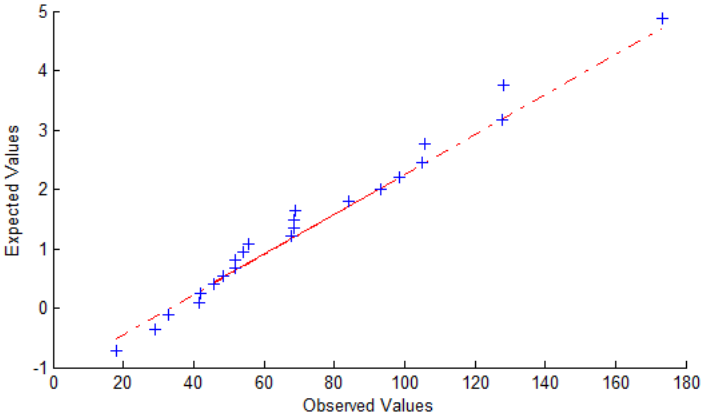

Figure 2. *JFST Q-Q plot of the ball bearing data*

Now, we consider the progressively Type-II censored samples with the following censoring schemes, see Table 4.

Table 4. *Progressively censored samples.*

n	m	Scheme	Censored data
23	23	(0^{*23})	17.88, 28.92, 33.0, 41.52, 42.12, 45.60, 48.40, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.4
23	18	$(3,0^{*16},2)$	17.88, 42.12, 45.60, 48.40, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92
23	15	$(8,0^{*14})$	17.88, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.4

For each censoring schemes, we obtain the profile likelihood estimates of a and b. Based on these estimate values of the shape parameters, the location parameter μ and the scale parameter σ are obtained using ML and MML methodologies. We then compute the 95% of the CIs for μ and σ based on the ML and the MML estimates.

The corresponding estimates of μ and σ and the corresponding CIs are summarized in Table 5.

Table 5. *Point and interval estimates of μ and σ based on the ML and the MML methodologies for the ball bearing data.*

		ML				MML			
n	m	a	b	$\hat{\mu}_{ML}$	$\hat{\sigma}_{ML}$	a	b	$\hat{\mu}_{MML}$	$\hat{\sigma}_{MML}$
23	23	4.12	1.78	37.306 (24.807,49.804)	22.736 (14.059,31.414)	4.15	1.82	37.732 (25.008,50.456)	23.272 (14.891,31.652)
23	20	7.9	2.3	25.146 (8.823,41.468)	20.1491 (12.267,28.021)	7.8	2.3	25.415 (9.487,41.343)	20.541 (12.911,28.171)
23	15	5.6	3.1	57.835 (40.471,75.199)	28.248 (16.775,39.720)	5.8	3.2	56.501 (39.768,73.234)	29.157 (18.451,39.865)

It is clear from Table 5 that the estimate values of μ and σ obtained using the ML and MML methodologies are very close to each other for all the censoring schemes, see also Figure 3-5. From these figures it can easily be said that the JFST distribution provides good modeling performance for these complete and censored data sets.

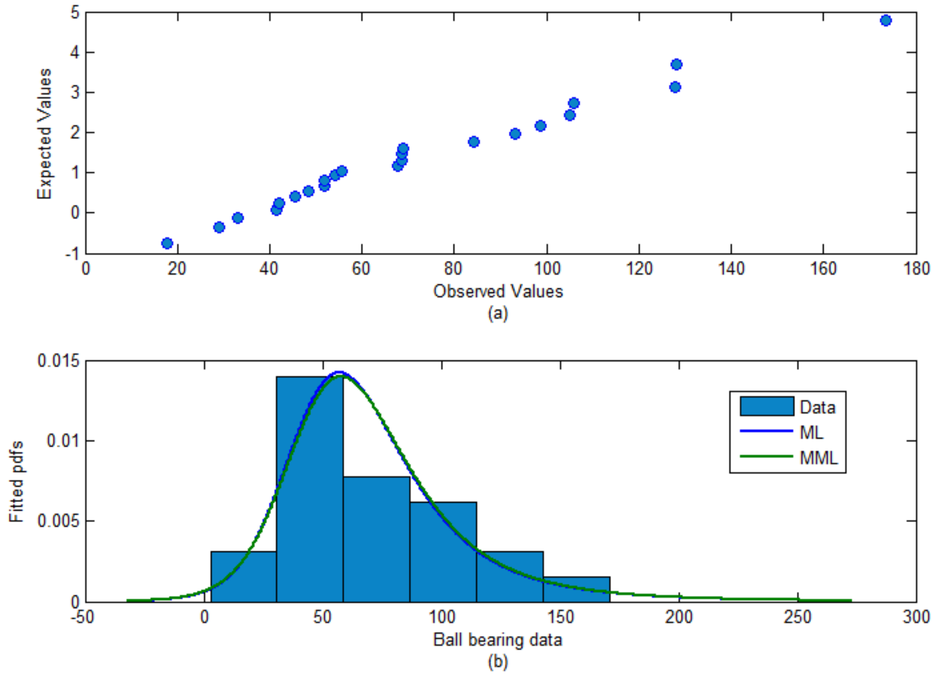

Figure 3. *Scatter plot of the ball bearing data (a) and the fitted pdfs (b) n =23, m =23*

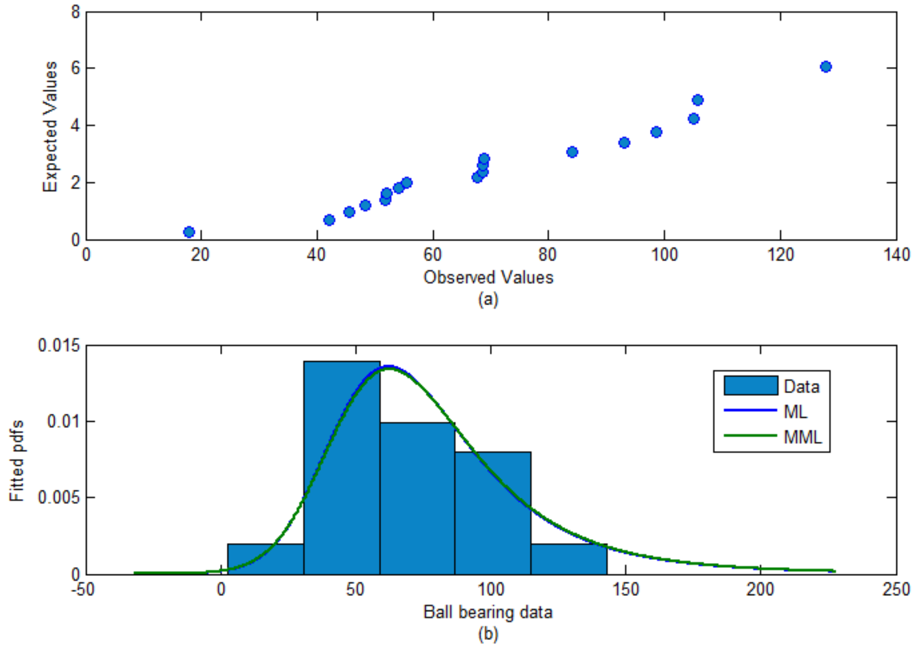

Figure 4. *Scatter plot of the ball bearing data (a) and the fitted pdfs (b) n =23, m =18*

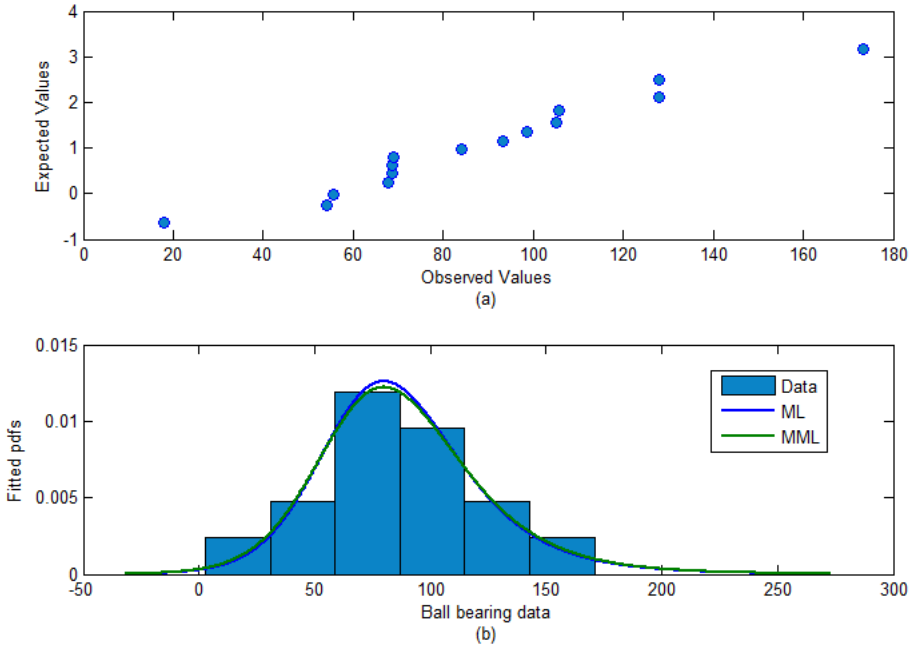

Figure 5. *Scatter plot of the ball bearing data (a) and the fitted pdfs (b)* n *=23,* m *=15*

The lengths of the CIs, based on the observed Fisher information matrices obtained using ML and MML estimates, are more or less the same. These results are in agreement with the simulation results.

6. CONCLUSIONS

In this paper, we obtain the point and interval estimators for the location and scale parameters of JFST distribution based on progressively Type-II censored samples. In the estimation procedure, the ML and MML methodologies are used. It should be noted that the MML estimates for the parameters μ and σ are used as initial values to obtain the ML estimates of μ and σ iteratively. We use pivotal quantities for constructing CIs for the parameters μ and σ. The performances of the proposed estimators are compared via Monte-Carlo simulation study. It seen that the ML and MML estimators are very similar in terms of bias and MSE criteria when the sample sizes increase as expected. The CPs based on the ML and the MML estimators of μ and σ are found to be unsatisfactory, especially when m/n is small. In spite of this, the corresponding CPs are close to the expected value of 95% when m/n is getting large. Moreover, the results of the real data analysis coincide with the simulation study.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors

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