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Some new results on ⋆-metric spaces

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Abstract — The concept of ⋆-metric, based on the relaxation of triangle inequality of metric axioms by using a t-definer, was introduced by Khatami and Mirzavaziri. This paper extends and generalizes some well-known results of classical metric space. Considering the definition of ⋆-metric space, it studies the notion of a closed ball. The paper proves some results related to closed sets, convergent sequences, Cauchy sequences, and the diameter of a set. This paper contains the study on the metrizability of ⋆-metric space and provides an alternative approach to the proof of metrizability for ⋆-metric space using the famous 'Niemytski and Wilson's metrization theorem'.

Subject Classification (2020): 54E15, 54E35

1. Introduction

As the concept of Metric Space and Banach contraction principle was brought to light, researchers attempted to elaborate that concept by generalizing the metric axioms or the contraction principle. The study of generalization with the distance functions is mainly followed by two approaches: increasing the number of components of the metric function and relaxing the triangle inequality of metric axioms. D-metric [1], G-metric [2], and S-metric [3] spaces are examples of the first type, whereas b-metric [4], rectangular metric [5], and ϕ-metric [6] are examples of the second type of generalization. The idea of metric functions was extended to fuzzy sets [7] by introducing the notion of fuzzy metric space [8]. Cone [9] and parametric metric [10] are another kind of generalization of distance function to abstract spaces. Thus far, researchers tried to establish several metric fixed point theorems in the setting of these generalized metric spaces. For more details, see [11–15].

Following the relaxation approach of the triangle inequality, Khatami and Mirzavaziri [16] introduced another generalized metric space, namely, ⋆-metric space. They used the concept of t-conorm function [17] to the 'triangle inequality' and developed this new structure. t-conorm $\star$ is function defined from $[0,1]^2$ to $[0,1]$ satisfying $\alpha \star 0 = \alpha$, for all $\alpha \in [0,1]$. Khatami and Mirzavaziri [16] extended this notion by defining a binary operation, triangular definer (or t-definer) on $[0,\infty)$ which has been used in the axiom $(\rho^\star 3)$ of ⋆-metric space. Later, they studied the topological space induced by a ⋆-metric, defined open ball, constructed some non-trivial examples of ⋆-metrizable topological spaces,

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and extended some topological concepts of metric spaces. Recently, He et al. [18] exercised on the metrizability of the topological space induced by a $\star$-metric, and concluded that every $\star$-metric space is metrizable. They define the notion of convergence of sequence, and proved some other results on compactness, total boundedness, and completeness.

This manuscript aims to extend of some well-known concepts in metric space, such as the notion of closed ball, results related to convergent sequence, etc. Lastly, we prove the metrizability of $\star$-metric space via 'Niemytski and Wilson's metrization theorem'. Although He et al. [18] have worked on the metrizability of $\star$-metric space but our new proof is a bit easier to understand than the existing one.

The article is organized as follows. Section 2 consists of preliminary results which are required for the main work of this article. Section 3 introduces some new metric results in the $\star$-metric space.

2. Preliminaries

This section consists of some definitions, and results on $\star$-metric space. The subsequent definitions are of t-definer and $\star$-metric.

Definition 2.1. [16] A t-definer $\star : [0,\infty)^2 \to [0,\infty)$ is a function satisfying the following conditions:

(T1) $\alpha \star \mu = \mu \star \alpha$

(T2) $\alpha \star (\mu \star \eta) = (\alpha \star \mu) \star \eta$

(T3) $\alpha \leq \mu \Rightarrow [\alpha \star \eta \leq \mu \star \eta$ and $\eta \star \alpha \leq \eta \star \mu]$

(T4) $\alpha \star 0 = \alpha$

(T5) $\star$ is continuous in its first component with respect to the Euclidean topology.

for all $\alpha, \mu, \eta \in [0,\infty)$.

Remark 2.2. [16] Observe that

i. Continuity in the first component of $\star$ implies the continuity in the second component, because of the condition (T1) of a t-definer.

ii. Moreover, from [17] we know that the t-definer is a non-decreasing function. Hence, the continuity of the first component is equivalent to its continuity.

Followings are some examples of t-definer.

Example 2.3. [16]

i. Lukasiewicz t-definer: $\alpha \star \mu = \alpha + \mu$, for all $\alpha, \mu \in [0,\infty)$

ii. Maximum t-definer: $\alpha \star \mu = \max\{\alpha, \mu\}$, for all $\alpha, \mu \in [0,\infty)$

iii. $\alpha \star \mu = (\sqrt{\alpha} + \sqrt{\mu})^2$, for all $\alpha, \mu \in [0,\infty)$

Definition 2.4. [16] Let Ω be a nonempty set and $\star$ be a t-definer. A pair $(\Omega, \rho^\star)$ is known to be as a $\star$-metric space if $\rho^\star : \Omega \times \Omega \to [0,\infty)$ is a function satisfying the following properties:

$(\rho^\star 1)$ $\rho^\star(\gamma_1, \gamma_2) \geq 0$, for all $\gamma_1, \gamma_2 \in \Omega$ and $\rho^\star(\gamma_1, \gamma_2) = 0 \Leftrightarrow \gamma_1 = \gamma_2$

$(\rho^\star 2)$ $\rho^\star(\gamma_1, \gamma_2) = \rho^\star(\gamma_2, \gamma_1)$, for all $\gamma_1, \gamma_2 \in \Omega$

$(\rho^\star 3)$ $\rho^\star(\gamma_1, \gamma_2) \leq \rho^\star(\gamma_1, \gamma_3) \star \rho^\star(\gamma_3, \gamma_2)$, for all $\gamma_1, \gamma_2, \gamma_3 \in \Omega$

Remark 2.5. From the Example 2.4 of [16], it can be concluded that $\star$-metric, in general, is not a metric. But observe that a $\star$-metric with respect to the Lukasiewicz t-definer reduced to a metric.

Definition 2.6. [16] Let $\star$ be a t-definer. The residuum of $\star$ is defined by $\alpha \dot{\to} \beta = \inf\{\gamma : \gamma \star \alpha \geq \beta\}$. Moreover, for any $\alpha, \beta, \gamma \in [0, \infty)$, $\gamma \geq \alpha \dot{\to} \beta$ if and only if $\gamma \star \alpha \geq \beta$, which is known as the residuation property of $\star$ and $\dot{\to}$.

Lemma 2.7. [16] Let $\star$ be a t-definer, and $\dot{\to}$ be its residuum. Then,

i. $\alpha \dot{\to} \beta = \min\{\gamma : \gamma \star \alpha \geq \beta\}$

ii. $0 \dot{\to} \alpha = \alpha$

iii. $\alpha \dot{\to} \beta = 0 \Leftrightarrow \alpha \geq \beta$

iv. $\alpha \star (\alpha \dot{\to} \beta) = \max\{\alpha, \beta\}$

v. $\alpha \dot{\to} \beta \geq (\alpha \dot{\to} \gamma) \dot{\to} (\gamma \dot{\to} \beta)$

vi. $\alpha \dot{\to} \beta \leq (\alpha \dot{\to} \gamma) \star (\gamma \dot{\to} \beta)$

In the following, we recall the definition of open ball, and some other related results on $\star$-metric space.

Definition 2.8. [16] Let $(\Omega, \rho^\star)$ be a $\star$-metric space.

i. For any $a \in \Omega$ and $t > 0$, the open ball with center a, and radius t is defined as

$$N_t(a) = \{\eta \in \Omega : \rho^\star(a, \eta) < t\}$$

ii. A point $x \in B \subseteq \Omega$ is called an interior point of B if there exists $\varepsilon > 0$ such that $N_\varepsilon(x) \subset B$.

iii. $B \subseteq \Omega$ is said to be an open set if each point of B is an interior point of its.

iv. The $\star$-metric topology is defined by the set $\tau_{\rho^\star} = \{A \subseteq \Omega : A \text{ is an open set in } \Omega\}$.

Remark 2.9. The definitions of closed set, limit point, interior, and closure of a set, derived set, continuous function are same as in usual metric space.

Proposition 2.10. [16] In a $\star$-metric space,

i. Every open ball is an open set.

ii. Every $\star$-metric space is

(a) Hausdorff.

(b) First countable.

(c) Normal.

Remark 2.11. [18] For $t > 0$, there exists $t_1 > 0$ such that $[0, t_1) \star [0, t_1) \subseteq [0, t)$.

Theorem 2.12. [18] Every $\star$-metric space $(\Omega, \rho^\star)$ with the topology $\tau_{\rho^\star}$ induced by $\rho^\star$ is metrizable.

Definition 2.13. [18] Let $(\Omega, \rho^\star)$ be a $\star$-metric space and $\{\alpha_n\}$ be a sequence in $(\Omega, \rho^\star)$.

i. If there exists an element $\alpha \in (\Omega, \rho^\star)$ such that for every $r > 0$, there exists $N \in \mathbb{N}$ such that $\rho^\star(\alpha_n, \alpha) < r$, for all $n \geq N$, then $\{\alpha_n\}$ is said to converge to α under ρ^* and written by $\alpha_n \to \alpha$.

ii. $\{\alpha_n\}$ is said to be a Cauchy sequence if for every $r > 0$, there exists $N \in \mathbb{N}$ such that $\rho^\star(\alpha_n, \alpha_m) < r$, for all $m, n \geq N$.

iii. $(\Omega, \rho^\star)$ is complete if every Cauchy sequence in Ω converges to some member of it.

Definition 2.14. [18] For a subset B of a $\star$-metric space $(\Omega, \rho^\star)$, the diameter of the set B is defined as $\delta(B) = \sup\{\rho^\star(\mu, \xi) : \mu, \xi \in \Omega\}$.

Moreover, we recall the metrization theorem by Niemytski and Wilson.

Theorem 2.15. [19] Let Ω be a topological space. If a distance function $\chi : \Omega \times \Omega \to [0, \infty)$ satisfies the followings

i. $\chi(\xi, \zeta) = 0 \Leftrightarrow \xi = \zeta$, for all $\xi, \zeta \in \Omega$

ii. $\chi(\xi, \zeta) = \chi(\zeta, \xi)$, for all $\xi, \zeta \in \Omega$

and one of the following conditions is holds

(a) given a point $\xi \in \Omega$ and a number $\varepsilon > 0$, there exists $f(\xi, \varepsilon) > 0$ such that if $\chi(\xi, \zeta) < f(\xi, \varepsilon)$ and $\chi(\zeta, \mu) < f(\xi, \varepsilon)$, then $\chi(\xi, \mu) < \varepsilon$;

(b) if $\xi \in \Omega$, and $\{\zeta_n\}$ and $\{\mu_n\}$ are two sequences in Ω such that $\chi(\zeta_n, \xi) \to 0$ and $\chi(\zeta_n, \mu_n) \to 0$ as $n \to \infty$, then $\chi(\mu_n, \xi) \to 0$ as $n \to \infty$;

(c) for each point $\xi \in \Omega$ and $s > 0$; there is $r > 0$ such that if $\zeta \in \Omega$ for which $\chi(\xi, \zeta) \geq s$ and μ is any point in Ω, then $\chi(\xi, \mu) + \chi(\zeta, \mu) \geq r$,

then the topological space Ω is metrizable.

Niemytski [20] showed the equivalence of the conditions (a) and (b) of Theorem 2.15. Later, Wilson [21] proved that a space is metrizable if there exists a distance function satisfying the conditions *i*, *ii*, and (c).

3. Main Result

In this section, we introduce the concept of closed ball, and prove some results related to the notion of convergence of sequence.

Definition 3.1. Let $(\Omega, \rho^\star)$ be a $\star$-metric space. For any $a \in \Omega$ and $t > 0$, we define the closed ball with center a and radius t by the set $\bar{N}_t(a) = \{\eta \in \Omega : \rho^\star(a, \eta) \leq t\}$.

Example 3.2. We consider the $\star$-metric $\rho^\star(a, b) = (\sqrt{a} - \sqrt{b})^2$ on $[0, \infty)$ of the Example 2.4 of [16] where the underlying t-definer is taken as $a \star b = (\sqrt{a} + \sqrt{b})^2$, for all $a, b \in [0, \infty)$.

Then, for any $a \in [0, \infty)$ and $t > 0$, the closed ball $\bar{N}_t(a)$ is of the form

$$\bar{N}_t(a) = \{\eta \in [0, \infty) : \rho^\star(a, \eta) \leq t\} = \{\eta \in [0, \infty) : (\sqrt{a} - \sqrt{\eta})^2 \leq t\}$$

In particular, let $a = 4$ and $t = 1$. Then,

$$\{\eta \in [0, \infty) : (2 - \sqrt{\eta})^2 \leq 1\} = \{\eta \in [0, \infty) : -1 \leq 2 - \sqrt{\eta} \leq 1\} = \{\eta \in [0, \infty) : 1 \leq \sqrt{\eta} \leq 3\} = [1, 9]$$

Proposition 3.3. In a $\star$-metric space $(\Omega, \rho^\star)$, every closed ball is closed set.

Proof.

Consider a closed ball $\bar{N}_r(a)$, with $a \in \Omega$ and $r > 0$. We will show that $\Omega \setminus \bar{N}_r(a)$ is an open set. Let us take $x \in \Omega \setminus \bar{N}_r(a)$. Therefore, $\rho^\star(x, a) > r$. Let $r_1 = r \dot{\to} \rho^\star(x, a)$. We claim that $\bar{N}_{r_1}(x) \subseteq \Omega \setminus \bar{N}_r(a)$. Let $b \in \bar{N}_{r_1}(x)$. Thus, $\rho^\star(x, b) < r_1$ which implies

$$\rho^\star(x, b) < r \dot{\to} \rho^\star(x, a)$$

or

$$\rho^\star(x, b) < \inf\{c : c \star r \geq \rho^\star(x, a)\}$$

or

$$\rho^\star(x, b) \star r < \rho^\star(x, a)$$

Again from $(\rho^\star 3)$, we have $\rho^\star(x,a) \leq \rho^\star(a,b) \star \rho^\star(b,x)$. Then,

$$\begin{aligned}\rho^\star(x,b) \star r < \rho^\star(a,b) \star \rho^\star(b,x) &\Rightarrow \rho^\star(a,b) > r \\ &\Rightarrow b \in \Omega \setminus \bar{N}_r(a)\end{aligned}$$

Hence, $\bar{N}_{r_1}(x) \subseteq \Omega \setminus \bar{N}_r(a)$ which implies $\Omega \setminus \bar{N}_r(a)$ is an open set. Consequently $\bar{N}_r(a)$ is a closed set. □

Proposition 3.4. Let $(\Omega, \rho^\star)$ be a $\star$-metric space, and $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $(\Omega, \rho^\star)$ that converge to α and β in $(\Omega, \rho^\star)$, respectively. Then, $\{\rho^\star(\alpha_n, \beta_n)\}$ converges to $\rho^\star(\alpha, \beta)$.

Proof.

Since $\alpha_n \to \alpha$ and $\beta_n \to \beta$ as $n \to \infty$, then $\underset{n\to\infty}{lim}\ \rho^\star(\alpha_n, \alpha) = 0$ and $\underset{n\to\infty}{lim}\ \rho^\star(\beta_n, \beta) = 0$. Using $(\rho^\star 3)$, for all $n \in \mathbb{N}$, we have

$$\rho^\star(\alpha,\beta) \leq \rho^\star(\alpha,\alpha_n) \star \rho^\star(\alpha_n,\beta) \leq \rho^\star(\alpha,\alpha_n) \star \rho^\star(\alpha_n,\beta_n) \star \rho^\star(\beta_n,\beta) \tag{3.1}$$

and

$$\rho^\star(\alpha_n,\beta_n) \leq \rho^\star(\alpha_n,\alpha) \star \rho^\star(\alpha,\beta_n) \leq \rho^\star(\alpha_n,\alpha) \star \rho^\star(\alpha,\beta) \star \rho^\star(\beta,\beta_n). \tag{3.2}$$

Since $\star$ is continuous, passing limit as $n \to \infty$ on the both side of (3.1) and (3.2), we have

$$\rho^\star(\alpha,\beta) \leq \underset{n\to\infty}{lim}\ \rho^\star(\alpha,\alpha_n) \star \underset{n\to\infty}{lim}\ \rho^\star(\alpha_n,\beta_n) \star \underset{n\to\infty}{lim}\ \rho^\star(\beta_n,\beta) \leq 0 \star \underset{n\to\infty}{lim}\ \rho^\star(\alpha_n,\beta_n) \star 0 = \underset{n\to\infty}{lim}\ \rho^\star(\alpha_n,\beta_n)$$

and

$$\underset{n\to\infty}{lim}\ \rho^\star(\alpha_n,\beta_n) \leq \underset{n\to\infty}{lim}\ \rho^\star(\alpha_n,\alpha)\ \star\ \rho^\star(\alpha,\beta)\ \star \underset{n\to\infty}{lim}\ \rho^\star(\beta_n,\beta) \leq 0\ \star\ \rho^\star(\alpha,\beta)\ \star\ 0 = \rho^\star(\alpha,\beta).$$

This implies $\underset{n\to\infty}{lim}\ \rho^\star(\alpha_n,\beta_n) = \rho^\star(\alpha,\beta)$. □

Proposition 3.5. A necessary and sufficient condition that a sequence $\{x_n\} \subseteq \Omega$ converges to x is that every neighborhood $N_r(x)$ of x contains all points of the sequence except perhaps a finite number.

Proof.

The proof is same as the proof in classical metric spaces. □

Theorem 3.6. In a $\star$-metric space $(\Omega, \rho^\star)$, for $A, B \subseteq \Omega$, the results are holds.

i. $A \subseteq \overline{A}$

ii. $A = \overline{A}$ if A is closed

iii. $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$

iv. $\overline{(A \cup B)} = \overline{A} \cup \overline{B}$

v. $\overline{\overline{A}} = \overline{A}$

Proof.

The proofs of *i*, *ii*, *iii*, and *iv* are straightforward. We only prove the last one.

Let $B = \overline{A}$. Then, by *i*, $B \subseteq \overline{B}$ which implies $\overline{A} \subseteq \overline{\overline{A}}$. Moreover, let $p \in \overline{\overline{A}}$ and $K = N_r(p),\ r > 0$. Then, $K \cap \overline{A} \neq \phi$. Hence, there exists $q \in K \cap \overline{A}$.

Let $L = N_{r_1}(q)$, neighborhood of q where $0 < r_1 < \rho^\star(p,q) \dot{\to} r$ and $\eta \in L$. Then,

$$\begin{aligned}\rho^\star(\eta,q) < r_1 &\Rightarrow \rho^\star(\eta,q) < \rho^\star(p,q) \dot{\to} r \\ &\Rightarrow \rho^\star(\eta,q) < inf\{c : c \star \rho^\star(p,q) \geq r\} \\ &\Rightarrow \rho^\star(\eta,q) \star \rho^\star(p,q) < r\end{aligned}$$

Therefore, $\rho^\star(\eta,p) < \rho^\star(\eta,q) \star \rho^\star(q,p) < r$ implies $\eta \in N_r(p)$. Hence, $\eta \in K$ which implies $L \subset K$. Since $q \in \overline{A}$ and L is any neighborhood of q, then $L \cap A \neq \phi$. Thus, $\alpha \in L \cap A$ implies $\alpha \in L \subset K$. Hence, $K \cap A \neq \phi$. Since K is any neighborhood of p, then $p \in \overline{A}$. Therefore, $\overline{\overline{A}} \subset \overline{A}$. Consequently, $\overline{\overline{A}} = \overline{A}$. □

Theorem 3.7. Limit of a convergent sequence in a $\star$-metric space $(\Omega, \rho^\star)$ is unique.

Proof.

Let $\{\alpha_n\} \subseteq \Omega$ be a convergent sequence in $(\Omega, \rho^\star)$ that converge to some $\alpha, \beta \in \Omega$. Let $\varepsilon > 0$ be given. Then, the continuity of '$\star$' ensures the existence of some $\varepsilon_1 > 0$ satisfying $\varepsilon_1 \star \varepsilon_1 < \varepsilon$. For this ε_1, there exists $N_1, N_2 \in \mathbb{N}$ such that

$$\rho^\star(\alpha_n, \alpha) < \varepsilon_1, \quad \text{for all} \quad n \geq N_1 \quad \text{and} \quad \rho^\star(\alpha_n, \beta) < \varepsilon_1, \quad \text{for all} \quad n \geq N_2$$

Let $N = \max\{N_1, N_2\}$. Then,

$$\rho^\star(\alpha, \beta) \leq \rho^\star(\alpha, \alpha_n) \star \rho^\star(\alpha_n, \beta) < \varepsilon_1 \star \varepsilon_1 < \varepsilon, \quad \text{for all} \quad n \geq N$$

Since $\varepsilon > 0$ is chosen arbitrarily, we obtain $\rho^\star(\alpha, \beta) = 0$, i.e, $\alpha = \beta$. □

Theorem 3.8. In a $\star$-metric space, every convergent sequence is Cauchy.

Proof.

Let $\{\alpha_n\}$ be a convergent sequence in a $\star$-metric space $(\Omega, \rho^\star)$ and converges to $\alpha \in \Omega$. Let $\varepsilon > 0$ be given. Then, by the continuity of '$\star$', there exists $\delta > 0$ satisfying $\delta \star \delta < \varepsilon$. For δ, there exists $N \in \mathbb{N}$ such that $\rho^\star(\alpha_n, \alpha) < \delta$ and $\rho^\star(\alpha_m, \alpha) < \delta$, for all $n, m \geq N$. Hence,

$$\rho^\star(\alpha_n, \alpha_m) \leq \rho^\star(\alpha_n, \alpha) \star \rho^\star(\alpha, \alpha_m) < \delta \star \delta < \varepsilon, \quad \text{for all} \quad m, n \geq N$$

Therefore, $\{\alpha_n\}$ is a Cauchy sequence in $(\Omega, \rho^\star)$. □

Remark 3.9. The converse of Theorem 3.8 does not hold in general. Note that every $\star$-metric space is a metric space if in particular we consider the Lukasiewicz t-definer. Since Cauchy sequence in usual metric space is not convergent in general, thus same in $\star$- metric space.

Theorem 3.10. Let $(\Omega, \rho^\star)$ be a $\star$-metric space. If $\{x_n\} \subset \Omega$ is a sequence such that $\{x_n\}$ converges to x, then $\{x_n\}$ is bounded in the sense that, for every fixed element α of Ω, the sequence $\{\rho^\star(x_n, \alpha)\}$ is bounded.

Proof.

From the inequality $(\rho^\star 3)$ gives $\rho^\star(x_n, \alpha) \leq \rho^\star(x_n, x) \star \rho^\star(x, \alpha)$, for all $n \in \mathbb{N}$. This implies

$$\lim_{n\to\infty} \rho^\star(x_n, \alpha) \leq \lim_{n\to\infty} \rho^\star(x_n, x) \star \lim_{n\to\infty} \rho^\star(x, \alpha) = 0 \star \rho^\star(x, \alpha), \quad \text{i.e.,} \quad \lim_{n\to\infty} \rho^\star(x_n, \alpha) \leq \rho^\star(x, \alpha)$$

Again using $(\rho^\star 3)$, $\rho^\star(x, \alpha) \leq \rho^\star(x, x_n) \star \rho^\star(x_n, \alpha)$, for all $n \in \mathbb{N}$ implies

$$\rho^\star(x, \alpha) \leq \lim_{n\to\infty} \rho^\star(x, x_n) \star \lim_{n\to\infty} \rho^\star(x_n, \alpha) = 0 \star \lim_{n\to\infty} \rho^\star(x_n, \alpha), \quad \text{i.e.,} \quad \rho^\star(x, \alpha) \leq \lim_{n\to\infty} \rho^\star(x_n, \alpha)$$

Then, $\{\rho^\star(x_n, \alpha)\}$ converges to $\rho^\star(x, \alpha)$. Thus, $\{\rho^\star(x_n, \alpha)\}$ is a convergent sequence of real numbers. Hence, $\{\rho^\star(x_n, \alpha)\}$ is bounded. □

Proposition 3.11. Let $(\Omega, \rho^\star)$ be a $\star$-metric space and $S \subseteq \Omega$ be bounded. Then, $\delta(S) = \delta(\overline{S})$.

Proof.

For any $S \subseteq \Omega$, $\delta(S) \leq \delta(\overline{S})$. Suppose that $\delta(\overline{S}) > k$. Then, $\sup\{\rho^\star(x, y) : x, y \in \overline{S}\} > k$ implies there exists $x_0, y_0 \in \overline{S}$ such that $\rho^\star(x_0, y_0) > k$. Since $x_0, y_0 \in \bar{S}$, then there exists $\{x_n\}, \{y_n\} \in S$

such that $x_n \to x_0$, $y_n \to y_0$ as $n \to \infty$. Hence, we can write, $\underset{n\to\infty}{lim}\rho^\star(x_n, y_n) = \rho^\star(x_0, y_0)$ which implies $\underset{n\to\infty}{lim}\ \rho^\star(x_n, y_n) > k$. That is, there exists a natural number N_0 such that $\rho^\star(x_n, y_n) > k$, for all $n \geq N_0$. Thus,

$$\begin{aligned} \sup\{\rho^\star(x_n, y_n)\} \geq k, \text{ for all } n \geq N_0 \Rightarrow & \sup_{x,y\in S}\{\rho^\star(x, y)\} \geq k \\ \Rightarrow & \delta(S) \geq k \\ \Rightarrow & \delta(S) \geq \delta(\overline{S}) \end{aligned}$$

Therefore, $\delta(S) = \delta(\overline{S})$. □

We establish the metrizability of $\star$-metric space via 'Niemytski and Wilson's metrization theorem'.

Theorem 3.12. Every $\star$-metric space is metrizable.

Proof.

Suppose $\rho^\star$ be a $\star$-metric on a non-empty set Ω. Then, clearly the function $\rho^\star$ satisfies the conditions i and ii of Theorem 2.15. Next consider a point a, and $\{\eta_n\}$ and $\{\nu_n\}$ be two sequences in $(\Omega, \rho^\star)$ such that $\{\rho^\star(\eta_n, a)\}$ and $\{\rho^\star(\nu_n, \eta_n)\}$ tends to 0 as $n \to \infty$. By the inequality ($\rho^\star3$), we have $\rho^\star(a, \nu_n) \leq \rho^\star(a, \eta_n) \ \star \ \rho^\star(\eta_n, \nu_n)$, for all $n \in \mathbb{N}$.

This implies

$$\underset{n\to\infty}{lim}\ \rho^\star(a, \nu_n) \leq \underset{n\to\infty}{lim}\ \rho^\star(a, \eta_n) \ \star \ \underset{n\to\infty}{lim}\ \rho^\star(\eta_n, \nu_n) = 0 \ \star \ 0, \quad \text{i.e.,} \quad \underset{n\to\infty}{lim}\ \rho^\star(a, \nu_n) = 0$$

Hence, by Niemytski and Wilson's metrization theorem (Theorem 2.15), $(\Omega, \rho^\star)$ is metrizable. □

4. Conclusion

In this article, we extended some well-known results of metric spaces in $\star$-metric space. We defined closed ball, and proved some results related to convergent and Cauchy sequences. All the results were proved with respect to the general t-definer '$\star$'. Lastly, we proved the metrizability of $\star$-metric space using the well-known 'Niemytski and Wilson's metrization theorem', which is simpler and easier to understand than the proof of metrizability by He et al. [18].

There is a vast scope of research on this new structure. The study in this new setting seems very interesting in the presence of a general t-definer '$\star$'. Though $\star$-metric space is metrizable, generalizing metric fixed point theories, especially for non-linear contraction mappings, are the open problems in $\star$-metric spaces. We hope our theoretical results of this manuscript may help researchers for further development in such spaces.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

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