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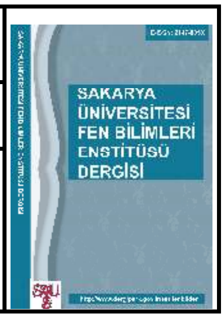

Laplacian spectral properties of nilpotent graphs over ring $\mathbb{Z}_n$

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ABSTRACT

We consider a ring $\boldsymbol{R}$ with unity. The nilpotent graph of $\boldsymbol{R}$ is a graph with vertex set $\boldsymbol{V_N(R)^* = \{0 \neq x \in R: xy \in N(R)\ for\ some\ 0 \neq y \in R\}}$; and two distinct vertices $\boldsymbol{x}$ and $\boldsymbol{y}$ are adjacent iff $\boldsymbol{xy \in N(R)}$, where $\boldsymbol{N(R)}$ is the set of all nilpotent elements of $\boldsymbol{R}$ and it is denoted by $\boldsymbol{\Gamma_N(R)}$. In this paper we study Laplacian spectral properties of the nilpotent graph over the ring $\mathbb{Z}_{\boldsymbol{n}}$.

Keywords: Nilpotent Graph, Laplacian matrix, Spectrum

$\mathbb{Z}_n$ Halkası üzerinde nilpotent grafların laplasyan spektral özellikleri

ÖZ

$\boldsymbol{R}$ birimli bir halka olsun. $\boldsymbol{R}$' nin $\boldsymbol{\Gamma_N(R)}$ ile gösterilen nilpotent grafı, $\boldsymbol{V_N(R)^* = \{0 \neq x \in R: xy \in N(R)}$ $\boldsymbol{baz}$□ $\boldsymbol{0 \neq y \in R}$ $\boldsymbol{için\}}$ noktalar kümesi ve $\boldsymbol{N(R)}$, $\boldsymbol{R}$ halkasının bütün nilpotent elemanlarının kümesi olmak üzere; "iki farklı $\boldsymbol{x}$ ve $\boldsymbol{y}$ noktaları komşudur $\Leftrightarrow$ $\boldsymbol{xy \in N(R)}$" önermesini sağlayan kenarlar kümesinden oluşur. Bu makalede $\mathbb{Z}_{\boldsymbol{n}}$ halkası üzerinde tanımlanan nilpotent grafın Laplasyan spektral özellikleri incelenmektedir.

Anahtar Kelimeler: Nilpotent graf, Laplasyan matris, Spektrum

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1. INTRODUCTION

Let $G = (V, E)$ be a graph with vertex set $V(G) = \{u_1, u_2, u_3, \dots, u_n\}$ and edge set $E(G)$. The degree of a vertex u_i is the number of vertices which adjacent to u_i and denoted by d_{u_i}. The set of neighbors of u_i is denoted by N_{u_i}. Let $\max_{1\le i\le n}\{d_{u_i}\} = \Delta$ and $\min_{1\le i\le n}\{d_{u_i}\} = \delta$. When $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are given graphs on disjoint sets of vertices, their union is the graph $G_1 + G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. The join, $G_1 \vee G_2$, is the graph obtained from $G_1 + G_2$ by adding new edges from each vertex of G_1 to every vertex of G_2. A complete split graph is a graph on n vertices consisting of a clique on $n - \alpha$ vertices and a stable set on the remaining vertices α in which each vertex of the clique is adjacent to each vertex of the independent set and denoted by $CS(n, n - \alpha)$.

The Laplacian matrix of a graph is denoted by $L(G) = D(G) - A(G)$, where $D(G) = diag(d_1, d_2, \dots, d_n)$ and $A(G)$ are the diagonal matrix of vertex degrees and (0,1) adjacency matrix of G, respectively. It is well known that $L(G)$ is positive semidefinite symmetric and singular. Its eigenvalues of $L(G)$ are denoted by μ_i, $1 \le i \le n$ and these eigenvalues can be ordered by $\mu_1 \ge \dots \ge \mu_{n-1} \ge \mu_n = 0$. The Laplacian spectrum of the graph and the Laplacian characteristic polynomial are defined by $S(G) = \{\mu_1(G), \mu_2(G), \dots, \mu_n(G)\}$ and $\Phi_L(G)$, respectively.

Recently, one of the interesting research topic has become the combination some algebraic structures and graph. There can be found a lot of papers on connecting a graph to a ring [1-7]. We consider a ring R with unity. The nilpotent graph of a ring R is introduced in [5]. That is; the nilpotent graph of a ring R is a graph with vertex set $V_N(R)^* = \{0 \neq x \in R : xy \in N(R) \text{ for some } 0 \neq y \in R\}$ such that two distinct vertices x and y are adjacent iff $xy \in N(R)$ and denoted by $\Gamma_N(R)$.

When given the ring $R = \mathbb{Z}_n$, it is well known that it has a nonzero nilpotent element if and only if n is divisible by the square of some primes. From this fact, $\mathbb{Z}_n$ does not have any non-zero nilpotent element when n is prime or $n = p_1 p_2 \dots p_t$. Also, it is easily seen that

$$N(\mathbb{Z}_n) = \{\bar{0}, \bar{p}, 2\bar{p}, \dots, (p^{m-1} - 1)\bar{p}\} \quad (1)$$

when $n = p^m, m > 1$ and

$$N(\mathbb{Z}_n) = \left\{ \begin{array}{l} \overline{(p_1 p_2 \dots p_t)}, 2\overline{(p_1 p_2 \dots p_t)}, \dots \\ , (\prod_{i=1}^{t} p_i^{s_i - 1} - 1)\overline{(p_1 p_2 \dots p_t)} \end{array} \right\} \quad (2)$$

when $n = \prod_{i=1}^{t} p_i^{s_i}$, $t \ge 2$.

In literature, although there are a few studies related to graph parameters such as diameter, girth, etc. on the nilpotent graph, there is no study on the (Laplacian) spectral properties of the nilpotent graphs. In this paper we use the nilpotent graph over the $\mathbb{Z}_n$ ring for all n and examine Laplacian spectral properties of it. We have seen that almost all Laplacian eigenvalues of the graph are exactly the degrees of the vertex (of course, integers).

2. MAIN RESULTS

Lemma 2.1. Let $\mathbb{Z}_n$ be integer ring, where $n = p^m$ and p is a prime number. Then, the vertex set of $\Gamma_N(\mathbb{Z}_n)$ is

$$V_N(\mathbb{Z}_{p^m})^* = \mathbb{Z}_{p^m}^* \quad (3)$$

Moreover, we have $d_i = p^m - 2$ for $i \in N(\mathbb{Z}_{p^m}^*)$ and $d_i = p^{m-1} - 1$ for $i \notin N(\mathbb{Z}_{p^m}^*)$.

Proof. We have $N(\mathbb{Z}_{p^m}) = \{\bar{0}, \bar{p}, \overline{2p}, \overline{3p}, \dots, \overline{(p^{m-1} - 1)p}\}$, where $\bar{p}$ is a prime number such that $\bar{p}^2 | n$.

Let $i \in \mathbb{Z}_{p^m}^*$.

Then, $i \in V_N(\mathbb{Z}_{p^m})^*$ since ij is nilpotent element of $\mathbb{Z}_{p^m}^*$ for all $j \in N(\mathbb{Z}_{p^m}^*)$, we get $i \in V_N(\mathbb{Z}_{p^m})^*$. Hence $\mathbb{Z}_{p^m}^* \subset V_N(\mathbb{Z}_{p^m})^*$. It is clear that $V_N(\mathbb{Z}_{p^m})^* \subset \mathbb{Z}_{p^m}^*$ from the definition of the vertex set of a nilpotent graph. Hence, we get $V_N(\mathbb{Z}_{p^m})^* = \mathbb{Z}_{p^m}^*$.

It is easy to see $d_i = p^m - 2$ for $i \in N(\mathbb{Z}_{p^m}^*)$ by the same arguments above. Now let us show that $d_i = p^{m-1} - 1$ for $i \notin N(\mathbb{Z}_{p^m}^*)$.

Let $i \notin N(\mathbb{Z}_{p^m}^*)$ and any $j \in \mathbb{Z}_{p^m}^*$. There are two cases.

Case 1: $j \in N(\mathbb{Z}_{p^m}^*)$.

In this case, we say that ij is nilpotent $(i \sim j)$ for all $j \in \mathbb{Z}_{p^m}^*$. Then, we have

$N_i = \{j : j \in N(\mathbb{Z}_{p^m}^*)\}$

i.e.

$d_i = p^{m-1} - 1$.

Case 2: $j \notin N(\mathbb{Z}_{p^m}^*)$.

Now we consider that $ij \in N(\mathbb{Z}_{p^m}^*)$. Then, $ij = pk$, where $1 \le k \le p^{m-1} - 1$. Hence $p | ij \Rightarrow p | i$ or $p | j$. If $p | i$, then $i \in N(\mathbb{Z}_{p^m}^*)$. But this is a contradiction that $i \notin N(\mathbb{Z}_{p^m}^*)$. Similarly, if $p | j$,

then we get the same contradiction to $j \notin N(\mathbb{Z}^*_{p^m})$. Therefore, we have $ij \notin N(\mathbb{Z}^*_{p^m})$, that's; $i \nsim j$. Consequently, we get
$N_i = \{k: k \in N(\mathbb{Z}^*_{p^m})\}$
i.e.
$$d_i = p^{m-1} - 1. \quad (4)$$
for all $i \notin N(\mathbb{Z}^*_{p^m})$.

Remark 2.2. By Lemma 2.1, we see that $\Gamma_N(\mathbb{Z}_{p^m})$ has two distinct degrees such that $\Delta = p^m - 2$ and $\delta = p^{m-1} - 1$.

Theorem 2.3. If p is a prime number then
$$S\left(\Gamma_N(\mathbb{Z}_{p^m})\right) = (0, (\delta)^{(\Delta-\delta)}, (\Delta+1)^{\delta}) \quad (5)$$
where $\Delta = p^m - 2$ and $\delta = p^{m-1} - 1$.

Proof. Let $X = \left(x_{\overline{1}}, \dots, x_{\overline{p}}, \dots, x_{\overline{2p}}, \dots, x_{\overline{p^m-1}}\right)^T$ be eigenvector corresponding to any eigenvalue μ for $L(\Gamma_N(\mathbb{Z}_{p^m}))$. We have
$$LX = \mu X \quad (6)$$
i.e.
$$\mu x_i = d_i x_i - \sum_{j: j\sim i} x_j \quad (7)$$
for $i = \overline{1}, \dots, \overline{p}, \dots, \overline{p^m - 1}$. Now we consider vectors $x_i = d_i$ such that $i \in N(\mathbb{Z}_{p^m}{}^*)$ and $x_j = 1$ for all other $j \in \mathbb{Z}_{p^m}{}^*$ and hence we get $(p^{m-1} - 1)$-vectors as
$$(-1, \dots, -1, \underbrace{d_{\overline{p}}}_{(p)^{th}}, -1, \dots, -1)^T,$$
$$(-1, \dots, -1, \underbrace{d_{\overline{2p}}}_{(2p)^{th}}, -1, \dots, -1)^T, \dots,$$
$$(-1, \dots, -1, \underbrace{d_{\overline{p^{m-1}-1}}}_{(p^{m-1}-1)^{th}}, -1, \dots, -1)^T.$$
These vectors are the characteristic eigenvectors corresponding eigenvalue $p^m - 1$ of $L\left(\Gamma_N(\mathbb{Z}_{p^m})\right)$ and each provide the equality in (7). Since all vectors above are linear independent, the algebraic multiplicity of the eigenvalue $p^m - 1$ is $(p^{m-1} - 1)$. By Remark 2.2, we get that $\Delta - \delta + 1$ is the eigenvalue with $\delta - 1$ multiplicity.

Similarly, we consider vectors as $x_i = 1$ such that $i \notin N(\mathbb{Z}_{p^m})$ and $x_J = -1$ for only one $j, j \nsim i$. We get
$$\sum_{j, j\nsim i} 1 = p^m - p^{m-1} - 1$$
Therefore we choose $(p^m - p^{m-1} - 1) -$ linear independent vectors as
$$\left(1, 0, \dots, 0, \underbrace{-1}_{(j)^{th}}, 0, \dots, 0\right)^T,$$
$$\dots, \left(0, \dots, 0, \underbrace{1}_{(i)^{th}}, 0, \dots, 0, \underbrace{-1}_{(j)^{th}}, 0, \dots, 0\right)^T \quad (8)$$
Each vectors provide the equality in (7) and they are also the characteristic eigenvectors corresponding eigenvalue $(p^{m-1} - 1)$ of $L\left(\Gamma_N(\mathbb{Z}_{p^m})\right)$. By again Remark 2.2, we get that $\Delta + 1$ is the eigenvalue with $\Delta - \delta + 1$ multiplicity.

On the other hand, it is well known that vector $e = (1, 1, \dots, 1)$ is an eigenvector corresponding eigenvalue 0. Hence we get the result of the theorem.

Lemma 2.4. [8] Let G be a connected graph on n vertices, then G has three distinct eigenvalues $0, r, \dots, r, n, \dots, n$ and r, s with multiplicities p and q if and only if

i. $(n - r - 1)|p$,
ii. $q + 1 \geq \frac{p}{n-r-1}$,
iii. $G = K_{q+1-\frac{p}{n-r-1}} \vee G_1 \vee \dots \vee G_1$, where G_1 is $n - r$ isolated vertices with multiplicity $\frac{p}{n-r-1}$.

Remark 2.5. The graph $\mathbf{\Gamma_N}(\mathbb{Z}_{p^m})$ has three distinct eigenvalues. Hence, by Lemma 2.4., we get

$\mathbf{\Gamma_N}(\mathbb{Z}_{p^m}) \cong \boldsymbol{CS}(\boldsymbol{p^m - 1}, \boldsymbol{p^{m-1} - 1})$.

Lemma 2.6. Let $\Gamma_N(\mathbb{Z}_n)$be graph , where $n = \prod_{i=1}^{t} p_i{}^{s_i}$, $t \geq 2$.

i. If $s_i = 1$ for each i, then
$$V_N(\mathbb{Z}_n)^* = \bigcup_{i\in I} S_{p_i} \quad (9)$$
where $S_{p_i} = \{p_i k: 1 \leq k \leq \frac{n}{p_i} - 1\}$ and $I = \{1, 2, \dots, t\}$.

ii. If $s_i \geq 2$ for at least i, then
$$V_N(\mathbb{Z}_n)^* = \mathbb{Z}_n{}^* \quad (10)$$

Proof.

i. Let $s_i = 1$ for each i. In this case, there is no non zero nilpotent element of $\mathbb{Z}_n$. That's; $N(\mathbb{Z}_n) = \{\overline{0}\}$. Let $0 \neq x \in V_N(\mathbb{Z}_n)^*$. From the definition of the vertices set for nilpotent graphs, there is a non-zero element $\overline{y} \in \mathbb{Z}_n$ such that $\overline{xy} = \overline{0}$.
i.e.
$$\overline{xy} | p_1 p_2 \dots p_t \Longrightarrow \overline{x} | p_1 p_2 \dots p_t \wedge \overline{y} | p_1 p_2 \dots p_t \quad (11)$$
By (11) , we get $\overline{x}|p_i$ and hence $V_N(\mathbb{Z}_n)^* \subset \bigcup_{i\in I} S_{p_i}$. It is also clear that $\bigcup_{i\in I} S_{p_i} \subset V_N(\mathbb{Z}_n)^*$. From these inclusions, we have the required result.

ii. If $s_i \geq 2$ for at least i, then we have
$N(\mathbb{Z}_n{}^*) = \{\overline{(p_1 p_2 \dots p_t)}, \overline{2(p_1 p_2 \dots p_t)}, \dots, \overline{(p_1{}^{s_1-1} p_2{}^{s_2-1} \dots p_t{}^{s_t-1} - 1)(p_1 p_2 \dots p_t)}\}$
by (2). Then, it is easily shown that each non-zero class of $\mathbb{Z}_n$ is adjacent to any nilpotent element. This completes the result.

Remark 2.7. From Lemma 2.6. (i)., we can easily see that the number of vertices of $\Gamma_N(\mathbb{Z}_n)$ is $n - 1 - \tau(n)$, where $\tau(n)$ is the number of positive divisors of n. Particularly, $\Gamma_N(\mathbb{Z}_{pq}) \cong K_{p-1,q-1}$ for p, q primes and $\Gamma_N(\mathbb{Z}_p) \cong K_0$, where K_0 is a null graph.

Lemma 2.8. Let's consider the graph $\Gamma_N(\mathbb{Z}_n)$, where $n = \prod_{i=1}^{t} {p_i}^{s_i}, t \geq 2$.

i. $d_i = n - 2$ for all $i \in N(\mathbb{Z}_n)$

ii. If $\prod_{m=1}^{r} p_{l_m} | i$ for $i \in V_N(\mathbb{Z}_n)^*$ such that $l_1, \ldots, l_r \in A$ for $1 \leq r \leq k$ and $A = \{1,2,\ldots,k\}$, then we get

$$N_i = \left\{z, 2z, \ldots, \left(\frac{n}{z} - 1\right) z\right\} \quad (12)$$

i.e.

$$d_i = \frac{n}{z} - 1 \quad (13)$$

where $z = \prod_j p_j$ for every $j \in A - \{l_1, \ldots, l_r\}$; d_i and N_i are the degree of vertex i and the set of neighbors of i, respectively.

iii. If $(i, p_k) = 1$ for all $1 \leq k \leq t$, then we get

$$N_i = N({\mathbb{Z}_n}^*) = \{\overline{(p_1 p_2 \ldots p_t)}, 2\overline{(p_1 p_2 \ldots p_t)}, \ldots, (\prod_{i=1}^{t} {p_i}^{s_i - 1} - 1)(\overline{(p_1 p_2 \ldots p_t)}\} \quad (14)$$

i.e.

$$d_i = \prod_{i=1}^{t} {p_i}^{s_i - 1} - 1 \quad (15)$$

Proof.

i. By similar method in Lemma 2.1., for all $j \in V_N({\mathbb{Z}_n}^*)$, ij is also nilpotent element of ${\mathbb{Z}_n}^*$ such that $i \in N(\mathbb{Z}_n)$. Then; $i \sim j$ for every j. That is;

$$N_i = \{j : j \in {\mathbb{Z}_n}^*\} \quad (16)$$

i.e.

$$d_i = n - 2 \quad (17)$$

ii. It is clear that $i \sim z$ under the mentioned conditions.

iii. If $(i, p_k) = 1$, $x \sim i$ for any $x \in {\mathbb{Z}_n}^*$ if and only if $x \in N({\mathbb{Z}_n}^*)$. From this fact, we get the result directly.

Theorem 2.9. Let $\mathbb{Z}_n$ be a ring , where $n = {p_1}^{s_1} {p_2}^{s_2} \ldots {p_t}^{s_t}$. Then we have

$$\Phi_{L(\Gamma_N(\mathbb{Z}_n))}(x) = x\left(x - d_q\right)^{\phi(n)} (x - n + 2)^{\prod_{i=1}^{t} {p_i}^{s_i - 1} - 1} \alpha_1(x)\alpha_2(x) \ldots \alpha_{t-1}(x) f(x)$$

where

$\alpha_1(x) = \prod_{k=1}^{t} (x - d_{p_k})^{W_{p_k} - 1}$,

$\alpha_2(x) = \prod_{k=1, k \neq l}^{t} (x - d_{p_k p_l})^{W_{p_k p_l} - 1}$,

…,

$\alpha_{t-1}(x) = \prod_{k=1, k \neq l \neq \ldots}^{t} (x - d_{p_k \ldots p_{t-1}})^{W_{p_k \ldots p_{t-1}} - 1}$

and $f(x)$ is any polynomial such that W_u is the cardinality of the set $\{v \in {\mathbb{Z}_n}^* : d_u = d_v\}$; $(q, p_i) = 1$ for $1 \leq i \leq t$ and ϕ is Euler's totient function.

Proof.

We consider at least $s_i \geq 2$. Let X be eigenvector corresponding to μ for $L(\Gamma_N(\mathbb{Z}_{p^m}))$. We have

$LX = \mu X$

i.e.

$$\mu x_i = d_i x_i - \sum_{j : j \sim i} x_j \quad (19)$$

for $i \in V_N(\mathbb{Z}_n)^*$.

There are three cases for vertex i from Lemma 2.8. Let $i \in N(\mathbb{Z}_n)$. We consider $x_i = -1$ and $x_j = 1$ for one $j \in N(\mathbb{Z}_n)$. Hence these $({p_1}^{s_1 - 1} {p_2}^{s_2 - 1} \ldots {p_t}^{s_t - 1} - 1)$-vectors are the characteristic vectors corresponding to eigenvalue $d_i = n - 2$ (by Lemma 2.8.(i).) and provide equality in (19).

Let $(i, p_k) = 1$, $1 \leq k \leq t$. Then we get $d_i = {p_1}^{s_1 - 1} {p_2}^{s_2 - 1} \ldots {p_t}^{s_t - 1} - 1$ from Lemma 2.7.iii.. Similarly, if we consider vectors $x_i = 1$ and $x_j = -1$ for one $j \in V_N(\mathbb{Z}_n)^*$, such that $d_i = d_j$ and $(j, p_k) = 1$. Since $n = {p_1}^{s_1} {p_2}^{s_2} \ldots {p_t}^{s_t}$, hence there are $\phi(n)$-class which are relative prime.

The other arguments can be shown by the same methods.

APPENDICES

Now we give Laplacian spectrum of the nilpotent graph over ring $\mathbb{Z}_n$, for $4 \leq n \leq 100$. As seen below table, almost all Laplacian eigenvalues of the graphs are the integers.

n	Laplacian Spectrum of $\Gamma_N(\mathbb{Z}_n)$
4	$\{0, 1, 3\}$
8	$\{0, 3^{(3)}, 7^{(3)}\}$
9	$\{0, 2^{(5)}, 8^{(2)}\}$
12	$\{0, 1^{(4)}, 5^{(1)}, 7^{(1)}, 11^{(1)}\}$
16	$\{0, 7^{(7)}, 15^{(7)}\}$
18	$\{0, 2^{(6)}, 5^{(5)}, 8^{(2)}, 11^{(1)}, 17^{(2)}\}$
20	$\{0, 1^{(8)}, 3^{(7)}, 9^{(1)}, 11^{(1)}, 19^{(1)}\}$
24	$\{0, 3^{(8)}, 7^{(7)}, 11^{(3)}, 15^{(1)}, 23^{(3)}\}$
25	$\{0, 4^{(19)}, 24^{(4)}\}$
27	$\{0, 8^{(17)}, 26^{(8)}\}$
28	$\{0, 1^{(12)}, 3^{(11)}, 13^{(1)}, 15^{(1)}, 27^{(1)}\}$
32	$\{0, 15^{(15)}, 31^{(15)}\}$
36	$\{0, 5^{(12)}, 11^{(11)}, 17^{(5)}, 15^{(1)}, 23^{(1)}, 35^{(5)}\}$
40	$\{0, 3^{(16)}, 7^{(15)}, 19^{(3)}, 23^{(1)}, 39^{(3)}\}$
44	$\{0, 1^{(20)}, 3^{(19)}, 21^{(1)}, 23^{(1)}, 43^{(1)}\}$
45	$\{0, 2^{(24)}, 8^{(11)}, 14^{(5)}, 20^{(1)}, 44^{(2)}\}$
48	$\{0, 7^{(16)}, 15^{(15)}, 23^{(7)}, 31^{(1)}, 47^{(7)}\}$
49	$\{0, 6^{(41)}, 48^{(6)}\}$
n	Laplacian Spectrum of $\Gamma_N(\mathbb{Z}_n)$
50	$\{0, 4^{(20)}, 9^{(19)}, 24^{(4)}, 29^{(1)}, 49^{(4)}\}$
54	$\{0, 8^{(18)}, 17^{(17)}, 26^{(8)}, 35^{(1)}, 53^{(8)}\}$
56	$\{0, 3^{(24)}, 7^{(23)}, 27^{(3)}, 31^{(1)}, 55^{(3)}\}$
60	$\{0, 1^{(16)}, 2.2, 3^{(15)}, 4.4, 5^{(7)}, 9^{(3)}, 11^{(7)}, 13.3, 19^{(3)}, 23.6, 29^{(1)}, 31.26, 59^{(1)}\}$
63	$\{0, 2^{(36)}, 8^{(17)}, 20^{(5)}, 26^{(1)}, 62^{(2)}\}$
72	$\{0, 11^{(24)}, 23^{(23)}, 35^{(11)}, 47^{(1)}, 71^{(11)}\}$
75	$\{0, 4^{(40)}, 14^{(19)}, 24^{(9)}, 34^{(1)}, 74^{(4)}\}$

80	$\{0, 7^{(31)}, 15^{(31)}, 39^{(7)}, 47^{(1)}, 78^{(7)}\}$
81	$\{0, 25^{(55)}, 80^{(24)}\}$
88	$\{0, 3^{(37)}, 7^{(39)}, 43^{(3)}, 47^{(1)}, 87^{(3)}\}$
90	$\{0, 2^{(24)}, 3.94, 5^{(23)}, 7.16, 8^{(11)}, 14^{(5)}, 17^{(11)}, 20.45, 29^{(5)}, 36.04, 44^{(2)}, 47.39, 89^{(2)}\}$
98	$\{0, 6^{(42)}, 13^{(41)}, 48^{(6)}, 55^{(1)}, 97^{(6)}\}$
99	$\{0, 2^{(60)}, 8^{(29)}, 32^{(5)}, 38^{(1)}, 98^{(2)}\}$
100	$\{0, 9^{(40)}, 19^{(39)}, 49^{(9)}, 59^{(1)}, 99^{(9)}\}$

REFERENCES

[1] D.F. Anderson, P.S. Livingston, "The zero-divisor graph of a commutative ring", J. Algebra, vol. 217, pp. 434-447, 1999.

[2] D.F. Anderson, A. Badawi, "The total graph of a commutative ring", J. Algebra, vol. 320, pp. 2706-2719, 2008.

[3] M.J.Nikmehr, S.Khojasteh, "On the nilpotent graph of a ring", Turkish Journal of Mathematics, vol. 37, pp. 553-559, 2013.

[4] P. Sharma, A.Sharma, R.K. Vats, "Analysis of Adjacency Matrix and Neighborhood Associated with Zero Divisor Graph of Finite Commutative Rings", International Journalof Computer Applications, vol. 13 pp. 38-42, 2011.

[5] P.W. Chen, A kind of graph structure of rings, Algebra Colloq., vol. 10 (2),pp. 229-238, 2003.

[6] S. Akbari, A. Mohammadian, "Zero-divisor graphs of non-commutative rings", J. Algebra, vol. 296, pp. 462-479, 2006.

[7] S.Akbari, A. Mohammadian, "On zero-divisor graphs of finite rings", J. Algebra, vol. 13, pp. 168-184, 2007.

[8] S. J. Kirkland, "Completion of Laplacian integral graphs via edge additions", Discrete Math, vol. 295, pp. 75-90, 2005.