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Research Article

The Class of Demi-Strongly Order Bounded Operators

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ABSTRACT

In this paper, we introduce the class of demi-strongly order bounded operators on a Riesz space generalization of strongly order bounded operators. Let M be a Riesz space, an operator H from M into M is said to be a demi-strongly order bounded operator if for every net $\{u_\alpha\}$ in M^+ whenever $0 \leq u_\alpha \uparrow \leq u''$, u'' in $M^{\sim\sim}$ and $\{u_\alpha - H(u_\alpha)\}$ is order bounded in M, then $\{u_\alpha\}$ is order bounded in M. We obtain a characterization of the b-property by the term of demi-strongly order bounded operators. In addition, we study the relationship between strongly order bounded operators and demi-strongly order bounded operators. Finally, we also investigate some properties of the class of demi-strongly order bounded operators.

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1. Introduction

In order to give our results, we need the following definitions and notations: Let M be a Riesz space. The order dual of M is denoted by $M^{\sim}$, and the order bidual of M is denoted by $M^{\sim\sim}$. A subset A of a Riesz space M is said to be order bounded if there exist x in M^+ such that $|y| \leq x$ for all $y \in A$ [1]. The canonical embedding $Q_M: M \to M^{\sim\sim}$ is defined by $Q_M(x) = \hat{x}$, $\hat{x}(f) = f(x)$, $f \in M^{\sim}$, $x \in M$ [1]. Let A be a subset of the Riesz space M, if $Q_M(A)$ is order bounded in $M^{\sim\sim}$, then A is called b-order bounded in M [2]. A Riesz space M is said to have b-property if every b-order bounded subset of M is order bounded in M [3]. An operator $H: M \to M$ between two Riesz spaces is said to be regular if it can be written as a difference of two positive operators [1].

Alpay S. and Altın B. introduced the strongly order bounded operators and recall from [4], let M, N be two Riesz spaces and H be an operator from M into N. H is said to be a strongly order bounded operator if for each net $0 \leq u_\alpha \uparrow u''$ for some u'' in $M^{\sim\sim}$, then $H(u_\alpha)$ is order bounded in N; in the other words, an operator $H: M \to N$ is a strongly order bounded if it maps b-bounded subsets of M into order bounded subsets of N, and the class of all strongly order bounded operators will be denoted by $L_{sb}(M,N)$, the operator H is said to be the pre-regular operator if $Q_N H$ is order bounded operator from M into $N^{\sim\sim}$ [5]. An operator H from M into N is pre-regular if and only if $H(A)$ is b-order bounded subset of N for each b-order bounded subset A of M [3]. The class of all pre-regular operators from M into N and on M are denoted by $L_{pr}(M,N)$ and $L_{pr}(M)$ respectively. The class of all linear operator on M is denoted by $L(M)$.

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The demi notation was used firstly in [6] by Petryshyn. The class of weakly demicompact operators was studied in [7]. After that, the class of demi Dunford-Pettis operators was introduced in [8], and the class of order weakly demicompact operators was studied in [9]. More recently, another study on the demi class was studied by Benkhaled H. in [10].

In this study, we will introduce the class of demi-strongly order bounded operators which are a generalization of strongly order bounded operators given by [4], and we also investigate some properties of the class of demi-strongly order bounded operators.

In addition, we assume all Riesz spaces in this note have sperating order duals. For all other undefined terms and notations, we will adhere to the conventions in [1].

2. Main Results

Let's start giving the definition of the demi-strongly order bounded operator.

Definition 1 Let M be a Riesz space. An operator H from M into M is said to be a demi-strongly order bounded operator $(\mathrm{d-sobo})$ if for every net $\{u_\alpha\}$ in M^+ whenever $0 \leq u_\alpha \uparrow \leq u''$, u'' in $M^{\sim\sim}$ and $\{u_\alpha - H(u_\alpha)\}$ is order bounded in M, then $\{u_\alpha\}$ is order bounded in M. The class of all demi-strongly order bounded operators on M will be denoted by $DL_{sb}(M)$.

Example 1 Let M be a Riesz space, then for all $b \neq 1$, bI belongs to $DL_{sb}(M)$.

Assume that $b \neq 1$, $\{u_\alpha\}$ is a net in M^+, $0 \leq u_\alpha \uparrow \leq u''$, u'' in $M^{\sim\sim}$ and $\{u_\alpha - bI(u_\alpha)\}$ is order bounded in M. Therefore, there exists y in M such that

$$|(u_\alpha - bI(u_\alpha))| \leq y$$

and

$0 \leq |1-b| u_\alpha \leq y$

It follows that

$$0 \leq u_\alpha \leq \frac{y}{|1-b|}.$$

Thus, $\{u_\alpha\}$ is order bounded in M, and bI is a d-sobo.

The following theorem states that $DL_{sb}(M)$ includes $L_{sb}(M)$ for each Riesz space M.

Theorem 1 Every strongly order bounded operator is a d-sobo.

Proof. Let M be a Riesz space, $H \in L_{sb}(M)$, $\{u_\alpha\}$ be a net in M^+, $0 \leq u_\alpha \uparrow \leq u''$, $u'' \in M^{\sim\sim}$ and $\{u_\alpha - H(u_\alpha)\}$ be order bounded in M. Hence, there exists y_1 in M such that

$$|(u_\alpha - H(u_\alpha))| \leq y_1$$

for all α. Since H is in $L_{sb}(M)$, there exists y_2 in M such that

$$|H(u_\alpha)| \leq y_2$$

for all α. We can write it as follows:

$$0 \leq u_\alpha = u_\alpha - H(u_\alpha) + H(u_\alpha)$$

$$\leq |(u_\alpha - H(u_\alpha))| + |H(u_\alpha)|$$

$$\leq y_1 + y_2.$$

Hence, we conclude that

$$0 \leq u_\alpha \leq y_1 + y_2.$$

Therefore, $\{u_\alpha\}$ is order bounded in M, and H is a d-sobo.

The following example shows the converse of the above theorem is not generally true.

Example 2 Let an operator $H: c_0 \to c_0$ and $H = \frac{1}{2} I$ H is a d-sobo. Consider the sequence $\{u_n\}$, its first n terms are two, others are zero, and $u = (2,2,\ldots)$. It is satisfied $0 \leq u_n \uparrow \leq u$ in $c_0^{\sim\sim} = l_\infty$, but $H(u_n) = \frac{1}{2}(u_n)$ is not order bounded in c_0, hence H is not a strongly order bounded operator.

The next example shows that the inclusion $DL_{sb}(M) \subseteq L(M)$ can be proper.

Example 3 Let e_n be a sequence nth term is one, the other terms are zero, $k \in \mathbb{N}$, and $H_k: c_0 \to c_0$ be an operator defined by $H_k(x) = \sum_{i=1}^{k} x_i e_i$ for each $x = (x_i) \in c_0$ where c_0 is the set of all sequences of $\mathbb{R}$ which converge to zero. H_k is the strongly order bounded operator, consider the sequence $\{u_n\}$, its first n terms are one, others are zero, and $u = (1,1,\ldots)$. Then, we have $0 \le u_n \uparrow \le u$ in $c_0^{\sim\sim} = l_\infty$. Define $S_k = I + H_k$ for each $k \in \mathbb{N}$.

$$|I(u_n) - S_k(u_n)| = |I(u_n) - I(u_n) - H_k(u_n)| = |H_k(u_n)|.$$

$H_k(u_n)$ is order bounded in c_0, but $\{u_n\}$ is not order bounded in c_0. Hence, S_k is not in $DL_{sb}(c_0)$ for each $k \in \mathbb{N}$.

Theorem 2 Let M be a Riesz space, $P: M \to M$ be a d-sobo, and S be in $L_{sb}(M)$, then $P + S$ is a d-sobo.

Proof. Let $\{u_\alpha\}$ be a net in M^+, $0 \le u_\alpha \uparrow \le u''$, u'' in $M^{\sim\sim}$ and $\{u_\alpha - (P+S)(u_\alpha)\}$ be order bounded in M. Hence, there exists y_1 in M such that

$$|(u_\alpha - (P+S)(u_\alpha))| \le y_1$$

for all α. Since S is in $L_{sb}(M)$, there exists y_2 in M such that

$$|S(u_\alpha)| \le y_2$$

for all α. We can write also

$$|u_\alpha - P(u_\alpha)| = |(u_\alpha - P(u_\alpha) - S(u_\alpha) + S(u_\alpha))|$$

$$\le |(u_\alpha - (P+S)(u_\alpha)| + |S(u_\alpha)|$$

$$\le y_1 + y_2.$$

We obtain that $\{u_\alpha - P(u_\alpha)\}$ is order bounded in M. Since P belongs to $DL_{sb}(M)$, then $\{u_\alpha\}$ is order bounded in M. Hence, $P + S$ is a d-sobo. However, as the next example shows, $DL_{sb}(M)$ is not a vector space in general.

Example 4 Let H be an operator on c_0, defined as $H = \frac{1}{2}I$. H is a d-sobo, but $H + H = I$ is not a d-sobo.
The identity operator is not a d-sobo in general. For example, consider the identity operator I on $M = c_0$ and the sequence $\{u_n\}$, its first n terms are one, others are zero. It is obvious that $0 \le u_n \uparrow \le u = (1,1,\ldots)$ in $c_0^{\sim\sim} = l_\infty$, and $|I(u_n) - I(u_n)| = |(0)|$ is order bounded in M, but $\{u_n\}$ is not order bounded in M. Hence, I is not a d-sobo on M.

Recall that an operator $T: M \to N$ between two Riesz spaces is said to be order bounded if it maps order bounded subsets of M into order bounded subsets of N, and the class of all order bounded operators from M into N will be denoted by $L_b(M,N)$ [1].

The following theorem gives us a characterization of the b-property.

Let M and N be two normed Riesz spaces and $K = M \oplus N = \{(a,b): a \in M, b \in N\}$ if K is equipped with the coordinatewise order that is $(a_1,b_1) \le (a_2,b_2) \Leftrightarrow a_1 \le a_2$ and $b_1 \le b_2$ for each $(a_1,b_1),(a_2,b_2) \in K$ and the norm $\| (a,b) \|_K = \| a \|_M + \| b \|_N$, then K will be the normed Riesz space [1].

Theorem 3 Let M and N be two Riesz spaces. Then the following operators are d-sobo.

(i) All operators R on M which $(I-R)^{-1}$ exists and is order bounded.

(ii) (S_β) is the class of operator on K, defined by the matrix $\begin{pmatrix} 0 & 0 \\ R & \beta I \end{pmatrix}$ on K for the every order bounded operator R from M into N where $K = M \oplus N$ and β is a real number different to one.

Proof. (i) Assume that $\{u_\alpha\}$ is a net in M^+, $0 \le u_\alpha \uparrow\le u''$, u'' in $M^{\sim\sim}$ and $\{u_\alpha - R(u_\alpha)\}$ be order bounded in M. Thus, there exists y_0 in M such that

$$\{u_\alpha - R(u_\alpha) : \alpha \in \Lambda\} \subseteq [-y_0, y_0]$$

for all α. Since $[-y_0, y_0]$ is order bounded in M and $(I-R)^{-1}$ is order bounded, then there exists x_0 in M such that
$(I-R)^{-1}([-y_0, y_0]) \subseteq [-x_0, x_0]$.

Now, we can write

$$u_\alpha = (I-R)^{-1}(I-R)(u_\alpha)$$
$$\subseteq (I-R)^{-1}([-y_0, y_0])$$
$$\subseteq [-x_0, x_0].$$

Hence, $\{u_\alpha\}$ is order bounded; consequently, R belongs to $DL_{sb}(M)$.

(ii) Let $\{z_\alpha\}$ be a net in K, $0 \leq z_\alpha \uparrow \leq z''$, z'' in $K^{\sim\sim}$, $z_\alpha = (x_\alpha, y_\alpha) \in K$, $\{(I-S_\beta)(z_\alpha)\} \subseteq M \oplus N$ be order bounded. It will be shown that $\{z_\alpha\} = \{(x_\alpha, y_\alpha)\}$is order bounded in K.

$$\left(z_\alpha - S_\beta(z_\alpha)\right) = \big((x_\alpha, y_\alpha) - (0, R(x_\alpha) + \beta y_\alpha)\big)$$

$$= (x_\alpha,\, y_\alpha - R(x_\alpha) - \beta y_\alpha)$$

$$= (x_\alpha, (1-\beta)y_\alpha - R(x_\alpha))$$

Since $\{(I-S_\beta)(z_\alpha)\} \subseteq M \oplus N$ is order bounded and from the above equality, then $\{x_\alpha\}$ and $\{(1-\beta)y_\alpha - R(x_\alpha)\}$ are order bounded in M and N respectively. Hence, there exists y_1 in N such that
$|(1-\beta)y_\alpha - R(x_\alpha)| \leq y_1$
for all α. Since R is order bounded, then $\{R(x_\alpha)\}$ is order bounded in N. There exists y_2 in N such that

$$|R(x_\alpha)| \leq y_2$$

for all α. We can write also

$$|(1-\beta)y_\alpha| = |(1-\beta)y_\alpha - R(x_\alpha) + R(x_\alpha)|$$

$$\leq |(1-\beta)y_\alpha - R(x_\alpha)| + |R(x_\alpha)|.$$

Hence, we obtain that
$$|y_\alpha| \leq \frac{1}{1-\beta}(y_1 + y_2).$$

Therefore, $\{y_\alpha\}$ is order bounded in N, so $\{z_\alpha] = \{(x_\alpha, y_\alpha)\}$ is order bounded in K.

The following example gives that Theorem 3 is not valid in case $\beta = 1$.

Now, it is clear that $K = c_0 \oplus c_0$ is a Banach lattice with the coordinate wise order and the norm $\|x\| = \|a\| + \|b\|$ for each $x = (a, b) \in K$ [1].

Example 5 Let an operator $\mathrm{H}: \mathrm{c}_0 \to \mathrm{c}_0$, $\mathrm{K} = \mathrm{c}_0 \oplus \mathrm{c}_0$ equipped with coordinate wise order and operator S is defined by $\begin{pmatrix} 0 & 0 \\ \mathrm{H} & \mathrm{I} \end{pmatrix}$. S does not belong to $\mathrm{DL_{sb}}(K)$. Indeed, consider the sequence $\mathrm{z_n} = (0, \mathrm{u_n})$ and $\mathrm{u_n}$ the first n term equals one, and others are zero. It is clear that $0 \leq \mathrm{z_n} \uparrow$ and $\|\mathrm{z_n}\| = 1$ for each $\mathrm{n} \in \mathbb{N}$. Since $\{\mathrm{z_n}\}$ is increasing, norm bounded and K is a Banach lattice $\{\mathrm{z_n}\}$ is b-order bounded in K from Corollary 3.4 in [11]. Hence, there exists u'' in K'' such that $0 \leq \mathrm{z_n} \uparrow \leq \mathrm{u}''$ where K'' is the second norm dual of K.

It is clear that

$$(\mathrm{I}-\mathrm{S})(\mathrm{z_n}) = (\mathrm{z_n} - \mathrm{S}(\mathrm{z_n}))$$

$$= [(0, \mathrm{u_n}) - (0, \mathrm{I}(\mathrm{u_n}))] = (0,0).$$

Therefore, $\{(\mathrm{I}-\mathrm{S})(\mathrm{z_n})\}$ is order bounded in K, but S is not a d-sobo, since $\{\mathrm{z_n}\}$ is not order bounded in K.

By the term of d-sobo of identity operator it is obtain that a characterization of the b-property.

Theorem 4 Let M be a Riesz space, then the following assertions are equivalent.

(i) All pre-regular operators $\mathrm{H}: \mathrm{M} \to \mathrm{M}$ are d-sobo.

(ii) $\mathrm{I}: \mathrm{M} \to \mathrm{M}$ is d-sobo.

(iii) M has b-property.

Proof. (i) $\Rightarrow$ (ii) It is clear.

(ii) $\Rightarrow$ (iii) Assume that $\{\mathrm{u}_\alpha\}$ is a net in M^+, $0 \leq \mathrm{u}_\alpha \uparrow \leq \mathrm{u}''$, u'' in $M^{\sim\sim}$. Since $|(\mathrm{I}-\mathrm{I})(\mathrm{u}_\alpha)| = 0$ is order bounded in M and

$I \in DL_{sb}(M)$, then $\{u_\alpha\}$ is order bounded in M. Hence, M has b-property.

(iii) ⇒ (i) Let H be a pre-regular operator on M, $\{u_\alpha\}$ be a net M^+, $0 \le u_\alpha \uparrow \le u''$, u'' in $M^{\sim\sim}$. $H(u_\alpha)$ is b-order bounded in M from Proposition 1 in [3]. Since M has b-property, then $H(u_\alpha)$ is order bounded in M, and we get H is in $L_{sb}(M)$. Hence, H is d-sobo from Theorem 1.

Let H, S be two operators on the Riesz space M and $0 \le S \le H$. If H is a d-sobo, then S is not a d-sobo in general. Hence, the class of demi-strongly order bounded operators does not satisfy the domination property.

Example 6 Let H, S be two operators on c_0, $S = I$ and $H = 2I$. It holds $0 \le S \le H$ and H is a d-sobo, but S is not a d-sobo.

Domination property for d-sobo is satisfied under certain condition as follows.

Theorem 5 Let S, H be two positive operators on the Riesz space M and $0 \le S \le H \le I$. If H is a d-sobo, then S is also a d-sobo.

Proof. Let $\{u_\alpha\}$ be a net in M^+, $0 \le u_\alpha \uparrow \le u''$, u'' in $M^{\sim\sim}$ and $\{u_\alpha - S(u_\alpha)\}$ be order bounded in M. Hence, there exists y_1 in M such that

$$0 \le u_\alpha - S(u_\alpha) \le y_1$$

for all α. It is clear that

$$0 \le (I - H)(u_\alpha) \le (I - S)(u_\alpha) \le y_1.$$

Hence, $\{(I - H)(u_\alpha)\}$ is order bounded in M. Since H belongs to $DL_{sb}(M)$, then $\{u_\alpha\}$ is order bounded in M; consequently, S belongs to $DL_{sb}(M)$.

Theorem 6 Let M be a Riesz space, S and H be two operators on M and $I \le S \le H$. If S belongs to $DL_{sb}(M)$, then H belongs to $DL_{sb}(M)$.

Proof. Assume that $\{u_\alpha\}$ is a net in M^+, $0 \le u_\alpha \uparrow \le u''$, u'' in $M^{\sim\sim}$and $\{(H - I)(u_\alpha)\}$ is order bounded in M. Hence, there exists y_1 in M such that

$$(H - I)(u_\alpha) \le y_1$$

for all α. It implies that
$0 \le (S - I)(u_\alpha) \le (H - I)(u_\alpha) \le y_1$.

Thus, $\{(S - I)(u_\alpha)\}$ is order bounded in M. Since S is a d-sobo, then $\{u_\alpha\}$ is order bounded in M, so H is in $DL_{sb}(M)$.

Theorem 7 Let M be a Riesz space, $P, S, N: M \to M$ be three operators, and $N \le S \le P \le I + N$. If N is in $L_{sb}(M)$ and P is in $DL_{sb}(M)$, then S is in $DL_{sb}(M)$.

Proof. By the hypothesis $0 \le S - N \le P - N \le I$. Assume that N is in $L_{sb}(M)$, and P is in $DL_{sb}(M)$. We have $P - N$ is a d-sobo from Theorem 2 and $S - N$ is a d-sobo from Theorem 5. Since N is in $L_{sb}(M)$, then by Theorem 2, $S = S - N + N$ is in $DL_{sb}(M)$.

Remark 1

(1) An order bounded operator H may not be a d-sobo whenever its adjoint is a d-sobo in general. For example, choice $M = c_0$ and H as an identity operator on M. Since M' has b-property [3], then $I': M' \to M'$ is a demi-strongly order bounded operator, but $I: c_0 \to c_0$ is not a d-sobo.

(2) Since order dual of every Riesz space has b-property and every adjoint of pre-regular operator is order bounded [3], then the adjoint of every pre-regular operator is strongly order bounded. Hence, every adjoint of pre-regular operator is d-sobo.

The following example gives us that the set of all d-sobo on a Riesz space M is not a lattice in general.

Example 7 Let H be an operator on $M = C[-1,1]$ defined by $H(f)(k) = f\left(\sin\left(\frac{1}{k}\right)\right) - f\left(\sin\left(k + \frac{1}{k}\right)\right)$ if $0 < |k| \le 1$ and $H(f)(0) = 0$. H is an order bounded, but it is not regular operator from Example 1.16 in [1]. Since M has b-property, H is in $L_{sb}(M)$. Hence, H is a d-sobo, but $|H|$ does not exist, since H is not a regular operator. Consequently, $DL_{sb}(M)$ is not lattice.

3. Conclusion

In this study, the class of demi-strongly order bounded operators on a Riesz space which is a generalization of strongly order bounded operators, is defined. Furthermore, the relationship between strongly order bounded operators and demi-strongly order bounded operators is examined and the conclusion that demi-strongly order bounded operator includes strongly order bounded operator is obtained. It is observed that the demi-strongly order bounded operators are not generally a vector space. A characterization of the b-property is obtained by the term of demi-strongly order bounded operators. It is obtained that the class of demi-strongly order bounded operators does not satisfy the domination property, but the domination property is satisfied when it is bounded from above with the identity operator. It is concluded that the class of demi-strongly order bounded operators does not form generally a lattice.

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