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Research Article

A new class of ideal Connes amenability

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Abstract

In this paper, we introduce the notion of $\sigma-$ideally Connes amenable for dual Banach algebras and give some hereditary properties for this new notion. We also investigate $\sigma-$ideally Connes amenability of $\ell^1(G,\omega)$. We show that if ω is a diagonally bounded weight function on discrete group G and σ is isometrically isomorphism of $\ell^1(G,\omega)$, then $\ell^1(G,\omega)$ is $\sigma-$ideally Connes amenable and so it is ideally Connes amenable.

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1. Introduction

Let $\mathcal{A}$ be a dual Banach algebra, that is, $\mathcal{A} = (\mathcal{A}_*)^*$ for some a closed submodule $\mathcal{A}_*$ of $\mathcal{A}^*$. Let X be a dual Banach $\mathcal{A}$-bimodule such that the maps $a \mapsto a.x$ and $a \mapsto x.a$ from $\mathcal{A}$ into X are w^*-continuous. Dual Banach $\mathcal{A}$-bimodules of this type are said to be *normal*. For a w^*-continuous endomorphism σ of $\mathcal{A}$, a map $D : \mathcal{A} \to X$ is called a *w^*-continuous $\sigma-$derivation* if it is w^*-continuous and

$$D(ab) = D(a) \cdot \sigma(b) + \sigma(a) \cdot D(b)$$

for all $a, b \in \mathcal{A}$. Also, D is called an *inner $\sigma-$derivation* if there exists $x \in X$ such that

$$D(a) = \delta_x^\sigma(a) := \sigma(a) \cdot x - x \cdot \sigma(a)$$

for all $a \in \mathcal{A}$. The space of all w^*-continuous (inner) $\sigma-$derivations from $\mathcal{A}$ into X is denoted by $(\mathcal{N}^1_\sigma(\mathcal{A}, X)$, respectively) $\mathcal{Z}^1_{\sigma,w^*}(\mathcal{A}, X)$. Let

$$\mathcal{H}^1_{\sigma,w^*}(\mathcal{A}, X) = \frac{\mathcal{Z}^1_{\sigma,w^*}(\mathcal{A}, X)}{\mathcal{N}^1_\sigma(\mathcal{A}, X)}.$$

Similar to the concept of amenability, $\mathcal{A}$ is said to be *$\sigma-$Connes amenable* if for every normal dual module X,

$$\mathcal{H}^1_{\sigma,w^*}(\mathcal{A}, X) = \{0\};$$

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or equivalently, every w^*-continuous $\sigma-$derivation from $\mathcal{A}$ into X is an inner $\sigma-$derivation [13]. In this case, if X is a w^*-closed two-sided ideal $\mathcal{I}$ in $\mathcal{A}$, then $\mathcal{A}$ is called $\sigma-\mathcal{I}-$*Connes amenable*, and if for every w^*-closed two-sided ideal $\mathcal{I}$ in $\mathcal{A}$, the dual Banach algebra $\mathcal{A}$ is $\sigma-\mathcal{I}-$Connes amenable, then A is called $\sigma-$*ideally Connes amenable.*

The concept of normal dual Banach bimodule was introduced by Johnson, Kadison, and Ringrose [6]. They also have studied the $n-$dimensional normal cohomology group $\mathcal{H}^n_{w^*}(\mathcal{A}, X)$ and gave conditions that

$$\mathcal{H}^n_{w^*}(\mathcal{A}, X) = \{0\},$$

when $\mathcal{A}$ is a unital C^*-algebra. One can prove that every derivation from a von Neumann algebra generated by an increasing sequence of finite dimensional $*-$algebras to a normal dual Banach bimodule is a coboundary. The converse of this result was proved by Connes [3]. Also, Connes [2] called a von Neumann algebra $\mathcal{A}$ *amenable* if

$$\mathcal{H}^1_{w^*}(\mathcal{A}, X) = \{0\}$$

for all normal dual Banach $\mathcal{A}-$bimodule X. Later, Helemskii [4] used the word "*Connes amenable*" instead of "*amenable*". He proved that the operator C^*-algebra $\mathcal{A}$ is Connes amenable if and only if the Banach $\mathcal{A}-$bimodule $\bar{\mathcal{A}}_*$ is injective. The first author, Bodaghi and Ebrahimi Bagha [7] generalized the concept of Connes amenability and introduced the notion of ideally Connes amenability for dual Banach algebras. They proved that von Neumann algebras are ideally Connes amenable; see also [12]; for study of the notion of quotient ideal amenability of Banach algebras see [16].

Let $\mathcal{A}$ be a dual Banach algebra and $\mathcal{I}$ be a weak*-closed two-sided ideal of $\mathcal{A}$. Then $\mathcal{I}$ is a dual Banach algebra and also it is a normal Banach $\mathcal{A}-$bimodule. A dual Banach algebra $\mathcal{A}$ is $\mathcal{I}-$Connes amenable if $\mathcal{H}^1_{w^*}(\mathcal{A}, J) = \{0\}$ and is ideally Connes amenable if it is $\mathcal{I}$-Connes amenable for every weak*-closed two-sided ideal $\mathcal{I}$ in $\mathcal{A}$; see [7]. Note that $\mathcal{I}$ is a dual Banach space with predual $\mathcal{I}_* = \frac{\mathcal{A}_*}{^\perp\mathcal{I}}$. Indeed, $\mathcal{I}$ is the weak*-closed subspace of $\mathcal{A}$ and so

$$(\mathcal{I}_*)^* = (\frac{\mathcal{A}_*}{^\perp\mathcal{I}})^* = (^\perp\mathcal{I})^\perp = \mathcal{I}.$$

Also, $\mathcal{I}_*$ is a submodule of $\frac{\mathcal{A}^*}{\mathcal{I}^\perp} = \mathcal{I}^*$. Thus, $\mathcal{I}$ is a dual Banach algebra. Once more, $^\perp\mathcal{I}$ is a submodule of $\mathcal{I}^\perp = \left(\frac{\mathcal{A}}{\mathcal{I}}\right)^*$ and

$$(^\perp\mathcal{I})^* = \frac{(\mathcal{A}_*)^*}{(^\perp\mathcal{I})^\perp} = \frac{\mathcal{A}}{\mathcal{I}}.$$

So, $\frac{\mathcal{A}}{\mathcal{I}}$ is a dual Banach space. On the other hand, multiplication in $\mathcal{A}$ and $\frac{\mathcal{A}}{\mathcal{I}}$ is separately weak*-continuous and thus $\frac{\mathcal{A}}{\mathcal{I}}$ is a dual Banach algebra. For details on this and other important results, refer to [5,8,10,11] and the references therein.

In this paper, we introduce the notion $\sigma-$ ideally Connes amenability for dual Banach algebras and investigate it. In Section 2, we prove under certain conditions that the ideally Connes amenability and $\sigma-$ ideally Connes amenability are equivalent. We also prove some hereditary properties of $\sigma-$ ideally Connes amenability of dual Banach algebras. In Section 3, we give some examples to illustrate our results. In Section 4, we study $\sigma-$ ideally Connes amenability of the Banach algebra $\ell^1(G,\omega)$ and show that if ω is diagonally bounded and σ is an isometric isomorphism, then $\ell^1(G,\omega)$ is $\sigma-$ ideally Connes amenable. In particular, $\ell^1(G,\omega)$ is ideally-Connes amenable.

2. $\sigma-$ideally Connes amenability

Throughout this section, σ is a w^*-continuous endomorphism of a dual Banach algebra $\mathcal{A}$. Before we give the first our result, let us recall that a dual Banach algebra $\mathcal{A}$ is called *ideally Connes amenable* if it is $id_{\mathcal{A}}-$Connes amenable, where $id_{\mathcal{A}}$ is the identity map on $\mathcal{A}$.

Proposition 2.1. *Let $\mathcal{A}$ be a dual Banach algebra. Then the following statements hold.*

(i) *If $\mathcal{A}$ is $\sigma-$Connes amenable and σ is onto, then $\mathcal{A}$ has an identity.*

(ii) *If $\mathcal{A}$ be σ-ideally Connes amenable for a w^*-continuous endomorphism $\sigma : \mathcal{A} \to \mathcal{A}$ with w^*-dense range, then $\mathcal{A}$ is ideally Connes amenable.*

Proof. (i) First, note that $X = \mathcal{A}$ with the following actions is a normal dual Banach $\mathcal{A}-$bimodule.

$$a \cdot x = 0 \quad \text{and} \quad x \cdot a = xa \tag{2.1}$$

for all $a \in \mathcal{A}$ and $x \in X$. We define the w^*-continuous $\sigma-$derivation $D : \mathcal{A} \to X$ by $D(a) = \sigma(a)$. Since $\mathcal{A}$ is $\sigma-$Connes amenable, there exists $x \in X$ such that $D = \delta^\sigma_x$. Using the module actins defined in (2.1), for every $a \in \mathcal{A}$ we have

$$\begin{aligned} \sigma(a) &= \sigma(a) \cdot x - x \cdot \sigma(a) \\ &= 0 - x\sigma(a) \\ &= -x\sigma(a). \end{aligned}$$

It follows that $\sigma(\mathcal{A}) = \mathcal{A}$ has a left identity. Similarly, $\mathcal{A}$ has a right identity. So (i) holds.

(ii) Assume that $\mathcal{A}$ is σ-ideally Connes amenable. Let $\mathcal{I}$ be a w^*-closed ideal of $\mathcal{A}$ and $D : \mathcal{A} \to \mathcal{I}$ be a w^*-continuous derivation. It is easy to see that $D \circ \sigma : \mathcal{A} \to \mathcal{I}$ is a w^*-continuous $\sigma-$derivation. So $D \circ \sigma = \delta^\sigma_x$ for some $x \in \mathcal{I}$. Now, if $a \in \mathcal{A}$, then there exists a net $(a_\lambda)_\lambda$ in $\mathcal{A}$ such that $a = \lim_\lambda \sigma(a_\lambda)$. Hence

$$\begin{aligned} D(a) &= w^* - \lim_\lambda D(\sigma(a_\lambda)) \\ &= w^* - \lim_\lambda (\sigma(a_\lambda)x - x\sigma(a_\lambda)) \\ &= ax - xa \\ &= \delta^{id_\mathcal{A}}_x(a). \end{aligned}$$

Thus, D is inner. Therefore, $\mathcal{A}$ is ideally Connes amenable. □

Let $\mathcal{I}$ be a w^*-closed two sided ideal in dual Banach algebra $\mathcal{A}$. It is clear that $\mathcal{I}$ is a dual Banach algebra with predual $\mathcal{I}_*$. Then we say that $\mathcal{I}$ has *the $\sigma-$dual trace extension property* if every $\phi \in \mathcal{I}$ with $\delta^\sigma_\phi = 0$ has an extension τ to $\mathcal{A}$ such that $\delta^{id_\mathcal{A}}_\tau = 0$.

Theorem 2.2. *Let $\mathcal{I}$ be a w^*-closed two sided ideal in dual Banach algebra $\mathcal{A}$, and let $\sigma(\mathcal{I}) = \mathcal{I}$. Then the following statements hold.*

(i) *If $\mathcal{I}$ is $\sigma-$Connes amenable and $\frac{\mathcal{A}}{\mathcal{I}}$ is $\hat{\sigma}$-Connes amenable, where $\hat{\sigma}(a+\mathcal{I}) = \sigma(a)+\mathcal{I}$ for all $a \in \mathcal{A}$, then $\mathcal{A}$ is $\sigma-$Connes amenable.*

(ii) *If $\mathcal{A}$ is σ-ideally Connes amenable and $\mathcal{I}$ has the σ-dual trace extension property, then $\frac{\mathcal{A}}{\mathcal{I}}$ is σ-ideally Connes amenable dual Banach algebra.*

Proof. (i) Let X be a normal dual Banach $\mathcal{A}-$bimodule and $D : \mathcal{A} \to X$ be a w^*-continuous $\sigma-$derivation. It is obvious that $D|_\mathcal{I}$ is a w^*-continuous $\sigma-$derivation from $\mathcal{I}$ into X. By the $\sigma-$Connes amenability of $\mathcal{I}$, there exists $x_0 \in X$ such that $D|_\mathcal{I} = \delta^\sigma_{x_0}$. Set $D_1 = D - \delta^\sigma_{x_0}$. Then D_1 is a w^*-continuous $\sigma-$derivation vanishes on $\mathcal{I}$. Now let

$$X_0 = \overline{\text{span}\{x\sigma(a) + \sigma(b)y : a, b \in A, x, y \in X\}}^{w^*}.$$

Then $\frac{X}{X_0}$ with the following actions is a normal dual Banach $\frac{\mathcal{A}}{\mathcal{I}}-$bimodule.

$$(a + \mathcal{I})(x + X_0) = \sigma(a)x + X_0 \quad \text{and} \quad (x + X_0)(a + \mathcal{I}) = x\sigma(a) + X_0$$

for all $a \in \mathcal{A}$ and $x \in X$. We define the w^*-continuous map $\hat{D} : \frac{\mathcal{A}}{\mathcal{I}} \to \frac{X}{X_0}$ by

$$\langle g_*, \hat{D}(a + \mathcal{I})\rangle = \langle g_*, D_1(a)\rangle,$$

where $g_* \in (\frac{X}{X_0})_* =^{\perp} X_0$. Since $D_1|_{\mathfrak{I}} = 0$, it follows that $\hat{D}$ is well-defined. For every $a, b \in \mathcal{A}$, we have

$$\begin{aligned}
\langle g_*, \hat{D}((a+\mathfrak{I})(b+\mathfrak{I}))\rangle &= \langle g_*, D_1(ab)\rangle \\
&= \langle g_*, \sigma(a)D_1(b) + D_1(a)\sigma(b)\rangle \\
&= \langle g_*\sigma(a), D_1(b)\rangle + \langle \sigma(b)g_*, D_1(a)\rangle \\
&= \langle g_* \cdot (a+\mathfrak{I}), \widehat{D}(b+\mathfrak{I})\rangle + \langle (b+\mathfrak{I}) \cdot g_*, \widehat{D}(a+\mathfrak{I})\rangle \\
&= \langle g_*, (a+\mathfrak{I}) \cdot \widehat{D}(b+\mathfrak{I})\rangle + \langle g_*, \widehat{D}(a+\mathfrak{I}) \cdot (b+\mathfrak{I})\rangle.
\end{aligned}$$

This shows that $\hat{D}$ is a w^*-continuous $\hat{\sigma}-$derivation, where $\hat{\sigma}(a+\mathfrak{I}) = \sigma(a) + \mathfrak{I}$ for all $a \in A$. So there exists $t \in \frac{X}{X_0}$, such that $\hat{D} = \delta_t^{\hat{\sigma}}$. Thus we have

$$\begin{aligned}
\langle g_*, D_1(a)\rangle &= \langle g_*, \hat{D}(a+\mathfrak{I})\rangle \\
&= \langle g_*, \hat{\sigma}(a+\mathfrak{I}) \cdot t - t \cdot \hat{\sigma}(a+\mathfrak{I})\rangle \\
&= \langle g_* \cdot \sigma(a), t\rangle - \langle \sigma(a) \cdot g_*, t\rangle \\
&= \langle g_*, \delta_t^{\sigma}(a)\rangle.
\end{aligned}$$

This implies that $D_1 = D - \delta_t^{\sigma}$, and therefore $D = \delta_{x_0-t}^{\sigma}$.

(ii) Let $\frac{\mathcal{J}}{\mathfrak{I}}$ be a w^*-closed two sided ideal in $\frac{\mathcal{A}}{\mathfrak{I}}$. Then $\mathcal{J}$ is a w^*-closed two sided ideal in $\mathcal{A}$. We shall briefly outline the argument. Let $(a_\alpha)_\alpha$ be a net in $\mathcal{J}$, such that $a_\alpha \longrightarrow a$ in w^*-topology of $\mathcal{J}$, we must show that a is in $\mathcal{J}$. It is clear that $a_\alpha + \mathfrak{I} \longrightarrow a + \mathfrak{I}$, in w^*- topology of $\frac{\mathcal{J}}{\mathfrak{I}}$. Note that $(a_\alpha + \mathfrak{I})_\alpha$ is a net in $\frac{\mathcal{J}}{\mathfrak{I}}$. Since $\frac{\mathcal{J}}{\mathfrak{I}}$ is w^*-closed, $a + \mathfrak{I}$ is in $\frac{\mathcal{J}}{\mathfrak{I}}$. Thus a belongs to $\mathcal{J}$, so $\mathcal{J}$ is w^*-closed. Note that ${}^{\perp}\mathfrak{I}$ is a predual of $\frac{\mathcal{A}}{\mathfrak{I}}$ and it is also a closed $\mathcal{A}$-submodule of $\mathcal{J}_*$. Let $\pi_* : \mathcal{J}_* \to {}^{\perp}\mathfrak{I}$ be the natural projection $\mathcal{A}$-bimodule homomorphism and $q : \mathcal{A} \to \frac{\mathcal{A}}{\mathfrak{I}}$ be the natural quotient map. Now if $D : \frac{\mathcal{A}}{\mathfrak{I}} \to \frac{\mathcal{J}}{\mathfrak{I}}$ is a w^*-continuous σ-derivation, then $\tilde{D} := (\pi_*)^* \circ D \circ q : \mathcal{A} \to \mathcal{J}$ is a w^*-continuous σ-derivation. Indeed, if $a, b \in \mathcal{A}$ and $j_* \in \mathcal{J}_*$, then

$$\begin{aligned}
\langle j_*, \tilde{D}(ab)\rangle &= \langle j_*, (\pi_*)^*(D \circ q(ab))\rangle \\
&= \langle j_*, (\pi_*)^*(D((a+\mathfrak{I})(b+\mathfrak{I})))\rangle \\
&= \langle \pi_*(j_*), (\sigma(a)+\mathfrak{I}) \cdot D(b+\mathfrak{I}) + D(a+\mathfrak{I}).(\sigma(b)+\mathfrak{I})\rangle \\
&= \langle \pi_*(j_*) \cdot (\sigma(a)+\mathfrak{I}), D(b+\mathfrak{I})\rangle + \langle (\sigma(b)+\mathfrak{I}) \cdot \pi_*(j_*), D(a+\mathfrak{I})\rangle \\
&= \langle \pi_*(j_*) \cdot \sigma(a), D(b+\mathfrak{I})\rangle + \langle \sigma(b) \cdot \pi_*(j_*), D(a+\mathfrak{I})\rangle \\
&= \langle \pi_*(j_* \cdot \sigma(a), D(b+\mathfrak{I})\rangle + \langle \pi_*(\sigma(b) \cdot j_*), D(a+\mathfrak{I})\rangle \\
&= \langle j_*, \sigma(a) \cdot (\pi_*)^*(D \circ q(b)) + (\pi_*)^*(D \circ q(a)) \cdot \sigma(b)\rangle \\
&= \langle j_*, \sigma(a) \cdot \tilde{D}(b) + \tilde{D}(a) \cdot \sigma(b)\rangle.
\end{aligned}$$

So $\tilde{D}(a) = \delta_\lambda^\sigma$ for some $\lambda \in \mathcal{J}$. If $i_* \in \mathfrak{I}_* = \frac{\mathcal{A}_*}{{}^{\perp}\mathfrak{I}}$, then $i_* \notin^{\perp} \mathfrak{I}$. But π_* is the projection on ${}^{\perp}\mathfrak{I}$. Thus $\pi_*(i_*) = 0$. That is, $\pi_* = 0$ on $\mathfrak{I}_*$. Let m be the restriction of λ to $\mathfrak{I}_*$, then $m \in \mathfrak{I}$ and for $i_* \in \mathfrak{I}_*$, we have

$$\begin{aligned}
\langle i_*, \sigma(a) \cdot m - m \cdot \sigma(a)\rangle &= \langle i_* \cdot \sigma(a) - \sigma(a) \cdot i_*, m\rangle \\
&= \langle i_* \cdot \sigma(a) - \sigma(a) \cdot i_*, \lambda\rangle \\
&= \langle i_*, \sigma(a) \cdot \lambda - \lambda \cdot \sigma(a)\rangle \\
&= \langle i_*, (\pi_*)^* \circ D \circ q(a)\rangle \\
&= \langle \pi_*(i_*), D \circ q(a)\rangle \\
&= 0.
\end{aligned}$$

Therefore $\sigma(a) \cdot m = m \cdot \sigma(a)$ for all $a \in \mathcal{A}$. Since $\mathfrak{I}$ has the $\sigma-$dual trace extension property, there exist a $\kappa \in \mathcal{A}$ such that $\kappa|_{\mathfrak{I}_*} = m$ and $a \cdot \kappa - \kappa \cdot a = 0$ for all $a \in \mathcal{A}$. Let τ

be the restriction of κ to $\mathcal{J}_*$. Then $\tau \in \mathcal{J}$ and $\lambda - \tau = 0$ on $\mathfrak{I}_*$. Therefore $\lambda - \tau \in \frac{\mathcal{J}}{\mathfrak{I}}$. By the surjectivity of π_*, for every $x \in (\frac{\mathcal{J}}{\mathfrak{I}})_*$ there exists $j_* \in \mathcal{J}_*$ such that $\pi_*(j_*) = x$. So

$$\begin{aligned}\langle x, D(a+\mathfrak{I})\rangle &= \langle \pi_*(j_*), D(a+\mathfrak{I})\rangle\\ &= \langle j_*, \sigma(a)\cdot\lambda - (\sigma(a)\cdot\tau - \tau\cdot a) - \lambda\cdot\sigma(a)\rangle\\ &= \langle j_*, \sigma(a)\cdot\lambda - \sigma(a)\cdot\tau + \tau\cdot\sigma(a) - \lambda.\sigma(a)\rangle\\ &= \langle j_*, \sigma(a)\cdot(\lambda-\tau) - (\lambda-\tau)\cdot\sigma(a).\rangle\end{aligned}$$

If $j_* \in {}^\perp\mathfrak{I}$, then by the definition of π_*, we have $\pi_*(j_*) = j_*$. Thus

$$\begin{aligned}\langle j_*, \sigma(a)\cdot(\lambda-\tau) - (\lambda-\tau)\cdot\sigma(a)\rangle &= \langle \pi_*(j_*), \sigma(a)\cdot(\lambda-\tau) - (\lambda-\tau)\cdot\sigma(a)\rangle\\ &= \langle x, \sigma(a)\cdot(\lambda-\tau) - (\lambda-\tau)\cdot\sigma(a)\rangle.\end{aligned}$$

Hence

$$D(a+\mathfrak{I}) = \sigma(a)\cdot(\lambda-\tau) - (\lambda-\tau)\cdot\sigma(a).$$

This shows that D is an inner σ-derivation. If $j_* \notin^\perp \mathfrak{I}$, then $\pi_*(j_*) = 0$. This implies that D is also an inner σ-derivation. Therefore, $\frac{\mathcal{A}}{\mathfrak{I}}$ is σ-ideally Connes amenable. □

In the following, let $\mathcal{A}^\sharp$ be the unitization of $\mathcal{A}$. It is easy to see that the map $\tilde{\sigma} : A^\sharp \to A^\sharp$ defined by

$$\tilde{\sigma}(a+\alpha) = \sigma(a) + \alpha \quad (a \in \mathcal{A}, \alpha \in \mathbb{C})$$

is a w^*-continuous endomorphism.

Theorem 2.3. *Let $\mathcal{A}$ be a dual Banach algebra. Then the following statements hold.*

(i) *If $\mathcal{A}^\sharp$ is $\tilde{\sigma}-$ideally Connes amenable, then $\mathcal{A}$ is $\sigma-$ideally Connes amenable.*
(ii) *If $H^1_{\tilde{\sigma},w^*}(\mathcal{A}^\sharp, \mathcal{A}^\sharp) = \{0\}$, then $H^1_{\sigma,w^*}(\mathcal{A}, \mathcal{A}) = \{0\}$.*
(iii) *If σ is idempotent and $\mathfrak{I}$ is a w^*-closed two sided ideal of $\mathcal{A}$ with a bounded approximate identity and $\sigma(\mathfrak{I}) = \mathfrak{I}$, then $H^1_{\sigma,w^*}(\mathfrak{I}, \mathfrak{I}) = \{0\}$ if and only if $H^1_{\sigma,w^*}(\mathcal{A}, \mathfrak{I}) = \{0\}$.*

Proof. (i) Let $D : \mathcal{A} \to \mathfrak{I}$ be a w^*-continuous $\sigma-$derivation. Define the weak$^*-$continuous $\tilde{\sigma}-$derivation $\tilde{D} : A^\sharp \to \mathfrak{I}$ by $\tilde{D}(a+\alpha) = D(a)$. Since $\mathcal{A}^\sharp$ is $\tilde{\sigma}-$ideally Connes amenable, it follows that $\tilde{D} = \delta^{\tilde{\sigma}}_a$ for some $a \in \mathcal{A}$. Hence for every $b \in \mathcal{A}$, we have

$$\begin{aligned}D(b) &= \tilde{D}(b+\alpha)\\ &= \tilde{\sigma}(b+\alpha)\cdot a - a\cdot\tilde{\sigma}(b+\alpha)\\ &= \sigma(b)\cdot a - a\cdot\sigma(b).\end{aligned} \tag{2.2}$$

This shows that D is $\sigma-$inner. Thus $\mathcal{A}$ is $\sigma-$ideally Connes amenable.

(ii) This follows from (i) and the fact that $\mathcal{A}$ is a normal $\mathcal{A}^\sharp-$bimodule with the following module action.

$$(a+\alpha)\cdot b = a\cdot b + \alpha b \quad \text{and} \quad b\cdot(a+\alpha) = b\cdot a + \alpha b,$$

for all $a, b \in \mathcal{A}$ and $\alpha \in \mathbb{C}$.

(iii) Assume that $H^1_{\sigma,w^*}(\mathfrak{I}, \mathfrak{I}) = \{0\}$. Let $D : \mathcal{A} \to \mathfrak{I}$ be a w^*-continuous $\sigma-$derivation and $i : \mathfrak{I} \to \mathcal{A}$ be the inclusion map. Then $d = D|_{\mathfrak{I}} : \mathfrak{I} \to \mathfrak{I}$ is a w^*-continuous $\sigma-$derivation. So there exists $t_0 \in \mathfrak{I}$ such that $d = \delta^\sigma_{t_0}$. Since $\mathfrak{I}$ has a bounded approximate identity and $\sigma(\mathfrak{I}) = \mathfrak{I}$, we have

$$\overline{\sigma(\mathfrak{I}^2)} = \overline{\mathfrak{I}^2} = \mathfrak{I}.$$

On the other hand,

$$\mathfrak{I} = \sigma(\mathfrak{I})\cdot\mathfrak{I}\cdot\sigma(\mathfrak{I}).$$

Thus $\mathcal{I}_* = \sigma(\mathcal{I}) \cdot \mathcal{I}_* \cdot \sigma(\mathcal{I})$. So for every $i, j \in \mathcal{I}$ and $i_* \in \mathcal{I}_*$, we have

$$\begin{aligned}
\langle \sigma(i)i_*\sigma(j), D(a)\rangle &= \langle \sigma(i)i_*, \sigma(j)D(a)\rangle \\
&= \langle \sigma(i)i_*, D(ja) - D(j)\sigma(a)\rangle \\
&= \langle \sigma(i)i_*, \sigma(ja)t_0 - t_0\sigma(ja)\rangle \\
&- \langle \sigma(i)i_*, (\sigma(j)t_0 - t_0\sigma(j))\sigma(a)\rangle \\
&= \langle \sigma(i)i_*\sigma(j), \sigma(a)t_0 - t_0\sigma(a)\rangle \\
&= \langle \sigma(i)i_*\sigma(j), \delta^{\sigma}_{t_0}(a)\rangle.
\end{aligned}$$

It follows that $D = \delta^{\sigma}_{t_0}$. So D is $\sigma-$inner.

Conversly, let $\mathcal{H}^1_{\sigma,w^*}(\mathcal{A}, I) = \{0\}$, and $D : \mathcal{I} \to \mathcal{I}$ be a w^*-continuous $\sigma-$derivation. Note that $\mathcal{I}$ is neo-unital Banach $\mathcal{I}-$bimodule. So

$$\mathcal{I} = \sigma(\mathcal{I}) \cdot \mathcal{I} \cdot \sigma(\mathcal{I}).$$

In view of [[14], Proposition 4.14], there exists a $\sigma-$derivation $\hat{D} : \mathcal{A} \to \mathcal{I}$ such that $\hat{D}|_{\mathcal{I}} = D$. From hypothesis we infer that $\hat{D}$ is $\sigma-$inner. Thus $H^1_{\sigma,w^*}(\mathcal{I}, \mathcal{I}) = \{0\}$. □

Let $\mathcal{A}$ be a dual Banach algebra. Recall that $\mathcal{A}$ is called *Connes amenable* if it is $id_{\mathcal{A}}-$Connes amenable. Also, $\mathcal{A}$ is said to be *weakly amenable* if every continuous derivation from $\mathcal{A}$ into $\mathcal{A}^*$ is inner; for more details see [15].

Theorem 2.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be dual Banach algebras and $\phi : \mathcal{A} \to \mathcal{B}$ be a w^*-continuous epimorphism. If $\mathcal{A}$ is either Connes amenable or commutative weakly amenable dual Banach algebra, then $\mathcal{B}$ is $\bar{\sigma}-$ideally Connes amenable, where $\bar{\sigma}$ is a weak$^*-$continuous endomorphism of $\mathcal{B}$.*

Proof. Let $\mathcal{I}$ be a w^*-closed two sided ideal of $\mathcal{B}$. Then $\mathcal{I}$ is a normal dual $\mathcal{A}-$bimodule with the following actions.

$$a \cdot i = \bar{\sigma}(\phi(a)) \cdot i \quad \text{and} \quad i \cdot a = i \cdot \bar{\sigma}(\phi(a))$$

for all $a \in \mathcal{A}$ and $i \in \mathcal{I}$. It is easy to check that if $D : \mathcal{B} \to \mathcal{I}$ is a w^*-continuous $\bar{\sigma}-$derivation, then $D \circ \phi : \mathcal{A} \to \mathcal{I}$ is a w^*-continuous $\bar{\sigma} \circ \phi-$derivation.

If $\mathcal{A}$ is Connes amenable, then there exists $t \in \mathcal{I}$ such that

$$D \circ \phi(a) = \delta^{id_{\mathcal{A}}}_t(a) = \delta^{\bar{\sigma}\circ\phi}_t(a) = \delta^{\bar{\sigma}}_t(\phi(a)).$$

Since ϕ is an epimorphism, $D = \delta^{\bar{\sigma}}_t$. Therefore, D is a $\bar{\sigma}-$inner derivation. Thus $\mathcal{B}$ is $\bar{\sigma}-$ideally Connes amenable.

If $\mathcal{A}$ is commutative weakly amenable, then $\mathcal{B}$ is commutative and so $\mathcal{I}$ is a symmetric Banach $\mathcal{B}-$bimodule. Hence $\mathcal{I}$ is a symmetric Banach $\mathcal{A}-$bimodule and $\mathcal{H}^1(\mathcal{A}, I) = \{0\}$. So $D \circ \phi = 0$. Consequently $D = 0$. Therefore, $\mathcal{B}$ is $\sigma-$ideally Connes amenable. □

3. Some examples

In this section, we give some examples to illustrate the new notion of $\sigma-$ideally Connes amenability introduced in this work. These examples show that the notion of $\sigma-$ideally Connes amenability is different from ideally Connes amenable. In doing this, we give some examples of $\sigma-$ideally Connes amenable dual Banach algebras that are not ideally Connes amenable.

Example 3.1. Let $\mathcal{A}$ be a dual Banach algebra, and let φ be a non-zero linear functional on A. Let $\mathcal{A}_{\varphi}$ be the Banach algebra $\mathcal{A}$ equipped with the following product.

$$a \cdot b = \varphi(a)b.$$

Then $(\mathcal{A}_\varphi, \cdot)$ is a Banach algebra. Note that φ is a linear functional on A and thus $\varphi(a) \in \mathbb{C}$ for all $a \in A$. Hence

$$\begin{aligned} a \cdot (b \cdot c) &= a \cdot (\varphi(b)c) = \varphi(a)\varphi(b)c \\ &= \varphi(\varphi(a)b)c = \varphi(a \cdot b)c \\ &= (a \cdot b) \cdot c \end{aligned}$$

for all $a, b, c \in \mathcal{A}$. This shows that the multiplication is associative. Since the product "$\cdot$" is separately w^*-continuous, $\mathcal{A}_\varphi$ is a dual Banach algebra. It is clear that $\mathcal{A}_\varphi$ has a left identity, say e, but it does not have bounded right approximate identity. So $\mathcal{A}_\varphi$ is not ideally Connes amenable; see [[7], Proposition 2.3].

We define the w^*-continuous endomorphism $\sigma : \mathcal{A}_\varphi \to \mathcal{A}_\varphi$ by

$$\sigma(a) = \varphi(a)e.$$

For every $a \in \mathcal{A}$, we have

$$\sigma^2(a) = \sigma(\varphi(a)e) = \varphi(a)\sigma(e) = \sigma(a).$$

Thus σ is idempotent. Obviously, e is identity for $\sigma(\mathcal{A}_\varphi)$.

We claim that any non-trivial two-sided ideal of $\mathcal{A}_\varphi$ is contained in $\ker\varphi$, and that any closed subspace of $\ker\varphi$ is a closed two-sided ideal. Indeed, let $\mathcal{I} \trianglelefteq \mathcal{A}_\varphi$ be a non-trivial two-sided ideal, so for $a \in \mathcal{I}$, $b \in \mathcal{A}$ we have $\varphi(a)b = a \cdot b \in \mathcal{I}$. Letting b vary and using that $\mathcal{I} \neq \mathcal{A}$ shows that $\varphi(a) = 0$, so $\mathcal{I} \subseteq \ker\varphi$. Conversely, if $\mathcal{I} \subseteq \ker\varphi$ is a closed subspace, then $a \cdot b = 0$ for each $a \in \mathcal{I}$, $b \in \mathcal{A}$, while $b \cdot a = \varphi(b)a \in \mathcal{I}$, showing that $\mathcal{I}$ is a two-sided ideal.

Let $\tilde{D} : \mathcal{A}_\varphi \to \mathcal{A}_\varphi$ be a non-zero w^*-continuous $\sigma-$derivation. Then for every $a, b \in \mathcal{A}_\varphi$, we have

$$\tilde{D}(a \cdot b) = \sigma(a) \cdot \tilde{D}(b) + \tilde{D}(a) \cdot \sigma(b).$$

Hence

$$\begin{aligned} \varphi(a) \cdot \tilde{D}(b) &= \varphi(a) \cdot e \cdot \tilde{D}(b) + \tilde{D}(a) \cdot \varphi(b) \cdot e \\ &= \varphi(a) \cdot \tilde{D}(b) + \varphi(b) \cdot \tilde{D}(a) \cdot e. \end{aligned}$$

Thus $\varphi(b)\tilde{D}(a) \cdot e = 0$. Since $\varphi \neq 0$, we have $\tilde{D}(a) \cdot e = 0$. Thus $\varphi(\tilde{D}(a))e = 0$, so we conclude that $e = 0$, that is a contradiction. It means that every $\sigma-$derivation is zero, so it is inner. Thus $\mathcal{A}_\varphi$ is $\sigma-$ideally Connes amenable.

Example 3.2. Let $\mathcal{A} = \ell^1(\mathbb{N})$ be equipped with the product

$$f \cdot g = f(1)g$$

and the norm $\|.\|_1$; see [18]. It is easy to see that $\mathcal{A}$ does not have bounded approximate identity. So $\mathcal{A}$ is not ideally Connes amenable [7].

For $f \in \mathcal{A}$, define the mapping $\widetilde{f} : \mathbb{N} \to \mathbb{C}$, by $\widetilde{f}(1) = 0$ and $\widetilde{f}(n) = f(n)$ for $n \geq 2$. Then $f = f \cdot e + \widetilde{f}$, where $e \in \ell^1(\mathbb{N})$ is defined by

$$e_n = \begin{cases} 1 & n = 1 \\ 0 & n \neq 1. \end{cases}$$

Let $\mathcal{I}$ be a weak*-closed two-sided ideal of $\mathcal{A}$ with $\mathcal{I} \neq \mathcal{A}$. Then $\mathcal{I}$ is contained in

$$\{f \in \mathcal{A} : f(1) = 0\}.$$

We define the w^*-continuous idempotent endomorphism σ on $\mathcal{A}$, be such that for all $a \in \ell^1(\mathbb{N})$

$$\sigma(a)(1) = a(1).$$

Let $D : \mathcal{A} \to \mathcal{I}$ be a weak*-continuous $\sigma-$derivation. Then

$$D(f) = \sigma(f)(1)D(e) + D(\widetilde{f}),$$

Since $D(\widetilde{f}) \in \mathcal{I}$ and $D(\widetilde{f})(1) = 0$, it follows that

$$D(\widetilde{f}) \cdot \sigma(e) = D(\widetilde{f})(1)\sigma(e) = 0.$$

So for every $g \in \mathcal{A}_*$, we have

$$\langle D(\widetilde{f}), g\rangle = \langle D(\widetilde{f}), \sigma(e) \cdot g\rangle = \langle D(\widetilde{f}) \cdot \sigma(e), g\rangle = 0.$$

Hence $D(\widetilde{f}) = 0$. From $D(e) \in \mathcal{I}$ and $D(e)(1) = 0$ we infer that $D(e) \cdot \sigma(f) = 0$. So

$$\begin{aligned} D(f) &= \sigma(f)(1)D(e) \\ &= \sigma(f) \cdot D(e) \\ &= \sigma(f) \cdot D(e) - D(e) \cdot \sigma(f). \end{aligned}$$

Therefore $H^1_{\sigma,w^*}(\mathcal{A}, \mathcal{I}) = \{0\}$.

Let $a \in \ell^1(\mathbb{N})$. Then there is a sequence $\{a_n\}$ in $c_0(\mathbb{N})$ such that $a_n \to a$ in the w^*-topology. For $f \in c_0(\mathbb{N})^*$, define the linear functional $\hat{f} \in \ell^1(\mathbb{N})^*$ by

$$\langle a, \hat{f}\rangle := w^* - lim_n \langle a_n, f\rangle.$$

This enables us to define the left and right module actions of $\ell^1(\mathbb{N})$ on $c_0(\mathbb{N})^*$ by

$$a \cdot f = \langle a, \hat{f}\rangle e \quad \text{and} \quad f \cdot a = a(1)f.$$

It is easy to prove that $c_0(\mathbb{N})^*$ is an $\ell^1(\mathbb{N})-$bimodule. Let D be a weak*-continuous $\sigma-$derivation from $\ell^1(\mathbb{N})$ to $\ell^1(\mathbb{N}) \cong c_0(\mathbb{N})^*$. For all $a \in \ell^1(\mathbb{N})$, we have

$$\begin{aligned} a(1)D(a) &= D(a^2) \\ &= D(a) \cdot \sigma(a) + \sigma(a) \cdot D(a) \\ &= \sigma(a)(1)D(a) + \langle \sigma(a), D(a)\rangle e. \end{aligned}$$

This shows that

$$\langle \sigma(a), D(a)\rangle = 0.$$

So for every $a, b \in \ell^1(\mathbb{N})$, we have

$$\begin{aligned} 0 &= \langle \sigma(ab), D(ab)\rangle \\ &= \langle \sigma(ab), D(a) \cdot \sigma(b) + \sigma(a) \cdot D(b)\rangle \\ &= \langle \sigma(a).\sigma(b), \sigma(b)(1)D(a) \\ &+ \langle \sigma(a), D(b)e\rangle \\ &= \langle \sigma(a).\sigma(b), \sigma(b)(1)D(a)\rangle \\ &+ \langle \sigma(a), D(b)\rangle\langle \sigma(a)\sigma(b), e\rangle \\ &= \sigma(b)(1)\langle \sigma(a).\sigma(b), D(a)\rangle \\ &+ \sigma(a)(1)\sigma(b)(1)\langle \sigma(a), D(b)\rangle \\ &= \sigma(b)(1).\sigma(a)(1)\langle \sigma(b), D(a)\rangle \\ &+ \sigma(a)(1)\sigma(b)(1)\langle \sigma(a), D(b)\rangle. \end{aligned}$$

It follows that

$$\langle \sigma(a), D(b)\rangle = -\langle \sigma(b), D(a)\rangle.$$

Let $t \in \sigma(\mathcal{A})$. Then there exists $b \in \ell^1(\mathbb{N})$ such that $t = \sigma(b) = \sigma^2(b)$. Thus

$$\begin{aligned}
\langle t, D(a)\rangle &= \langle t, D(ea)\rangle \\
&= \langle t, D(e)\cdot\sigma(a)\rangle + \langle t, \sigma(e)\cdot D(a)\rangle \\
&= \langle \sigma(a).t, D(e)\rangle + \langle t.\sigma(e), D(a)\rangle \\
&= \langle \sigma(a).\sigma^2(b), D(e)\rangle + \langle \sigma^2(b).\sigma(e), D(a)\rangle \\
&= \langle \sigma(a).\sigma(b), D(e)\rangle + \langle \sigma(\sigma(b).e), D(a)\rangle \\
&= \langle \sigma(a).\sigma(b), D(e)\rangle - \langle \sigma(a), D(\sigma(b).e)\rangle \\
&= \langle \sigma(a).\sigma(b), D(e)\rangle - \langle \sigma(a), \sigma(b)(1)D(e)\rangle \\
&= \langle \sigma(b), D(e)\cdot\sigma(a)\rangle - \langle \sigma(a), D(e)\cdot\sigma(b)\rangle \\
&= \langle \sigma(b), D(e)\cdot\sigma(a)\rangle - \langle \sigma(b).\sigma(a), D(e)\rangle \\
&= \langle \sigma(b), D(e)\cdot\sigma(a)\rangle - \langle \sigma(b), \sigma(a)\cdot D(e)\rangle \\
&= \langle t, D(e)\cdot\sigma(a) - \sigma(a)\cdot D(e)\rangle.
\end{aligned}$$

Hence

$$D(a) = D(e)\cdot\sigma(a) - \sigma(a)\cdot D(e) = \delta^{\sigma}_{-D(e)}(a).$$

Therefore, $\ell^1(\mathbb{N})$ is $\sigma-$ideally Connes amenable.

Example 3.3. Let $\mathcal{A}$ be a non-ideally Connes amenable Banach algebra with a right approximate identity. It is known from [7] that $\mathcal{A}^\sharp$ is not ideally Connes amenable. Define the w^*-continuous map $\sigma : \mathcal{A}^\sharp \to \mathcal{A}^\sharp$ by

$$\sigma(a + \alpha) = \alpha.$$

Let $(e_\alpha)_{\alpha\in\Lambda}$ be a right approximate identity for $\mathcal{A}$, and let $\mathfrak{I}$ be a w^*-closed two-sided ideal in $\mathcal{A}^\sharp$. If $D : \mathcal{A}^\sharp \to \mathfrak{I}$ is a w^*-continuous $\sigma-$derivation, then a simple calculation shows that $D(ae_\alpha) = 0$, for all $a \in \mathcal{A}$ and $\alpha \in \Lambda$. Consequently, $D(a) = 0$. If $e_{\mathcal{A}^\sharp}$ denotes the identity element of $\mathcal{A}^\sharp$, then

$$D(a + \alpha e_{\mathcal{A}^\sharp}) = D(a) + \alpha D(e_{\mathcal{A}^\sharp}) = 0.$$

That is, $D = 0$ and so $\mathcal{A}^\sharp$ is σ-ideally Connes amenable.

4. $\sigma-$ideally Connes amenability of $\ell^1(G,\omega)$

Let us recall that a Banach space E is called an $L-$*embedded Banach space* if it is an l^1- summand in its bidual.

The following theorem is proved in [1] is needed to prove the main result of this section.

Theorem 4.1. *Let E be an $L-$embedded Banach space and F be a non-empty bounded subset of E. Then the family of isometry maps of E preserving F has a common fixed point in F.*

Let G be a discrete group and $\omega : G \to [1,\infty)$ be a weight function, i.e, $\omega(e) = 1$ and

$$\omega(xy) \le \omega(x)\omega(y)$$

for all $x, y \in G$. Let us recall that a weight function ω on G is called *diagonally bounded* if $\sup_{x\in G}(\omega(x)\omega(x^{-1}))$ is finite. Also, recall that $\ell^1(G,\omega)$ denotes the space of all complex-valued functions on G such that $\omega f \in \ell^1(G)$. For details on these algebras, refer to [9] and the references therein.

We know that $\ell^1(G)$ is $L-$embedded, and since $\ell^1(G,\omega)$ is isometrically isomorphic to $\ell^1(G)$ as a Banach space (although not as a Banach algebra), it too must be $L-$embedded. We show that a weak*-closed linear subspace of $\ell^1(G)$ is $L-$embedded. We shall briefly outline the argument. Let $i : c_0(G) \hookrightarrow \ell^\infty(G)$ be the canonical embedding, and let $p = i^*$. Then p is the projection $\ell^1(G)^{**} \longrightarrow \ell^1(G)$ witnessing its $L-$embeddedness, that is to say

$$\|\Phi\| = \|p(\Phi)\| + \|(id - p)(\Phi)\| \quad (\Phi \in \ell^1(G)^{**}). \tag{4.1}$$

Let I be a weak*-closed linear subspace of $\ell^1(G)$, and let $j : c_0(G)/I_\perp \longrightarrow l^\infty(G)/I^\perp$ be the map

$$j : x + I_\perp \longmapsto i(x) + I^\perp \quad (x \in c_0(G)).$$

Then j can be thought of an embedding $I_* \hookrightarrow I^*$. Let $q = j^* : I^{\perp\perp} \longrightarrow I$. Canonically $I^{\perp\perp} \cong I^{**}$ (isometrically) and we can check that $p|_{I^{\perp\perp}} = q$. A simple calculation using Equation (4.1) then shows that

$$\|\Phi\| = \|q(\Phi)\| + \|(id - q)(\Phi)\| \quad (\Phi \in I^{\perp\perp}),$$

so that I is $L-$embedded.

Theorem 4.2. *Let ω be a diagonally bounded weight function on a discrete group G and σ be an isometric isomorphism of $\ell^1(G,\omega)$. Then $\ell^1(G,\omega)$ is $\sigma-$ideally Connes amenable.*

Proof. Let ω be a weight function on G. Fix $a \in G$ and define the weight function ω_a on G by

$$\omega_a(x) = \omega(axa^{-1})$$

for all $x \in G$. Then for every $x \in G$, we have $\omega_a(x) \leq \omega(a)\omega(a^{-1})\omega(x)$ and

$$\begin{aligned} \omega(x) &= \omega(a^{-1}(axa^{-1})a) \\ &\leq \omega(a^{-1})\omega(a)\omega(axa^{-1}) = \omega(a^{-1})\omega(a)\omega_a(x). \end{aligned}$$

Now, define the weight function ω' on G by $\omega'(x) = \sup_{a\in G} \omega(axa^{-1})$. Since ω is diagonally bounded, there is a constant $m > 0$ such that $\omega(a)\omega(a^{-1}) \leq m$ for every $a \in G$. Hence $\omega(axa^{-1}) \leq \omega(x)\omega(a)\omega(a^{-1}) \leq m\omega(x)$ for every $a \in G$. Thus $\sup_{a\in G} \omega(axa^{-1}) \leq m\omega(x)$, therefore

$$\omega'(x) \leq m\omega(x) \tag{4.2}$$

On the other hand

$$\omega(x) = \omega(exe^{-1}) \leq \sup_{a\in G} \omega(axa^{-1}) = \omega'(x) \tag{4.3}$$

Due to relations (4.2) and (4.3) we conclude that ω and ω' are equivalent. Thus $\ell^1(G,\omega)$ and $\ell^1(G,\omega')$ are isometrically isomorphic.

Let D be a w^*-continuous derivation from $\ell^1(G,\omega')$ into w^*-closed two sided ideal $\mathfrak{I}$ of $\ell^1(G,\omega')$. Define the function $h : G \to \mathfrak{I}$ by $h(t) = D(\delta_t) * \sigma(\delta_{t^{-1}})$. Since ω is diagonally bounded, ω' does so. Thus h is bounded. Indeed, for every $t \in G$, we have

$$\begin{aligned} \| h(t) \| &= \| D(\delta_t) * \sigma(\delta_{t^{-1}}) \| \\ &\leq \| D \| \| \delta_t \|_{w'} \| \delta_{t^{-1}} \|_{w'} \\ &= \| D \| \, w'(t)w'(t^{-1}). \end{aligned}$$

For $t \in G$ and $g \in \mathfrak{I}$, define the action

$$t \cdot g = \sigma(\delta_t) * g * \sigma(\delta_{t^{-1}}).$$

Then

$$\begin{aligned} h(st) &= D(\delta_{st}) * \sigma(\delta_{(st)^{-1}}) = D(\delta_s * \delta_t) * \sigma(\delta_{t^{-1}} * \delta_{s^{-1}}) \\ &= D(\delta_s) * \sigma(\delta_t) + \sigma(\delta_s) * D(\delta_t) * \sigma(\delta_{t^{-1}}) * \sigma(\delta_{s^{-1}}) \\ &= D(\delta_s) * \sigma(\delta_{s^{-1}}) + \sigma(\delta_s) * D(\delta_t) * \sigma(\delta_{t^{-1}}) * \sigma(\delta_{s^{-1}}) \\ &= h(s) + s \cdot h(t). \end{aligned}$$

Using h we can define another action of G on $\mathfrak{I}$ as follows.

$$t \bullet g = t \cdot g + h(t) = \sigma(\delta_t) * g * \sigma(\delta_{t^{-1}}) + h(t)$$

for all $t \in G$ and $g \in \mathfrak{I}$. Since σ is an isometric isomorphism of $\ell^1(G, w')$, there exist a continuous character $\gamma : G \to \mathbb{T}$ and an automorphism ψ on G such that for every $t \in G$,

$$\sigma(\delta_t) = \frac{w(t)\gamma(t)}{w(\psi(t))}\delta_{\psi(t)};$$

see [[17] Theorem 2.4]. This implies that "$\bullet$" is isometry. Thus for every $g_1, g_2 \in \ell^1(G, w')$, we have

$$\begin{aligned} \| t \bullet (g_1 - g_2) \|_{1,w'} &= \| t \cdot (g_1 - g_2) \|_{1,w'} \\ &= \| \sigma(\delta_t) * (g_1 - g_2) * \sigma(\delta_{t^{-1}}) \|_{1,w'} \\ &= | \frac{w(t)\gamma(t)}{w(\psi(t))} | \, | \frac{w(t^{-1})\gamma(t^{-1})}{w(\psi(t^{-1}))} | \sum_{x \in G} (| (g_1 - g_2)(x) | \, w'(\psi(t^{-1})x\psi(t))) \\ &= \sum_{x \in G} | (g_1 - g_2)(x) | \, w'(x) \\ &= \| g_1 - g_2 \|_{1,w'} . \end{aligned}$$

But, for $t \in G$, we have

$$\begin{aligned} t \bullet h(G) &= \{t \bullet h(s) : s \in G\} \\ &= \{t \cdot h(s) + h(t) : h \in G\} \\ &= \{h(ts) : s \in G\} \\ &= h(G). \end{aligned}$$

These facts let us to apply Theorem 4.1 to $E = \mathfrak{I}$ and $F = h(G)$. So there exists $g \in \mathfrak{I}$ such that $t \bullet g = g$ for all $t \in G$. It follows that

$$\begin{aligned} D(\delta_t) * \sigma(\delta_{t^{-1}}) &= h(t) \\ &= t \bullet g - t \cdot g \\ &= g - t \cdot g \\ &= g - \sigma(\delta_t) * g * \sigma(\delta_{t^{-1}}). \end{aligned}$$

This shows that

$$D(\delta_t) = g * \sigma(\delta_t) - \sigma(\delta_t) * g.$$

Since span$\{\delta_t; t \in G\}$ is weak* dense in $\ell^1(G, w')$, we conclude that

$$D(f) = g * \sigma(f) - \sigma(f) * g = \delta_g^\sigma(f)$$

for all $f \in \ell^1(G, w')$. Thus $\ell^1(G, w')$ is $\sigma-$ideally Connes amenable. □

We finish this section with the following result which is an immediate consequence of Theorem 4.2.

Corollary 4.3. *Let ω be a diagonally bounded weight function on a discrete group G. Then $\ell^1(G, \omega)$ is ideally Connes amenable.*

5. Conclusion

In this paper, we introduced the concept of $\sigma-$ideally Connes amenable for dual Banach algebras. We gave some examples to illustrate this notion and showed that it is different from ideally Connes amenable. We also determined relation between $\sigma-$ideally Connes amenability of a dual Banach algebra with its unitization, quotient spaces and homomorphic images. Finally, we studied $\sigma-$ideally Connes amenability of weighted group algebra $\ell^1(G, \omega)$ and proved that if ω is a diagonally bounded weight function on discrete group G and σ is isometrically isomorphism of $\ell^1(G, \omega)$, then $\ell^1(G, \omega)$ is $\sigma-$ideally Connes amenable.

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