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A new approach for the bigeometric newton method

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ABSTRACT

In this study, quadratic convergent new bigeometric Newton's method (nBGNM) was developed. For this, the basic definitions and theorems of bigeometric analysis, which is one of the non-Newtonian analysis, were used. Using the bigeometric Taylor expansion, a convergence analysis of this new method was given. Also, the new bigeometric Newton method (nBGNM) was compared in detail with the geometric (multiplicative) Newton method (GNM) and the classical Newton method (NM).

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1. Introduction

Solutions of nonlinear functional (algebraic, differential integral etc.) equations are very important in applied sciences and engineering. Newton's method is one of the most widely used methods in classical analysis to analyze the solutions of such equations [1-4]. This method was introduced by Newton for finding the real valued function $F:\mathbb{R}\to\mathbb{R}$ of a real variable from the equation in the form of $F(t)=0$. As is known, Newton's method is the most common method used to approximate the real root of a function. The classical Newton's method is given by

$$t_{n+1}=t_n-\frac{F(t_n)}{F'(t_n)},\quad n=0,1,2,\ldots.$$

That is, the classical Newton's method has an iterative procedure that includes the function and its derivative.

In the last quarter of the 19th century, Grossman and Katz defined non-Newtonian analysis [5]. The importance of these new analysis, in which different arithmetic operations are used, has been better understood in recent years. Especially, Bashirov et al., managed to draw the attention of researchers to multiplicative analysis by their study in 2008 [6]. Thus, researchers have shown great interest in developing the theory and application of multiplicative (geometric) analysis [7-11,12-17,18-26]. In geometric analysis, important studies of numerical analysis were made by Bilgehan, Mısırlı, Gürefe, Riza, Ozyapici, Waseem [27-33]. Unal et al. examined Newton's method and cubic convergence in geometric analysis in a study they conducted in 2017 [34].Newton method was given for multiplicative and Volterra Calculi by Ozyapici et al in 2016 [31]. In this study, the Bigeometric Newton method will be given with the help of the Bigeometric Taylor expansion. Also, the convergence of the bigeometric Newton method will be examined. Then bigeometric Newton's method will be compared with Newton's methods in classical analysis and multiplicative analysis. Therefore some basic definitions and theorems of bigeometric analysis that can be found in the studies of [35-42] will be given.

Definition 1. *(see [35-38])* Let $f:\mathbb{R}_{exp}\to\mathbb{R}_{exp}$ be a function. The bigeometric derivative of the function f is given by

$$\frac{d^{\pi}f}{dt}(t)=f^{\pi}(t)=\lim_{h\to 0}\left\{\frac{f[(1+h)\cdot t]}{f(t)}\right\}^{1/h}$$

The relationship of the bigeometric derivative with the classical derivative and the geometric derivative is given as

$$f^{\pi}(t) = \exp\left\{t \cdot \frac{f'(t)}{f(t)}\right\} = f^{*}(t)^{t}$$

Theorem 1. *(see [35-38])* Let f, g and h be bigeometric differentiable functions and $k > 0$ is an arbitrary constant then the bigeometric differentiation rules are valid as,

i) $(k.f)^{\pi}(t) = f^{\pi}(t)$,

ii) $(f+g)^{\pi}(t) = f^{\pi}(t)^{\frac{f(t)}{f(t)+g(t)}} \cdot g^{\pi}(t)^{\frac{g(t)}{f(t)+g(t)}}$,

iii) $(f \cdot g)^{\pi}(t) = f^{\pi}(t) \cdot g^{\pi}(t)$,

iv) $\left(\frac{f}{g}\right)^{\pi}(t) = \frac{f^{\pi}(t)}{g^{\pi}(t)}$,

v) $(f^{h})^{\pi}(t) = f^{\pi}(t)^{h(t)} \cdot f(t)^{t \cdot h'(t)}$,

vi) $(f \circ g)^{\pi}(t) = f^{\pi}[g(t)]^{g'(t)}$.

Theorem 2. *(see [35-38]) (Bigeometric Taylor theorem)*

Let D be an open interval and let $f: D \to \mathbb{R}_{exp}$ is $(n+1)$ times bigeometric differentiable on D. Then for any $t, t+h \in D$, there exists a number $\theta \in (0,1)$ such that

$$f(t+h) = \prod_{i=0}^{n}\left[f^{\pi(i)}(t)\right]^{\frac{\ln\left(1+\frac{h}{t}\right)^{i}}{i!}} \cdot \left[\left(f^{\pi(n+1)}(t+\theta \cdot h)\right)^{\frac{\ln\left(1+\frac{h}{t}\right)^{n+1}}{(n+1)!}}\right]$$

Definition 2. *(see [4])* If the sequence $\{t_i\}$ tends to a limit r in such a way that

$$\lim_{i\to\infty}\frac{t_{i+1}-r}{(t_i-r)^{p}} = C$$

for some $C \neq 0$ and $p \geq 1$, p is called as the order of convergence of the sequence and C is known as the asymptotic error constant.

Definition 3. *(see [3])* Let r be a root of the function $f(t)$ and suppose that t_{i+1}, t_i and t_{i-1} are three consecutive iterations closer to the root r. Then, the computational order of convergence (COC) ρ can be approximated using the formula $\rho \approx \frac{\ln|(t_{i+1}-r)/(t_i-r)|}{\ln|(t_i-r)/(t_{i-1}-r)|}$.

2. New Bigeometric Newton Method

In bigeometric calculus, Taylor expansion of function in neighboord a point t_0 can be written [42] as;

$$f(t_0+h) = \prod_{i=0}^{\infty}\left[f^{\pi(i)}(t_0)\right]^{\frac{\left[\ln\left(1+\frac{h}{t_0}\right)\right]^{i}}{i!}} \tag{1}$$

Let $t_0 + h = t$. We can write following equation using the first two terms of above equation:

$$f(t) = f(t_0) \cdot [f^{\pi}(t_0)]^{\ln\left(\frac{t}{t_0}\right)}. \tag{2}$$

If we calculate logarithm of the equation (2), we have

$$\ln f(t) = \ln f(t_0) + \ln\left(\frac{t}{t_0}\right) \cdot \ln f^{\pi}(t_0).$$

If we consider that $f(r) = 1$, we get

$$r = t_0 \cdot e^{-\frac{\ln f(t_0)}{\ln f^{\pi}(t_0)}}$$

Hence, we obtain new bigeometric Newton method as

$$t_{n+1} = t_n \cdot e^{-\frac{\ln f(t_n)}{\ln f^{\pi}(t_n)}} \tag{3}$$

Equivalent in classical analysis of iterative formula (3) is also that

$$t_{n+1} = t_n \cdot e^{-\frac{\ln f(t_n)}{t_n \cdot (\ln f)'(t_n)}}. \tag{4}$$

Remark 1. The equation (4) can be written as

$$\frac{t_{n+1}}{t_n} = e^{-\frac{\ln f(t_n)}{t_n \cdot (\ln f)'(t_n)}}. \tag{5}$$

Since right on the handle of the equation (5) is positive for each t_n, markers of t_{n+1} and t_n are been same as. This imply that markers of root of function and initial point are been same as. That is; for positive root, we must choose a positive initial point. Similarly, for negative root, we must choose a negative initial point.

3. Convergence of the method

Iteration function of the bigeometric Newton Method is

$$g(t) = t \cdot e^{-\frac{\ln f(t)}{t \cdot (\ln f)'(t)}}$$

The derivative of the function $g(t)$ is

$$g'(t) = e^{-\frac{\ln f(t)}{t \cdot (\ln f)'(t)}} - \frac{t \cdot \left[\frac{f'(t)}{f(t)}\right]^2 - \left[\frac{f'(t)}{f(t)} + t \cdot \frac{f''(t) \cdot f(t) - (f'(t))^2}{(f(t))^2}\right] \cdot \ln f(t)}{t^2 \cdot \left[\frac{f'(t)}{f(t)}\right]^2} \cdot e^{-\frac{\ln f(t)}{t \cdot (\ln f)'(t)}} \cdot t.$$

Considering the classical Taylor expansion of order two of $g(t_n)$ neighborhood of fixed point r and $f(r) = 1$, we obtain

$$g(t_n) = g(r) + \frac{(t_n - r)^2}{2} \cdot g''(r). \quad (7)$$

If we rearrange to the equation (7), we have

$$t_{n+1} - r = \frac{(t_n - r)^2}{2} \cdot g''(r).$$

Let $e_{n+1} = t_{n+1} - r, e_n = t_n - r$. Hence,

$$e_{n+1} = \frac{(t_n - r)^2}{2} \cdot g''(r) = \frac{{e_n}^2}{2} \cdot g''(r) \quad (8)$$

is obtained. The equation (8) implies that the bigeometric Newton method is quadratically convergent.

It is clear that the number of function evaluation in per iteration for method (3) is two. According to the definition of efficiency index [4], the efficiency index of (3) is $\sqrt{2} \cong 1.414$. Hence, the efficiency index of bigeometric Newton method defined by (3) is same as the ones of the classical Newton method and multiplicative Newton method.

4. Numerical examples

In this section, the new Bigeometric Newton method (nBGNM) obtained in this work to solve some non-linear equation $f(t) = 0$, $f(t) + 1 = 1$ for bigeometric methods. Summing the number of evaluations of function f with the number of evaluations of its derivative, the number of function evaluation (NFE) in per iteration is found. The results obtained via these methods are showed in Table 1.
The results show that nBGNM converging quadratically can compete with Newton method (NM) and GNM. Besides, nBGNM method is better than other methods in some case where newton method and multiplicative newton method diverge, such as example (b), (c). Further, in certain problems such as example (a), (d) (f), (g) and (h), nBGNM gives better results compared to NM and GNM. Also if we look closely to example (h), nBGNM that is developed in this work reaches the root in one step. This is not actually an amazing result. This case originate from bigeometric calculus.

Test functions:

- **(a)** $f(t) = \ln t - \sin(t)$, geometric and bigeometric version: $f(t) = \ln t - \sin(t) + 1$
- **(b)** $f(t) = \arctan t$, geometric and bigeometric version: $f(t) = \arctan t + 1$
- **(c)** $f(t) = t^{1/3}$, geometric and bigeometric version: $f(t) = t^{1/3} + 1$
- **(d)** $f(t) = \ln t$, geometric and bigeometric version: $f(t) = \ln t + 1$
- **(e)** $f(t) = e^{t^3 + 7t - 30} - 1$, geometric and bigeometric version: $f(t) = e^{t^3 + 7t - 30}$
- **(f)** $f(t) = t \cdot e^t - 1$, geometric and bigeometric version: $f(t) = t \cdot e^t$
- **(g)** $f(t) = (t-1)^6 - 1$, geometric and bigeometric version: $f(t) = (t-1)^6$
- **(h)** $f(t) = 5t^6 - 1$, geometric and bigeometric version: $f(t) = 5t^6$

.

Table 1. *Comparison of iterative methods*

	n	NFE	COC	t_n
(a), $t_0 = 1.5$				
NM	5	10	1.99	2.219107148913746
GNM	5	10	1.99	2.219107148913746
nBGNM	4	8	2.03	2.219107148913746
(b), $t_0 = 3$				
NM	-	-	-	divergence
GNM	-	-	-	divergence
nBGNM	31	62	0.95	0.000000000000000
(c), $t_0 = 3$				
NM	-	-	-	divergence
GNM	-	-	-	divergence
nBGNM	12	24	0.91	0.000000000000000
(d), $t_0 = 3$				
NM	6	12	2.00	1
GNM	-	-	-	divergence
nBGNM	5	10	2.00	1
(e), $t_0 = 3$				
NM	23	46	1.99	2.374101653848877
GNM	5	10	1.99	2.374101653848877
nBGNM	5	10	1.99	2.374101653848877
(f), $t_0 = 2$				
NM	7	14	1.99	0.567143290409784
GNM	6	12	2.00	0.567143290409784
nBGNM	5	10	1.99	0.567143290409784
(g), $t_0 = 3.5$				
NM	10	20	1.99	2
GNM	7	14	2.00	2
nBGNM	5	10	2.00	2
(h), $t_0 = 2$				
NM	10	20	1.99	0.764724491331730
GNM	8	16	1.99	0.764724491331730
nBGNM	1	2	ND	0.764724491331730

NM- Newton method
GNM- Geometric Newton method
nBGNM- New Bigeometric Newton method
ND- Not defined

n- Number of iterations
NFE- Number of functional evaluations
COC- Computational order of convergence

5. Conclusion

In this study, bigeometric Newton method is given as an an alternative to the classical Newton method with the help of bigeometric Taylor expansion. In addition, the quadratic convergence of the method is shown.

In the test functions given in this study, it has been seen that the bigeometric Newton Method is a better convergent method than the multiplicative Newton method [17] and the classical Newton method.

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