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Research Article

More on the Vague Complex Subhypergroups

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Abstract: The concept of a vague complex set provides a more comprehensive perspective than that of a vague set. This article aims to investigate vague complex and anti-vague complex subhypergroups (H_v-subgroups), supported by various examples with the help of vague complex sets and hyperstructures. Moreover, we explore their properties and relationships with vague and anti-vague subhypergroups.

Keywords: Vague complex set, vague complex subhypergroup, anti-vague complex subhypergroup.

1. Introduction

The integration of mathematics with other scientific fields, such as computer science, is highly significant and has been a primary focus of global research conducted by experts in hyperstructure theory over recent decades. Marty [8] introduced algebraic hyperstructures, which are a broader concept than traditional algebraic structures. In traditional algebraic structures, the combination of two elements produces another element, whereas in algebraic hyperstructures, it results in a set. Since this innovation, many extensive works have focused on this area of study (see [2]). Today, hyperstructures have various applications in different branches of mathematics and computer science and are explored in numerous researches around the world.

Vougiouklis [13–15] has made significant contributions to the field of generalized algebraic hyperstructures or in the area of H_v-structures. In traditional hyperstructure axioms, the concept of equality is replaced by the condition of having a non-empty intersection. Recently, many studies have introduced H_v-structures. For example, Davvaz and et al. have conducted extensive research on both hyperstructures and H_v-structures [1, 3]. Moreover, Davvaz [4] introduced the concept of fuzzy subhypergroups (or H_v-subgroups) within the context of hypergroups (or H_v-groups). A concise overview of the theory of fuzzy algebraic hyperstructures can be found in [5].

In fuzzy theory, fuzzy mathematics is a field closely related to the theory of fuzzy sets

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and logic. This idea, developed by Zadeh [16], extends traditional set theory. Fuzzy sets include elements that have varying degrees of membership. Unlike conventional set theory, where an element either belongs to the set or does not, fuzzy sets enable a gradual evaluation of membership levels, providing a more detailed perspective. This variability is expressed through a membership function that assigns values within the real interval $[0, 1]$. After Zadeh introduced fuzzy sets, numerous applications in mathematics and related fields emerged, providing researchers with significant motivation to explore various concepts. This has led to the expansion of abstract algebra within the framework of fuzzy settings. The study of fuzzy algebraic structures began with Rosenfeld's description of fuzzy subgroups [9]. The exploration of fuzzy hyperstructures has become a fascinating research area within fuzzy sets, and a considerable amount of research has been focused on examining the connections between fuzzy sets and hyperstructures.

As for vague set theory, a vague set consists of elements, each assigned a membership level within a continuous subinterval of $[0, 1]$. These sets are defined by both an accuracy membership function and an inaccuracy membership function. The concept of vague sets was introduced by Gau and Buehrer [6] as an extension of fuzzy set theory. On the other hand, Ramot [10, 11] introduced complex fuzzy sets, which further extend the idea of fuzzy sets. Then, Singh [12] proposed the concept of complex vague sets based on a lattice structure. Additionally, Husban and Salleh [7] explored complex vague relations, extending the scope of vague relations by expanding the range of accuracy and inaccuracy membership functions from the interval $[0, 1]$ to the unit circle in the complex plane.

The key benefit of vague sets which generalizes the fuzzy sets compared to fuzzy sets is that they distinguish between the positive and negative evidence for an element's membership in the set. With vague sets, we not only have an estimate of the likelihood that an element belongs to the set, but we also have lower and upper bounds on this likelihood. In light of the aforementioned studies, we want to define subhyperstructures using complex vague sets in order to expand the features of subhyperstructures that work with fuzzy complex sets.

The main aim of this work is to investigate the algebraic structures of subhypergroups using vague complex sets. After an Introduction, in Section 2 we present some definitions related to vague sets, hyperstructures and vague, anti-vague subhypergroups. In Section 3, we define vague complex subhypergroups and investigate their algebraic properties supporting different examples. In Section 4, we study anti-vague complex subhypergroups and investigate the relationships between these subhypergroups and vague complex subhypergroups.

2. Preliminaries

This section will provide foundational knowledge about vague complex H_v-subgroups. First, we will present definitions and theorems related to hyperstructures and vague subhyperstructures.

Definition 2.1 *[3] Let S be a non-void set. A mapping $*: S \times S \to P^*(S)$ is called a binary hyperoperation on S, where $P^*(S)$ represents the family of all non-void subsets of S. In this definition, the $(S,*)$ is called a hypergroupoid.*

In the above definition, if K and L are two non-void subsets of S and $e \in S$, then we define:

1. $K * L = \bigcup_{\substack{k \in K \\ l \in L}} k * l,$
2. $e * K = \{e\} * K$ *and* $K * e = K * \{e\}$.

Definition 2.2 *[3] A hypergroupoid $(S,*)$ is called as:*

- *Semihypergroup: If for all $k, l, m \in S$, then $k * (l * m) = (k * l) * m$,*
- *Quasihypergroup: If for all $k \in S$, we have $k * S = S = S * k$ (This condition is known as the reproduction property),*
- *Hypergroup: If it satisfies both the conditions of a semihypergroup and a quasihypergroup,*
- *H_v-group: If it is a quasihypergroup and for all $k, l, m \in S$, we have $k*(l*m) \cap (k*l)*m \neq \varnothing$.*

Definition 2.3 *[3] Let $(S,*)$ be a hypergroup (or H_v-group) and $M \subseteq S$. The pair $(M,*)$ is considered as a subhypergroup (or H_v-subgroup) of $(S,*)$ if for every $m \in M$, the requirement $m * M = M = M * m$ holds.*

Definition 2.4 *[16] A fuzzy set, defined on a universe of discourse A is characterized by a membership function $\lambda_F(x)$ which assigns any element a grade of membership in F. The fuzzy set can be represented as follows:*

$$F = \{(x, \lambda_F(x)) : x \in A\},$$

where $\lambda_F(x)$ belongs to the interval $[0,1]$.

Definition 2.5 *[6] A vague set V in the universe of discourse S is characterized by two membership functions:*

$$\textit{An accuracy membership function } \; t_V : S \to [0,1],$$

$$\textit{An inaccurate membership function } \; f_V : S \to [0,1].$$

Here, $t_V(s)$ represents the lower bound of grade of membership of s derived from the "evidence for s" and $f_V(s)$ represents the lower bound of negation of membership of s derived from the "evidence against s".

Additionally, $0 \le t_V(s) + f_V(s) \le 1$. The grade of membership of s in the vague set V is bounded within the subinterval $[t_V(s), 1-f_V(s)]$ of $[0,1]$. This implies that if the actual grade of membership is represented by $\lambda_F(s)$, then $t_V(s) \le \lambda_F(s) \le 1 - f_V(s)$. Vague set V is denoted by:

$$V = \{(s, t_V(s), 1 - f_V(s)) : s \in S\}$$

In this representation, $[t_V(s), 1 - f_V(s)]$ means the "vague value" of s in V.

Definition 2.6 *[6] The complement of a vague set V is represented by V^C and is characterized as*

$$t_{V^C}(s) = f_V(s)$$

$$1 - f_{V^C}(s) = 1 - t_V(s)$$

Definition 2.7 *[6] A vague set U is contained in the other vague set V, $U \subseteq V$, if and only if $t_U \le t_V$ and $1 - f_U \le 1 - 1 - f_V$.*

Definition 2.8 *[6] Union of two vague sets K and L is a vague set M, written as $M = K \bigcup L$, whose accuracy membership and inaccurate membership functions are associated with those of K and L with*

$$t_M = \max(t_K, t_L)$$

$$1 - f_M = \max(1 - f_K, 1 - f_L) = 1 - \min(f_K, f_L)$$

The union of two vague sets K and L is the smallest vague set containing both K and L.

Definition 2.9 *[6] Intersection of two vague sets K and L is a vague set N, written as $N = K \bigcap L$, whose accuracy membership and inaccurate membership functions are associated with those of K and L with*

$$t_N = \min(t_K, t_L)$$

$$1 - f_N = \min(1 - f_K, 1 - f_L) = 1 - \max(f_K, f_L)$$

The intersection of two vague sets K and L is the largest vague set contained in both K and L.

Definition 2.10 *Suppose that $(S, *)$ is a hypergroup (H_v-group) and V is a vague subset of S with accuracy membership function $t_V(s) \in [0,1]$ and inaccurate membership function $f_V(s) \in [0,1]$. If the following conditions hold, V is called a vague subhypergroup (H_v-subgroup) of S :*

1. $min\{t_V(\gamma), t_V(\delta)\} \leq inf\{t_V(\epsilon) : \epsilon \in \gamma * \delta\}$ *for every* $\gamma, \delta \in S$,

2. $min\{1 - f_V(\gamma), 1 - f_V(\delta)\} \leq inf\{1 - f_V(\epsilon) : \epsilon \in \gamma * \delta\}$ *for every* $\gamma, \delta \in S$,

3. *For every* $d, \gamma \in S$, *there exists* $\eta \in S$ *such that* $d \in \gamma * \eta$ *and* $min\{t_V(d), t_V(\gamma)\} \leq t_V(\eta)$,

4. *For every* $d, \gamma \in S$, *there exists* $\eta \in S$ *such that* $d \in \gamma * \eta$ *and* $min\{1 - f_V(d), 1 - f_V(\gamma)\} \leq 1 - f_V(\eta)$,

5. *For every* $d, \gamma \in S$, *there exists* $\rho \in S$ *such that* $d \in \rho * \gamma$ *and* $min\{t_V(d), t_V(\gamma)\} \leq t_V(\rho)$,

6. *For every* $d, \gamma \in S$, *there exists* $\rho \in S$ *such that* $d \in \rho * \gamma$ *and* $min\{1 - f_V(d), 1 - f_V(\gamma)\} \leq 1 - f_V(\rho)$.

Lemma 2.11 *Assume that* $(S, *)$ *is a hypergroup* (H_v*−group) and*

$$V = \{(s, t_V(s), 1 - f_V(s)) : s \in S\}$$

is a vague subhypergroup (H_v*−subgroup) of* S. *Then, for every* $g_1, g_2, \ldots, g_n \in S$

$$min\{t_V(g_1), t_V(g_2), \ldots, t_V(g_n)\} \leq inf\{t_V(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\}$$

and

$$min\{1 - f_V(g_1), 1 - f_V(g_2), \ldots, 1 - f_V(g_n)\} \leq inf\{1 - f_V(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\}.$$

Definition 2.12 *Suppose that* $(S, *)$ *is a hypergroup* (H_v*−group) and* V *is a vague subset of* S *with accuracy membership function* $t_V(s) \in [0, 1]$ *and inaccurate membership function* $f_V(s) \in [0, 1]$. *If the following conditions hold,* V *is called an anti-vague subhypergroup* (H_v*−subgroup) of* S*:*

1. $sup\{t_V(\epsilon) : \epsilon \in \gamma * \delta\} \leq max\{t_V(\gamma), t_V(\delta)\}$ *for every* $\gamma, \delta \in S$,

2. $sup\{1 - f_V(\epsilon) : \epsilon \in \gamma * \delta\} \leq max\{1 - f_V(\gamma), 1 - f_V(\delta)\}$ *for every* $\gamma, \delta \in S$,

3. *For every* $d, \gamma \in S$, *there exists* $\eta \in S$ *such that* $d \in \gamma * \eta$ *and* $t_V(\eta) \leq max\{t_V(d), t_V(\gamma)\}$,

4. *For every* $d, \gamma \in S$, *there exists* $\eta \in S$ *such that* $d \in \gamma * \eta$ *and* $1 - f_V(\eta) \leq max\{1 - f_V(d), 1 - f_V(\gamma)\}$,

5. *For every* $d, \gamma \in S$, *there exists* $\rho \in S$ *such that* $d \in \rho * \gamma$ *and* $t_V(\rho) \leq max\{t_V(d), t_V(\gamma)\}$,

6. For every $d, \gamma \in S$, there exists $\rho \in S$ such that $d \in \rho * \gamma$ and $1 - f_V(\rho) \leq max\{1 - f_V(d), 1 - f_V(\gamma)\}$.

Lemma 2.13 *Assume that $(S, *)$ is a hypergroup (H_v-group) and V an anti-vague subhypergroup (H_v-subgroup) of S. Then, for every $g_1, g_2, \ldots, g_n \in S$*

$$max\{t_V(g_1), t_V(g_2), \ldots, t_V(g_n)\} \geq sup\{t_V(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\}$$

and

$$max\{1 - f_V(g_1), 1 - f_V(g_2), \ldots, 1 - f_V(g_n)\} \geq sup\{1 - f_V(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\}.$$

Theorem 2.14 *Assume that $(S, *)$ is a hypergroup (H_v-group) and V a vague subset of S. V is a vague subhypergroup (H_v-subgroup) of S if and only if V^C is an anti-vague subypergroup (H_v-subgroup) of S.*

3. Vague Complex Subhypergroups

In this part, we will present properties of the notion of vague complex subsets to clarify and characterize vague complex subhypergroups (H_v-subgroups). Then, we will explain the properties of vague complex subhypergroups.

Definition 3.1 *Suppose that $V = \{(s, t_V(s), 1 - f_V(s)) : s \in S\}$ is a vague set. Then, the $\pi-$vague set V_π is defined as $V_\pi = \{(s, 2\pi t_V(s), 2\pi(1 - f_V(s)) : s \in S\}$, where it must satisfies $0 \leq 2\pi t_V(s) + 2\pi(1 - f_V(s)) \leq 2\pi$.*

Proposition 3.2 *Suppose that $(S, *)$ is a hypergroup (H_v-group). V is a vague subhypergroup (H_v-subgroup) of S if and only if a $\pi-$vague set V_π is a π-vague subhypergroup (H_v-subgroup) of S.*

Proof The proof is clear. □

Definition 3.3 *[7] A vague complex set V in the universe of discourse S is characterized with two complex membership functions:*
A complex accuracy membership function

$$\widehat{t}_V(s) : S \to \{\alpha : \alpha \in \mathbb{C}, |\alpha| \leq 1\} \text{ and } \widehat{t}_V(s) = t_V(s) e^{i\alpha w_V^t(s)}.$$

A complex inaccurate membership function

$$\widehat{f}_V(s) : S \to \{\alpha : \alpha \in \mathbb{C}, |\alpha| \leq 1\} \text{ and } \widehat{f}_V(s) = 1 - f_V(s) e^{i\alpha(2\pi - w_V^f(s))}.$$

Here, $\widehat{t_V}(s)$ *stands for the lower bound of the complex grade of membership* s *derived from "evidence for* s *"and* $\widehat{f_V}(s)$ *stands for the lower bound on the negation of* s *derived from the "evidence against* s *".*

In this definition, $0 \leq t_V(s) + f_V(s) \leq 1$ *with both* $t_V(s)$ *and* $f_V(s)$ *is real valued, belonging to the interval* $[0,1]$*. Additionally,* $\alpha \in [0, 2\pi]$ *and* $w_V^t(s), w_V^f(s) \in [0,1]$*. The complex vague set* V *is symbolized as*

$$V = \{(s, \widehat{t_V}(s) = t_V(s)e^{i\alpha w_V^t(s)}, \widehat{f_V}(s) = 1 - f_V(s)e^{i\alpha(2\pi - w_V^f(s))}) : s \in S\}.$$

In this paper, $[t_V(s)e^{i\alpha w_V^t(s)}, 1 - f_V(s)e^{i\alpha(2\pi - w_V^f(s))}]$ *means vague complex rate of* s *in* V*.*

Definition 3.4 *[7] Let* $V = \{(s, t_V(s)e^{i\alpha w_V^t(s)}, 1 - f_V(s)e^{i\alpha(2\pi - w_V^f(s))}) : s \in S\}$ *be a vague complex subset. The complement of* V *is characterized as*

$$V^C = \{(s, f_V(s)e^{i\alpha w_V^f(s)}, 1 - t_V(s)e^{i\alpha(2\pi - w_V^t(s))}) : s \in S\}.$$

Definition 3.5 *[7] Let* $K = \{(s, t_K(s)e^{i\alpha w_K^t(s)}, 1 - f_K(s)e^{i\alpha(2\pi - w_K^f(s))}) : s \in S\}$ *and* $L = \{(s, t_L(s)e^{i\alpha w_L^t(s)}, 1 - f_L(s)e^{i\alpha(2\pi - w_L^f(s))}) : s \in S\}$ *be two vague complex subsets of the identical universe* S*. Union of* K *and* L*, represented by* $M = K \bigcup L$*, is determined as:*

$$M = \{(s, \widetilde{t_M}(s), \widetilde{f_M}(s)) : s \in S\},$$

where

$$\widetilde{t_M}(s) = \max(t_K(s), t_L(s))e^{i\alpha \max(w_K^t(s), w_L^t(s))}$$

and

$$\widetilde{f_M}(s) = \max(1 - f_K(s), 1 - f_L(s))e^{i\alpha \max(2\pi - w_K^f(s), 2\pi - w_L^f(s))}.$$

Definition 3.6 *[7] Let* $K = \{(s, t_K(s)e^{i\alpha w_K^t(s)}, 1 - f_K(s)e^{i\alpha(2\pi - w_K^f(s))}) : s \in S\}$ *and* $L = \{(s, t_L(s)e^{i\alpha w_L^t(s)}, 1 - f_L(s)e^{i\alpha(2\pi - w_L^f(s))}) : s \in S\}$ *be two vague complex subsets of the identical universe* S*. Intersection of* K *and* L*, represented by* $N = K \bigcap L$*, is determined as:*

$$N = \{(s, \widetilde{\widetilde{t_N}}(s), \widetilde{\widetilde{f_N}}(s)) : s \in S\},$$

where

$$\widetilde{\widetilde{t_N}}(s) = \min(t_K(s), t_L(s))e^{i\alpha \min(w_K^t(s), w_L^t(s))}$$

and

$$\widetilde{\widetilde{\widetilde{f_N}}}(s) = \min(1 - f_K(s), 1 - f_L(s))e^{i\alpha \min(2\pi - w_K^f(s), 2\pi - w_L^f(s))}.$$

Definition 3.7 *Let* $K = \{(s, t_K(s)e^{i\alpha w_K^t(s)}, 1 - f_K(s)e^{i\alpha(2\pi - w_K^f(s))}) : s \in S\}$ *and* $L = \{(s, t_L(s)e^{i\alpha w_L^t(s)}, 1 - f_L(s)e^{i\alpha(2\pi - w_L^f(s))}) : s \in S\}$ *be two vague complex sets of a non-void set* S *with the accuracy membership functions* $t_K(s)e^{i\alpha w_K^t(s)}$ *and* $t_L(s)e^{i\alpha w_L^t(s)}$ *and inaccurate membership functions* $1 - f_K(s)e^{i\alpha(2\pi - w_K^f(s))}$ *and* $1 - f_L(s)e^{i\alpha(2\pi - w_L^f(s))}$ *respectively. Then*

1. *A vague complex subset* K *is called homogeneous if for every* $m, s \in S$*, we have that*

$$t_K(m) \leq t_K(s) \text{ if and only if } w_K^t(m) \leq w_K^t(s)$$

and

$1 - f_K(m) \leq 1 - f_K(s)$ *if and only if* $2\pi - w_K^f(m) \leq 2\pi - w_K^f(s)$.

2. *A vague complex subset* K *is called homogeneous by* L *if for every* $m, s \in S$*, we have that*

$$t_K(m) \leq t_L(s) \text{ if and only if } w_K^t(m) \leq w_L^t(s)$$

and

$$1 - f_K(m) \leq 1 - f_L(s) \text{ if and only if } 2\pi - w_K^f(m) \leq 2\pi - w_L^f(s).$$

Notation 3.8 *Let* $K = \{(s, t_K(s)e^{i\alpha w_K^t(s)}, 1 - f_K(s)e^{i\alpha(2\pi - w_K^f(s))}) : s \in S\}$ *and* $L = \{(s, t_L(s)e^{i\alpha w_L^t(s)}, 1 - f_L(s)e^{i\alpha(2\pi - w_L^f(s))}) : s \in S\}$ *be two vague complex sets of a non-empty set* S *with the accuracy membership functions* $t_K(s)e^{i\alpha w_K^t(s)}$ *and* $t_L(s)e^{i\alpha w_L^t(s)}$ *and inaccurate membership functions* $1 - f_K(s)e^{i\alpha(2\pi - w_K^f(s))}$ *and* $1 - f_L(s)e^{i\alpha(2\pi - w_L^f(s))}$ *respectively.*

- *With* $t_K(s)e^{i\alpha w_K^t(s)} \leq t_L(s)e^{i\alpha w_L^t(s)}$, *it refers that* $t_K(s) \leq t_L(s)$ *and* $w_K^t(s) \leq w_L^t(s)$.
- *With* $1 - f_K(s)e^{i\alpha(2\pi - w_K^f(s))} \leq 1 - f_L(s)e^{i\alpha(2\pi - w_L^f(s))}$, *it refers that* $1 - f_K(s) \leq 1 - f_L(s)$ *and* $2\pi - w_K^f(s) \leq 2\pi - w_L^f(s)$.

In this work, all vague complex sets are considered homogeneous.

Definition 3.9 *Suppose that* $(S, *)$ *is a hypergroup (*H_v*−group) and*

$$V = \{(s, \widehat{t_V}(s) = t_V(s)e^{i\alpha w_V^t(s)}, \widehat{f_V}(s) = 1 - f_V(s)e^{i\alpha(2\pi - w_V^f(s))}) : s \in S\}$$

is a (homogeneous) vague complex subset of S*. If the following conditions are satisfied,* V *is called a vague complex subhypergroup (*H_v-*subgroup) of* S*:*

1. $min\{\widehat{t_V}(\gamma), \widehat{t_V}(\delta)\} \leq inf\{\widehat{t_V}(\epsilon) : \epsilon \in \gamma * \delta\}$ *for every* $\gamma, \delta \in S$,
2. $min\{\widehat{f_V}(\gamma), \widehat{f_V}(\delta)\} \leq \ inf\{\widehat{f_V}(\epsilon) : \epsilon \in \gamma * \delta\}$ *for every* $\gamma, \delta \in S$,
3. *For every* $d, \gamma \in S$*, there exists* $\eta \in S$ *such that* $d \in \gamma * \eta$ *and* $min\{\widehat{t_V}(d), \widehat{t_V}(\gamma)\} \leq \widehat{t_V}(\eta)$,
4. *For every* $d, \gamma \in S$*, there exists* $\eta \in S$ *such that* $d \in \gamma * \eta$ *and* $min\{\widehat{f_V}(d), v_A(\gamma)\} \leq \widehat{f_V}(\eta)$,
5. *For every* $d, \gamma \in S$*, there exists* $\rho \in S$ *such that* $d \in \rho * \gamma$ *and* $min\{\widehat{t_V}(d), \widehat{t_V}(\gamma)\} \leq \widehat{t_V}(\rho)$,
6. *For every* $d, \gamma \in S$*, there exists* $\rho \in S$ *such that* $d \in \rho * \gamma$ *and* $min\{\widehat{f_V}(d), v_A(\gamma)\} \leq \widehat{f_V}(\rho)$.

Example 3.10 *Let* $S = \{\gamma, \delta\}$ *and define hypergroup* $(S, *)$ *by next table:*

$*$	γ	δ
γ	γ	S
δ	S	δ

Describe a vague complex subset V *of* S *as:*

$$t_V(\gamma)e^{i\alpha w_V^t(\gamma)} = 0.3e^{i0} \quad and \ 1 - f_V(\gamma)e^{i\alpha(2\pi - w_V^f(\gamma))} = 0.6e^{i\pi},$$

$$t_V(\delta)e^{i\alpha w_V^t(\delta)} = 0.4e^{i\pi} \quad and \ 1 - f_V(\delta)e^{i\alpha(2\pi - w_V^f(\delta))} = 0.7e^{i3\pi/2}.$$

In that case, V *is homogeneous vague complex subhypergroup of* S*.*

Example 3.11 *Let* $S = \{\gamma, \delta, \epsilon\}$ *and define the hypergroup* $(S, *)$ *as follows:*

$*$	γ	δ	ϵ
γ	$\{\gamma\}$	$\{\delta\}$	$\{\epsilon\}$
δ	$\{\delta\}$	$\{\gamma\}$	$\{\delta, \epsilon\}$
ϵ	$\{\epsilon\}$	$\{\delta, \epsilon\}$	$\{\gamma\}$

Describe a vague complex subset V *of* S *as:*

$$t_V(\gamma)e^{i\alpha w_V^t(\gamma)} = 0.4e^{i\pi/2} \quad and \ 1 - f_V(\gamma)e^{i\alpha(2\pi - w_V^f(\gamma))} = 0.4e^{i\pi/2},$$

$$t_V(\delta)e^{i\alpha w_V^t(\delta)} = 0.5e^{i\pi} \quad and \ 1 - f_V(\delta)e^{i\alpha(2\pi - w_V^f(\delta))} = 0.6e^{i\pi},$$

$$t_V(\epsilon)e^{i\alpha w_V^t(\epsilon)} = 0.5e^{i3\pi/2} \quad and \ 1 - f_V(\epsilon)e^{i\alpha(2\pi - w_V^f(\epsilon))} = 0.7e^{i5\pi/3}.$$

In that case, V is homogeneous vague complex subhypergroup of S.

Theorem 3.12 *Suppose that $(S,*)$ is a hypergroup (H_v-group), V is a (homogeneous) vague complex subset of S. V is a vague complex subhypergroup (H_v-subgroup) of S if and only if t_V and $1-f_V$ are vague subhypergroups (or H_v-subgroups) of S and w_V^t and $2\pi - w_V^f$ are $\pi-$vague subhypergroups (or H_v-subgroups) of S.*

Proof $(\Longrightarrow)$: Let V be a vague complex subhypergroup (H_v-subgroup) of S. We want to demostrate requirements of description of vague subhypergroups are satisfied for t_V, $1-f_V$ and w_V^t, $2\pi - w_V^f$. For every $\gamma, \delta \in S$, we have

$$min\{\widehat{t_V}(\gamma), \widehat{t_V}(\delta)\} \leq inf\{\widehat{t_V}(\epsilon) : \epsilon \in \gamma * \delta\},$$

$$min\{\widehat{f_V}(\gamma), \widehat{f_V}(\delta)\} \leq inf\{\widehat{f_V}(\epsilon) : \epsilon \in \gamma * \delta\}.$$

Notation 3.8 means that

$$inf\{t_V(\epsilon) : \epsilon \in \gamma * \delta\} \geq min\{t_V(\gamma), t_V(\delta)\},$$

$$inf\{1-f_V(\epsilon) : \epsilon \in \gamma * \delta\} \geq min\{1-f_V(\gamma), 1-f_V(\delta)\}$$

and

$$inf\{w_V^t(\epsilon) : \epsilon \in \gamma * \delta\} \geq min\{w_V^t(\gamma), w_V^t(\delta)\},$$

$$inf\{2\pi - w_V^f(\epsilon) : \epsilon \in \gamma * \delta\} \geq min\{2\pi - w_V^f(\gamma), 2\pi - w_V^f(\delta)\}.$$

Let $\gamma, d \in S$. There exist $\eta, \rho \in S$ such that $d \in \gamma * \eta$ and $d \in \rho * \gamma$. Then,

$$min\{\widehat{t_V}(\gamma)\widehat{t_V}(d)\} \leq \widehat{t_V}(\eta)$$

$$min\{\widehat{f_V}(\gamma), \widehat{f_V}(d)\} \leq \widehat{f_V}(\eta)$$

and

$$min\{\widehat{t_V}(\gamma), \widehat{t_V}(d)\} \leq \widehat{t_V}(\rho)$$

$$min\{\widehat{f_V}(\gamma), \widehat{f_V}(d)\} \leq \widehat{f_V}(\rho).$$

Notation 3.8 means that the conditions $3-6$ of definition of vague subhypergroup are satisfied for both t_V , $1-f_V$ and w_V^t , $2\pi - w_V^f$.

$(\Longleftarrow)$: Let t_V and $1-f_V$ be vague subhypergroups (or H_v-subgroups) of S and w_V^t , $2\pi - w_V^f$ be $\pi-$vague subhypergroups (or H_v-subgroups) of S. We want to show that the conditions of vague complex subhypergroups are provided. For every $\gamma, \delta \in S$, we have

$$inf\{t_V(\epsilon) : \epsilon \in \gamma * \delta\} \geq min\{t_V(\gamma), t_V(\delta)\},$$

$$inf\{1 - f_V(\epsilon) : \epsilon \in \gamma * \delta\} \geq min\{1 - f_V(\gamma), 1 - f_V(\delta)\}$$

and

$$inf\{w_V^t(\epsilon) : \epsilon \in \gamma * \delta\} \geq min\{w_V^t(\gamma) w_V^t(\delta)\},$$

$$inf\{2\pi - w_V^f(\epsilon) : \epsilon \in \gamma * \delta\} \geq min\{2\pi - w_V^f(\gamma), 2\pi - w_V^f(\delta)\}.$$

Notation 3.8 means that

$$min\{\widehat{t_V}(\gamma), \widehat{t_V}(\delta)\} \leq inf\{\widehat{t_V}(\epsilon) : \epsilon \in \gamma * \delta\},$$

$$min\{\widehat{f_V}(\gamma), \widehat{f_V}(\delta)\} \leq inf\{\widehat{f_V}(\epsilon) : \epsilon \in \gamma * \delta\}.$$

Suppose that $\gamma, d \in S$. There exist $\eta, \rho \in S$ such that $d \in \gamma * \eta$ and $d \in \rho * \gamma$

$$min\{t_V(\gamma), t_V(d)\} \leq t_V(\eta) \text{ and } min\{w_V^t(\gamma), w_V^t(d)\} \leq w_V^t(\eta),$$

$$min\{1 - f_V(\gamma), 1 - f_V(d)\} \leq 1 - f_V(\eta) \text{ and } min\{2\pi - w_V^f(\gamma), 2\pi - w_V^f(d)\} \leq 2\pi - w_V^f(\eta)$$

and

$$min\{t_V(\gamma), t_V(d)\} \leq t_V(\rho) \text{ and } min\{w_V^t(\gamma), w_V^t(d)\} \leq w_V^t(\rho),$$

$$min\{1 - f_V(\gamma), 1 - f_V(d)\} \leq 1 - f_V(\rho) \text{ and } min\{2\pi - w_V^f(\gamma), 2\pi - w_V^f(d)\} \leq 2\pi - w_V^f(\rho).$$

Notation 3.8 means that the conditions $3 - 6$ of definition of vague complex subhypergroup are satisfied for V. □

Lemma 3.13 *Suppose that $(S, *)$ is a hypergroup (H_v− group) and V is a (homogeneous) vague complex subhypergroup (H_v− subgroup) of S. For every $g_1, g_2, \ldots, g_n \in S$*

$$min\{\widehat{t_V}(g_1), \widehat{t_V}(g_2), \ldots, \widehat{t_V}(g_n)\} \leq inf\{\widehat{t_V}(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\}$$

and

$$min\{\widehat{f_V}(g_1), \widehat{f_V}(g_2), \ldots, \widehat{f_V}(g_n)\} \leq inf\{\widehat{f_V}(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\}.$$

Proof Let $g_1, g_2, \ldots, g_n \in S$ and

$$\widehat{t_V}(s) = t_V(s) e^{i\alpha w_V^t(s)}, \widehat{f_V}(s) = 1 - f_V(s) e^{i\alpha(2\pi - w_V^f(s))}.$$

To demonstrate the lemma, it suffices to indicate that

$$min\{t_V(g_1), t_V(g_2), \ldots, t_V(g_n)\} \leq inf\{t_V(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\},$$

$$min\{w_V^t(g_1), w_V^t(g_2), \ldots, w_V^t(g_n)\} \leq inf\{w_V^t(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\}$$

and

$$min\{1 - f_V(g_1), 1 - f_V(g_2), \ldots, 1 - f_V(g_n)\} \leq inf\{1 - f_V(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\},$$

$$min\{2\pi - w_V^f(g_1), 2\pi - w_V^f(g_2), \ldots 2\pi - w_V^f(g_n)\} \leq inf\{2\pi - w_V^f(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\}.$$

Because V is homogeneous, it suffices to indicate that

$$min\{t_V(g_1), t_V(g_2), \ldots, t_V(g_n)\} \leq inf\{t_V(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\},$$

$$min\{1 - f_V(g_1), 1 - f_V(g_2), \ldots, 1 - f_V(g_n)\} \leq inf\{1 - f_V(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\}.$$

Theorem 3.12 claims that t_V and $1 - f_V$ are vague subhypergroups (or H_v-subgroups) of S. Using Lemma 2.11 , the proof is completed. □

Definition 3.14 *Assume that*

$$V = \{(s, \widehat{t}_V(s) = t_V(s)e^{i\alpha w_V^t(s)}, \widehat{f}_V(s) = 1 - f_V(s)e^{i\alpha(2\pi - w_V^f(s))}) : s \in S\}$$

is a (homogeneous) vague complex subset of a non-empty set S. Level subset $V_{(r,t)}$ of S is described as $V_{(r,t)} = \{s \in S : \widehat{t}_V(s) \geq r$ and $\widehat{f}_V(s) \geq t\}$, where $r = me^{i\varphi}, t = ne^{i\psi}$ such that $m, n \in [0, 1]$ and $\varphi, \psi \in [0, 2\pi]$.

Remark 3.15 *$\widehat{t}_V(x) \geq r$ means that $x \in \widehat{t}_r$. Similarly, $\widehat{f}_V(x) \geq t$ means that $x \in \widehat{f}_t$.*

Remark 3.16 *Let $V = \{(s, \widehat{t}_V(s) = t_V(s)e^{i\alpha w_V^t(s)}, \widehat{f}_V(s) = 1 - f_V(s)e^{i\alpha(2\pi - w_V^f(s))}) : s \in S\}$ be a (homogeneous) vague complex subset of a non-empty set S. Followings are true:*

i. Provided that $r_1 \leq r_2$, then $\widehat{t}_{r_2} \subseteq \widehat{t}_{r_1}$,

ii. $\widehat{t}_{0e^{0i}} = S$,

iii. Provided that $t_1 \leq t_2$, then $\widehat{f}_{t_2} \subseteq \widehat{f}_{t_1}$,

iv. $\widehat{f}_{0e^{0i}} = S$.

Theorem 3.17 *Suppose that $(S, *)$ is a hypergroup (H_v-group) and V is a (homogeneous) vague complex subset of S. V is a vague complex subhypergroup (H_v-subgroup) of S if and only if for all $r = ke^{i\psi}$, $t = me^{i\varphi}$ such that $k, m \in [0, 1]$ and $\psi, \varphi \in [0, 2\pi]$, $V_{(r,t)} \neq \varnothing$ is a subhypergroup (H_v-subgroup) of S.*

Proof Let V be a vague complex subhypergroup (H_v-subgroup) of S and $\gamma, \delta \in V_{(r,t)} \neq \emptyset$. For every $\epsilon \in \gamma * \delta$, we have

$$\widehat{t_V}(\epsilon) \geq min\{\widehat{t_V}(\gamma), \widehat{t_V}(\delta)\} \geq r,$$

$$\widehat{f_V}(\epsilon) \geq min\{\widehat{f_V}(\gamma), \widehat{f_V}(\delta)\} \geq t.$$

Hence, $\epsilon \in \gamma * \delta \subseteq V_{(r,t)}$ and for all $\epsilon \in V_{(r,t)}$, we have $\epsilon * V_{(r,t)} \subseteq V_{(r,t)}$. Moreover, let $\gamma \in V_{(r,t)}$, using definition of vague complex subhypergroup, there exists $\eta \in S$ such that $\gamma \in \epsilon * \eta$ and

$$\widehat{t_V}(\eta) \geq min\{\widehat{t_V}(\epsilon), \widehat{t_V}(\gamma)\} \geq r,$$

$$\widehat{f_V}(\eta) \geq min\{\widehat{f_V}(\epsilon), \widehat{f_V}(\gamma)\} \geq t.$$

Therefore, this means $\eta \in V_{(r,t)}$. We can get $V_{(r,t)} * \epsilon \subseteq V_{(r,t)}$ using definition of vague complex subhypergroup.

For the converse, let $r = ke^{i\psi}, t = me^{i\varphi}$ such that $k, m \in [0,1]$ and $\psi, \varphi \in [0, 2\pi]$ and $V_{(r,t)} \neq \emptyset$ be a subhypergroup (H_v-subgroup) of S. Suppose that

$$r_0 = k_0 e^{i\psi_0} = min\{\widehat{t_V}(\gamma), \widehat{t_V}(\delta)\},$$

$$t_0 = m_0 e^{i\varphi_0} = min\{\widehat{f_V}(\gamma), \widehat{f_V}(\delta)\}.$$

Then,

$$k_0 = \ min\{t_V(\gamma)t_V(\delta)\},$$

$$\psi_0 = min\{w_V^t(\gamma), w_V^t(\delta)\}$$

and

$$m_0 = \ min\{1 - f_V(\gamma), 1 - f_V(\delta)\},$$

$$\varphi_0 = \ min\{2\pi - w_V^f(\gamma), 2\pi - w_V^f(\delta)\}.$$

Because $\gamma, \delta \in V_{(r_0,t_0)}$ and $V_{(r_0,t_0)}$ is a a subhypergroup (H_v-subgroup) of S, we have $\gamma * \delta \subseteq V_{(r_0,t_0)}$. Thus, for all $\epsilon \in \gamma * \delta$

$$\widehat{t_V}(\epsilon) \geq r_0 = min\{\widehat{t_V}(\gamma), \widehat{t_V}(\delta)\},$$

$$\widehat{f_V}(\epsilon) \geq t_0 = min\{\widehat{f_V}(\gamma), \widehat{f_V}(\delta)\}$$

and hence, conditions $1, 2$ of the definition of vague complex subhypergroup are satisfied. Similarly, we will prove conditions $3 - 6$ are obtained. For every $\epsilon, \gamma \in S$, setting by

$$r_1 = k_1 e^{i\psi_1} = min\{\widehat{t_V}(\gamma), \widehat{t_V}(\epsilon)\},$$

$$t_1 = m_1 e^{i\varphi_1} = min\{\widehat{f_V}(\gamma), \widehat{f_V}(\epsilon)\}.$$

Then, $\gamma, \epsilon \in V_{(r_1,t_1)}$. With $V_{(r_1,t_1)}$ is a subhypergroup (H_v-subgroup) of S refers $\epsilon * V_{(r_1,t_1)} = V_{(r_1,t_1)}$. The latter means there exists $\delta \in V_{(r_1,t_1)}$ such that $\gamma \in \epsilon * \delta$. Hence,

$$\widehat{t_V}(\delta) \geq r_1 = min\{\widehat{t_V}(\gamma), \widehat{t_V}(\epsilon)\},$$

$$\widehat{f_V}(\delta) \geq t_1 = min\{\widehat{f_V}(\gamma), \widehat{f_V}(\epsilon)\}.$$

□

Corollary 3.18 *Assume that $(S, *)$ is a hypergroup (H_v-group) and V is a (homogeneous) vague complex subhypergroup (H_v-subgroup) of S with*

$$V = \{(s, \widehat{t_V}(s) = t_V(s)e^{i\alpha w_V^t(s)}, \widehat{f_V}(s) = 1 - f_V(s)e^{i\alpha(2\pi - w_V^f(s))}) : s \in S\}.$$

If $0e^{0i} \leq r_1 = s_1e^{i\theta_1} < r_2 = s_2e^{i\theta_2} \leq 1e^{2\pi i}$, then $\widehat{t}_{r_1} = \widehat{t}_{r_2}$ if and only if there is no $\sigma \in S$ such that $r_1 \leq \widehat{t_V}(\sigma) < r_2$ and similarly if $0e^{0i} \leq m_1 = n_1e^{i\varphi_1} < m_2 = n_2e^{i\varphi_2} \leq 1e^{2\pi i}$, then $\widehat{f}_{m_1} = \widehat{f}_{m_2}$ if and only if there is no $\sigma \in S$ such that $m_1 \leq \widehat{f_V}(\sigma) < m_2$.

Proof Suppose that $0e^{0i} \leq r_1 = s_1e^{i\theta_1} < r_2 = s_2e^{i\theta_2} \leq 1e^{2\pi i}$ such that $\widehat{t}_{r_1} = \widehat{t}_{r_2}$. Assume that there exists $\sigma \in S$ such that $r_1 \leq \widehat{t_V}(\sigma) < r_2$. We have $\sigma \in \widehat{t}_{r_1} = \widehat{t}_{r_2}$. This means that $\widehat{t_V}(\sigma) \geq r_2$ and it is contradiction. Similarly, let $0e^{0i} \leq m_1 = n_1e^{i\varphi_1} < m_2 = n_2e^{i\varphi_2} \leq 1e^{2\pi i}$ such that $\widehat{f}_{m_1} = \widehat{f}_{m_2}$. Assume that there exists $\sigma \in S$ such that $m_1 \leq \widehat{f_V}(\sigma) < m_2$. We have $\sigma \in \widehat{f}_{m_1} = \widehat{f}_{m_2}$. This means that $\widehat{f_V}(\sigma) \geq m_2$ and it is a contradiction.

Because of $0e^{0i} \leq r_1 = s_1e^{i\theta_1} < r_2 = s_2e^{i\theta_2} \leq 1e^{2\pi i}$, it follows by Remark 3.16 that $\widehat{t}_{r_2} \subseteq \widehat{t}_{r_1}$. In order to show that $\widehat{t}_{r_1} \subseteq \widehat{t}_{r_2}$, let $\sigma \in \widehat{t}_{r_1}$. Then, $\widehat{t_V}(\sigma) \geq r_1$. Because there is no $\sigma \in S$ such that $r_1 \leq \widehat{t_V}(\sigma) < r_2$, it follows that $\widehat{t_V}(\sigma) \geq r_2$. Thus, $\sigma \in \widehat{t}_{r_2}$ and $\widehat{t}_{r_1} \subseteq \widehat{t}_{r_2}$. Similarly, since $0e^{0i} \leq m_1 = n_1e^{i\varphi_1} < m_2 = n_2e^{i\varphi_2} \leq 1e^{2\pi i}$, it follows by previous Remark 3.16 that $\widehat{f}_{m_2} \subseteq \widehat{f}_{m_1}$. In order to show that $\widehat{f}_{m_1} \subseteq \widehat{f}_{m_2}$, let $\sigma \in \widehat{f}_{m_1}$. Then, $\widehat{f_V}(\sigma) \geq m_1$. Because there is no $\sigma \in S$ such that $m_1 \leq \widehat{f_V}(\sigma) < m_2$, it follows that $\widehat{f_A}(\sigma) \geq m_2$. Thus, $\sigma \in \widehat{f}_{m_2}$ and $\widehat{f}_{m_1} \subseteq \widehat{f}_{m_2}$. □

Corollary 3.19 *Assume that $(S, *)$ is a hypergroup (H_v-group) and V is a (homogeneous) vague complex subhypergroup (or H_v-subgroup) of S with*

$$V = \{(s, \widehat{t_V}(s) = t_V(s)e^{i\alpha w_V^t(s)}, \widehat{f_V}(s) = 1 - f_V(s)e^{i\alpha(2\pi - w_V^f(s))}) : s \in S\}.$$

If the range of $\widehat{t_V}$ is finite set $\{r_1, r_2, \ldots, r_n\}$ and $\widehat{f_V}$ is finite set $\{m_1, m_2, \ldots, m_n\}$, then the sets $\{\widehat{t}_{r_i} : i = 1, 2, \ldots, n\}$ and $\{\widehat{f}_{m_i} : i = 1, 2, \ldots, n\}$ comprises whole the level subhypergroups (or

H_v-subgroups) of S. Besides of this, if $r_1 \geq r_2 \geq \ldots \geq r_n$ then whole the level subhypergroups of S create the chain $\widehat{t}_{r_1} \subseteq \widehat{t}_{r_2} \subseteq \ldots \subseteq \widehat{t}_{r_n}$. Similarly, if $m_1 \geq m_2 \geq \ldots \geq m_n$ then whole the level subhypergroups of S create the chain $\widehat{f}_{m_1} \subseteq \widehat{f}_{m_2} \subseteq \ldots \subseteq \widehat{f}_{m_n}$.

Proof Suppose that $\widehat{t}_q, \widehat{f}_q \neq \varnothing$ are level subhypergroups (or H_v-subgroups) of S such that $\widehat{t}_q \neq \widehat{t}_{r_i}$ and $\widehat{f}_q \neq \widehat{f}_{m_i}$ for all $1 \leq i \leq n$. Let r_z and m_z be closest complex numbers to q. There are two cases : $q < r_z$ and $q < m_z$, $q > r_z$ and $q > m_z$. We think the first case, the second case is like that of the first case. Because the ranges of $\widehat{t}_V$ and $\widehat{f}_V$ are finite sets $\{r_1, r_2, \ldots, r_n\}$ and $\{m_1, m_2, \ldots, m_n\}$ respectively, it follows that there is no $\epsilon \in S$ such that $q \leq \widehat{t}_V(\epsilon) < r_z$ and $q \leq \widehat{f}_V(\epsilon) < m_z$. Using Corollary 3.18, we have a contradiction. □

Proposition 3.20 *Suppose that $(S, *)$ is the biset hypergroup like $\gamma * \delta = \{\gamma, \delta\}$ for all $\gamma, \delta \in S$ and V is any homogeneous vague complex subset of S. Then V is a vague complex subhypergroup of S.*

Proof Assume that $r = me^{i\psi}$ and $t = ne^{i\varphi}$ such that $m, n \in [0,1]$ and $\psi, \varphi \in [0, 2\pi]$. By Theorem 3.17, it suffices to indicate that $V_{(r,t)} \neq \varnothing$ is a subhypergroup of S. We have $V_{(r,t)} \subseteq \upsilon * V_{(r,t)}$ as for all $\tau \in V_{(r,t)}$, $\tau \in \upsilon * \tau = \{\upsilon, \tau\}$. Besides of this, $\upsilon * V_{(r,t)} = V_{(r,t)} * \upsilon = \{\tau * \upsilon : \tau \in V_{(r,t)}\} = \{\tau, \upsilon\} \subseteq V_{(r,t)}$ for every $\upsilon \in V_{(r,t)}$. Therefore, $V_{(r,t)}$ is a subhypergroup of S. □

Proposition 3.21 *Suppose that $(S, *)$ is the total hypergroup like $\gamma * \delta = S$ for every $\gamma, \delta \in S$ and V is any homogeneous vague complex subset of S. Then V is a vague complex subhypergroup of S if and only if $\widehat{t}_V$ and $\widehat{f}_V$ are stable complex functions.*

Proof It is clear that if $\widehat{t}_V$ and $\widehat{f}_V$ are constant complex functions, then V is a vague complex subhypergroup of S. Let V be a vague complex subhypergroup of S and $\widehat{t}_V$ be not a constant complex function. We may find $\gamma, \delta \in S$, $r = se^{i\phi}$ such that $s \in [0,1]$ and $\phi \in [0, 2\pi]$ such that $\widehat{t}_V \ (\gamma) < \widehat{t}_V(\delta) = r$. It is clear that $\gamma \notin \widehat{t}_r$ and $\delta \in \widehat{t}_r$. Because $V_{(r,t)} \neq \varnothing$ is a subhypergroup of S, it follows that $S = \delta * \delta \subseteq \widehat{t}_r$. Similarly, let V be a vague complex subhypergroup of S and $\widehat{f}_V$ be not a constant complex function. We may find $\gamma, \delta \in S$, $u = ke^{i\varphi}$ such that $k \in [0,1]$ and $\varphi \in [0, 2\pi]$ such that $\widehat{f}_V \ (\gamma) < \widehat{f}_V(\delta) = u$. It is clear that $\gamma \notin \widehat{f}_u$ and $\delta \in \widehat{f}_u$. Because $V_{(r,t)} \neq \varnothing$ is a subhypergroup of S, it follows that $S = \delta * \delta \subseteq \widehat{f}_u$. These are contradictions. Hence, $\widehat{t}_V$ and $\widehat{f}_V$ are constant complex functions. □

Proposition 3.22 *Suppose that $(S, *)$ is a hypergroup (H_v-group) and V is a (homogeneous) vague complex subset of S. V is a vague complex subhypergroup (H_v-subgroup) of S if and only* *if for all* $r = ke^{i\psi}$ *and* $t = me^{i\varphi}$ *such that* $k, m \in [0,1]$ *and* $\psi, \varphi \in [0, 2\pi]$, *the following assertions are provided:*

i. $V_{(r,t)} * V_{(r,t)} \subseteq V_{(r,t)}$,

ii. $\upsilon * (S - V_{(r,t)}) - (S - V_{(r,t)}) \subseteq \upsilon * V_{(r,t)}$, *for every* $\upsilon \in V_{(r,t)}$,

iii. $(S - V_{(r,t)}) * \upsilon - (S - V_{(r,t)}) \subseteq V_{(r,t)} * \upsilon$, *for every* $\upsilon \in V_{(r,t)}$.

Proof Let V be a vague complex subhypergroup (H_v−subgroup) of S. Then, $V_{(r,t)}$ is a subhypergroup (H_v−subgroup) of S as $\upsilon * V_{(r,t)} = V_{(r,t)}$ for every $\upsilon \in V_{(r,t)}$. Hence, $V_{(r,t)} * V_{(r,t)} \subseteq V_{(r,t)}$. We need to indicate that $\upsilon * (S - V_{(r,t)}) - (S - V_{(r,t)}) \subseteq \upsilon * V_{(r,t)}$. Suppose that $\eta \in \upsilon * (S - V_{(r,t)}) - (S - V_{(r,t)})$. We have that η is not an element in $(S - V_{(r,t)})$. This implies that $\eta \in V_{(r,t)} = \upsilon * V_{(r,t)}$. Similarly, condition iii may be proved.

Conversely, let the conditions i, ii be satisfied. Using Theorem 3.17, it suffices to indicate $V_{(r,t)}$ is a subhypergroup (H_v−subgroup) of S as $\upsilon * V_{(r,t)} = V_{(r,t)} * \upsilon = V_{(r,t)}$ for every $\upsilon \in V_{(r,t)}$. Suppose that there exists $\gamma \in V_{(r,t)}$ such that γ is not an element in $\upsilon * V_{(r,t)}$. Using reproduction property of $(S, *)$, there exists $\delta \in S$ such that $\gamma \in \upsilon * \delta$. There are two cases for δ :

Case 1 : $\delta \in V_{(r,t)}$. We have that $\gamma \in \upsilon * \delta \subseteq \upsilon * V_{(r,t)}$. This is a contradiction.

Case 2 : $\delta \notin V_{(r,t)}$. We have that $\delta \in (S - V_{(r,t)})$. $\gamma \in \upsilon * \delta$ means $\gamma \in \upsilon * (S - V_{(r,t)})$. Because $\gamma \in V_{(r,t)}$, it follows that γ is not in $(S - V_{(r,t)})$. Hence, using by assumption we have $\gamma \in \upsilon * (S - V_{(r,t)}) - (S - V_{(r,t)}) \subseteq \upsilon * V_{(r,t)}$. This is a contradiction. Similarly, using condition iii, we can prove that $V_{(r,t)} * \upsilon = V_{(r,t)}$. □

Proposition 3.23 *Assume that* $(S, *)$ *is a hypergroup (H_v− group) and* V *is a (homogeneous) vague complex subhypergroup (H_v− subgroup) of* S *with*

$$V = \{(s, \widehat{t}_V(s) = t_V(s)e^{i\alpha w^t_V(s)}, \widehat{f}_V(s) = 1 - f_V(s)e^{i\alpha(2\pi - w^f_V(s))}) : s \in S\}.$$

Describe $\overline{\widehat{t}}$ *and* $\overline{\widehat{f}}$ *as follows:*

$$\overline{\widehat{t}} = \{s \in S : \widehat{t}_V(s) = 1e^{2\pi i}\} \text{ and } \overline{\widehat{f}} = \{s \in S : \widehat{f}_V(s) = 1e^{2\pi i}\}.$$

Then, $\overline{\widehat{t}}$ *and* $\overline{\widehat{f}}$ *are empty or subhypergroups (H_v− subgroups) of* S.

Proof We need to demonstrate that $\tau * \overline{\widehat{t}} = \overline{\widehat{t}} = \overline{\widehat{t}} * \tau$ for every $\tau \in \overline{\widehat{t}}$. Let $d \in \overline{\widehat{t}}$ and $\eta \in \tau * d$. Then, $\widehat{t}_V(\eta) \geq min\{\widehat{t}_V(\tau), \widehat{t}_V(d)\} = 1e^{2\pi i}$ implies that $\widehat{t}_V(\eta) = 1e^{2\pi i}$. Hence, $\eta \in \tau * d \subseteq \overline{\widehat{t}}$. For every $\tau, d \in \bar{\bar{t}}$, there exists $\rho \in S$ such that $d \in \tau * \rho$ and $\widehat{t}_V(\rho) \geq min\{\widehat{t}_V(\tau), \widehat{t}_V(d)\} = 1e^{2\pi i}$. This refers that $\widehat{t}_V(\rho) = 1e^{2\pi i}$ and $\rho \in \bar{\bar{t}}$.

Similarly, we want to indicate $\upsilon * \bar{\bar{f}} = \bar{\bar{f}} = \bar{\bar{f}} * \upsilon$ for every $\upsilon \in \bar{\bar{f}}$. Let $d \in \bar{\bar{f}}$ and $\eta \in \upsilon * d$. Then, $\widehat{f}_V(\eta) \geq min\{\widehat{f}_V(\upsilon), \widehat{f}_V(d)\} = 1e^{2\pi i}$ implies that $\widehat{f}_V(\eta) = 1e^{2\pi i}$. Hence, $\eta \in \upsilon * d \subseteq \bar{\bar{f}}$. For every $\upsilon, d \in \bar{\bar{f}}$, there exists $\rho \in S$ such that $d \in \upsilon * \rho$ and $\widehat{f}_V(\rho) \geq min\{\widehat{f}_V(\upsilon), \widehat{f}_V(d)\} = 1e^{2\pi i}$. This refers $\widehat{f}_V(\rho) = 0e^{0i}$ and $\rho \in \bar{\bar{f}}$. □

Proposition 3.24 *Suppose that $(S, *)$ is a hypergroup (H_v−group) and V is a (homogeneous) vague complex subhypergroup (H_v−subgroup) of S with*

$$V = \{(s, \widehat{t}_V(s) = t_V(s)e^{i\alpha w_V^t(s)}, \widehat{f}_V(s) = 1 - f_V(s)e^{i\alpha(2\pi - w_V^f(s))}) : s \in S\}.$$

Describe $supp(\widehat{t})$ and $supp(\widehat{f})$ as follows:

$$supp(\widehat{t}) = \{s \in S : \widehat{t}_V(s) > 0e^{0i}\} \text{ and } supp(\widehat{f}) = \{s \in S : \widehat{f}_V(s) > 0e^{0i}\}.$$

Then, $supp(\widehat{t})$ and $supp(\widehat{f})$ are empty or subhypergroups (H_v−subgroups) of S.

Proof We need to demonstrate $\tau * supp(\widehat{t}) = supp(\widehat{t}) = supp(\widehat{t}) * \tau$ for every $\tau \in supp(\widehat{t})$. Let $d \in supp(\widehat{t})$ and $\eta \in \tau * d$. Then, $\widehat{t}_V(\eta) \geq min\{\widehat{t}_V(\tau), \widehat{t}_V(d)\} > 0e^{0i}$ implies that $\widehat{t}_V(\eta) > 0e^{0i}$. Hence, $\eta \in \tau * d \subseteq supp(\widehat{t})$. For every $\tau, d \in supp(\widehat{t})$, there exists $\rho \in S$ such that $d \in \tau * \rho$ and $\widehat{t}_V(\rho) \geq min\{\widehat{t}_V(\tau), \widehat{t}_V(d)\} > 0e^{0i}$. This refers that $\widehat{t}_V(\rho) > 0e^{0i}$ and $\rho \in supp(\widehat{t})$.

Similarly, we want to indicate that $\upsilon * supp(\widehat{f}) = supp(\widehat{f}) = supp(\widehat{f}) * \upsilon$ for every $\upsilon \in supp(\widehat{f})$. Let $d \in supp(\widehat{f})$ and $\eta \in \upsilon * d$. Then, $\widehat{f}_V(\eta) \geq min\{\widehat{f}_V(\upsilon), \widehat{f}_V(d)\} > 0e^{0i}$ implies that $\widehat{f}_V(\eta) > 0e^{0i}$. Hence, $\eta \in \upsilon * d \subseteq supp(\widehat{f})$. For every $\upsilon, d \in supp(\widehat{f})$, there exists $\rho \in S$ such that $d \in \upsilon * \rho$ and $\widehat{f}_V(\rho) \geq min\{\widehat{f}_V(\upsilon), \widehat{f}_V(d) > 0e^{0i}$. This refers $\widehat{f}_V(\rho) > 0e^{0i}$ and $\rho \in supp(\widehat{f})$. □

Remark 3.25 *We defined the complement of the vague complex set in Definition 3.4. Now, we present some examples, where V and V^C are vague complex subhypergroups (This situation generally is not valid).*

Example 3.26 *Assume that $S = \{\gamma, \delta\}$ and describe the hypergroup $(S, *)$ by the next table:*

$*$	γ	δ
γ	γ	S
δ	S	δ

Describe a vague complex subset V of S as follows:

$$t_V(\gamma)e^{i\alpha w_V^t(\gamma)} = 0.3e^{i0} \text{ and } 1 - f_V(\gamma)e^{i\alpha(2\pi - w_V^f(\gamma))} = 0.6e^{i\pi},$$

$$t_V(\delta)e^{i\alpha w_V^t(\delta)} = 0.4e^{i\pi} \text{ and } 1 - f_V(\delta)e^{i\alpha(2\pi - w_V^f(\delta))} = 0.7e^{i3\pi/2}.$$

We have

$$t_{V^C}(\gamma)e^{i\alpha w_{V^C}^t(\gamma)} = 0.4e^{i\pi} \text{ and } 1 - f_{V^C}(\gamma)e^{i\alpha(2\pi - w_{V^C}^f(\gamma))} = 0.7e^{i2\pi},$$

$$t_{V^C}(\delta)e^{i\alpha w_{V^C}^t(\delta)} = 0.3e^{i\pi/2} \text{ and } 1 - f_{V^C}(\delta)e^{i\alpha(2\pi - w_{V^C}^f(\delta))} = 0.6e^{i\pi}.$$

In that case, V and V^C are homogeneous vague complex subhypergroups of S.

Example 3.27 *Suppose that $(S,*)$ is any hypergroup (H_v-group) with the vague complex subset V of S, which is described:*

$$V = \{(s, \widehat{t_V}(s) = t_V(s)e^{i\alpha w_V^t(s)}, \widehat{f_V}(s) = 1 - f_V(s)e^{i\alpha(2\pi - w_V^f(s))}) : s \in S\},$$

where $t_V(s)$, $f_V(s) \in [0,1]$ such that $0 \le t_V(s) + f_V(s) \le 1$ and $w_V^t(s), w_V^f(s) \in [0,1]$ and $\alpha \in [0,2\pi]$ are fixed real numbers. Therefore, V and V^C are homogeneous vague complex subhypergroups (H_v-subgroups) of S.

Remark 3.28 *Assume that $(S,*)$ is a hypergroup (H_v-group), V is a (homogeneous) vague complex subhypergroup (H_v-subgroup) of S. V^C is not necessarily a vague complex subhypergroup (or H_v-subgroup) of S.*

We will explain this previous situation with the next example.

Example 3.29 *Suppose that $S = \{\gamma, \delta, \epsilon\}$, as a H_v-group $(S,*)$ is defined as:*

$*$	γ	δ	ϵ
γ	γ	$\{\delta,\epsilon\}$	ϵ
δ	$\{\delta,\epsilon\}$	ϵ	γ
ϵ	ϵ	γ	δ

Describe vague complex subset V of S

$$t_V(\gamma)e^{i\alpha w_V^t(\gamma)} = 0.3e^{i\pi} \text{ and } 1 - f_V(\gamma)e^{i\alpha(2\pi - w_V^f(\gamma))} = 0.5e^{i3\pi/2},$$

$$t_V(\delta)e^{i\alpha w_V^t(\delta)} = t_V(\epsilon)e^{i\alpha w_V^t(\epsilon)} = 0.2e^{i\pi/2} \text{ and } 1 - f_V(\delta)e^{i\alpha(2\pi - w_V^f(\delta))} = 1 - f_V(\epsilon)e^{i\alpha(2\pi - w_V^f(\epsilon))} = 0.4e^{i\pi}.$$

We have,

$$\widehat{t_r} = \left\{ \begin{array}{ll} S, & if\ r \leq 0.2e^{i\pi/2} \\ \{\gamma\}, & if\ \ 0.2e^{i\pi/2} < r \leq 0.3e^{i\pi} \\ \varnothing, & otherwise \end{array} \right\} \ and\ \ \widehat{f_t} = \left\{ \begin{array}{ll} S, & if\ t \leq 0.4e^{i\pi} \\ \{\gamma\}, & if\ \ 0.4e^{i\pi} < t \leq 0.5e^{i3\pi/2} \\ \varnothing, & otherwise \end{array} \right\}$$

either empty sets or subhypergroups of S, which refers V is homogeneous vague complex subhypergroup of S. Because

$$0.5e^{i\pi/2} = \widehat{t}_{V^C}(\gamma) = \widehat{t}_{V^C}(\delta * \epsilon) < min\{\widehat{t}_{V^C}(\delta), \widehat{t}_{V^C}(\epsilon)\} = 0.6e^{i\pi},$$

$$0.7e^{i\pi} = \widehat{f}_{V^C}(\gamma) = \widehat{f}_{V^C}(\delta * \epsilon) < min\{\widehat{f}_{V^C}(\delta), \widehat{f}_{V^C}(\epsilon)\} = 0.8e^{i3\pi/2}.$$

This is a contradiction. Hence, V^C is not vague complex H_v-subgroup of S.

4. Anti-Vague Complex Subhypergroups

In this part, we will examine anti-vague complex subhypergroups (or H_v-subgroup) along with their properties. Additionally, we will elucidate the transitions between vague complex subhypergroups (or H_v-subgroup) and anti-vague complex subhypergroups (or H_v-subgroup).

Definition 4.1 *Suppose that $(S, *)$ is a hypergroup (H_v-group) and*

$$V = \{(s, \widehat{t}_V(s) = t_V(s)e^{i\alpha w_V^t(s)}, \widehat{f}_V(s) = 1 - f_V(s)e^{i\alpha(2\pi - w_V^f(s))}) : s \in S\}$$

is a (homogeneous) vague complex subset of S. If the following assertions are provided, V is called an anti-vague complex subhypergroup (H_v-subgroup) of S:

1. $sup\{\widehat{t}_V(\epsilon) : \epsilon \in \gamma * \delta\} \leq max\{\widehat{t}_V(\gamma), \widehat{t}_V(\delta)\}$ *for every* $\gamma, \delta \in S$,
2. $sup\{\widehat{f}_V(\epsilon) : \epsilon \in \gamma * \delta\} \leq max\{\widehat{f}_V(\gamma), \widehat{f}_V(\delta)\}$ *for every* $\gamma, \delta \in S$,
3. *For every $d, \gamma \in S$, there exists $\eta \in S$ such that $d \in \gamma * \eta$ and* $\widehat{t}_V(\eta) \leq max\{\widehat{t}_V(d), \widehat{t}_V(\gamma)\}$,
4. *For every $d, \gamma \in S$, there exists $\eta \in S$ such that $d \in \gamma * \eta$ and* $\widehat{f}_V(\eta) \leq max\{\widehat{f}_V(d), v_A(\gamma)\}$,
5. *For every $d, \gamma \in S$, there exists $\rho \in S$ such that $d \in \rho * \gamma$ and* $\widehat{t}_V(\rho) \leq max\{\widehat{t}_V(d), \widehat{t}_V(\gamma)\}$,
6. *For every $d, \gamma \in S$, there exists $\rho \in S$ such that $d \in \rho * \gamma$ and* $\widehat{f}_V(\rho) \leq max\{\widehat{f}_V(d), v_A(\gamma)\}$.

Now, we will give some examples of anti-vague complex H_v-subgroups.

Example 4.2 *Assume that* $S = \{\gamma, \delta\}$ *and characterize the hypergroup as* $(S, *)$*:*

$*$	γ	δ
γ	γ	S
δ	S	δ

Describe an vague complex subset V *of* S *as follows:*

$$t_V(\gamma)e^{i\alpha w_V^t(\gamma)} = 0.3e^{i0} \text{ and } 1 - f_V(\gamma)e^{i\alpha(2\pi - w_V^f(\gamma))} = 0.6e^{i\pi},$$

$$t_V(\delta)e^{i\alpha w_V^t(\delta)} = 0.4e^{i\pi} \text{ and } 1 - f_V(\delta)e^{i\alpha(2\pi - w_V^f(\delta))} = 0.7e^{i3\pi/2}.$$

We have

$$t_{V^C}(\gamma)e^{i\alpha w_{V^C}^t(\gamma)} = 0.4e^{i\pi} \text{ and } 1 - f_{V^C}(\gamma)e^{i\alpha(2\pi - w_{V^C}^f(\gamma))} = 0.7e^{i2\pi},$$

$$t_{V^C}(\delta)e^{i\alpha w_{V^C}^t(\delta)} = 0.3e^{i\pi/2} \text{ and } 1 - f_{V^C}(\delta)e^{i\alpha(2\pi - w_{V^C}^f(\delta))} = 0.6e^{i\pi}.$$

Then, V *and* V^C *are homogeneous anti-vague complex subhypergroups of* S.

Example 4.3 *Assume that* $S = \{\gamma, \delta, \epsilon\}$ *and define the hypergroup as* $(S, *)$*:*

$*$	γ	δ	ϵ
γ	$\{\gamma\}$	$\{\delta\}$	$\{\epsilon\}$
δ	$\{\delta\}$	$\{\delta, \epsilon\}$	S
ϵ	$\{\epsilon\}$	S	$\{\delta, \epsilon\}$

Describe a vague complex subset V *of* S *as:*

$$t_V(\gamma)e^{i\alpha w_V^t(\gamma)} = 0.6e^{i2\pi} \text{ and } 1 - f_V(\gamma)e^{i\alpha(2\pi - w_V^f(\gamma))} = 0.8e^{i3\pi/2},$$

$$t_V(\delta)e^{i\alpha w_V^t(\delta)} = 0.4e^{i3\pi/2} \text{ and } 1 - f_V(\delta)e^{i\alpha(2\pi - w_V^f(\delta))} = 0.7e^{i\pi},$$

$$t_V(\epsilon)e^{i\alpha w_V^t(\epsilon)} = 0.3e^{i\pi/2} \text{ and } 1 - f_V(\epsilon)e^{i\alpha(2\pi - w_V^f(\epsilon))} = 0.7e^{i\pi/2}.$$

In that case, V is homogeneous anti-vague complex subhypergroup of S.

Example 4.4 *Suppose that* $(S, *)$ *is any hypergroup (*H_v-*group) with the vague complex subset* V *of* S*, which is defined as:*

$$V = \{(s, \widehat{t}_V(s) = t_V(s)e^{i\alpha w_V^t(s)}, \widehat{f}_V(s) = 1 - f_V(s)e^{i\alpha(2\pi - w_V^f(s))}) : s \in S\},$$

where $t_V(s), f_V(s) \in [0,1]$ *such that* $0 \leq t_V(s) + f_V(s) \leq 1$ *and* $w_V^t(s), w_V^f(s) \in [0,1]$ *and* $\alpha \in [0, 2\pi]$ *are fixed real numbers. Therefore,* V *is a homogeneous anti-vague complex subhypergroup* (H_v-*subgroup) of* S.

Proposition 4.5 *Let* $(S, *)$ *be a hypergroup (or* H_v-*group). Then* V *be an anti-vague subhypergroup* (H_v-*subgroup) of* S *if and only if a* $\pi-$*vague set* V_π *is a* $\pi-$*anti-vague subhypergroup* (H_v-*subgroup) of* S.

Proof Proof is straightforward. □

Theorem 4.6 *Suppose that* $(S, *)$ *is a hypergroup* (H_v-*group) and* V *is a (homogeneous) vague complex subset of* S. *Then* V *is an anti-vague complex subhypergroup* (H_v-*subgroup) of* S *if and only if* t_V *and* $1 - f_V$ *are anti-vague subhypergroups* (H_v-*subgroups) of* S *and* w_V^t *and* $2\pi - w_V^f$ *are* $\pi-$*anti-vague subhypergroup* (H_v-*subgroups) of* S.

Proof $(\Longrightarrow)$: Let V be an anti-vague complex subhypergroup (H_v-subgroup) of S. We need to demonstrate that requirements of description of anti-vague subhypergroups are satisfied for t_V, $1 - f_V$ and w_V^t, $2\pi - w_V^f$. For every $\gamma, \delta \in S$, we have

$$sup\{\widehat{t_V}(\epsilon) : \epsilon \in \gamma * \delta\} \leq max\{\widehat{t_V}(\gamma), \widehat{t_V}(\delta)\},$$

$$sup\{\widehat{f_V}(\epsilon) : \epsilon \in \gamma * \delta\} \leq max\{\widehat{f_V}(\gamma), \widehat{f_V}(\delta)\}.$$

Notation 3.8 implies that

$$sup\{t_V(\epsilon) : \epsilon \in \gamma * \delta\} \leq max\{t_V(\gamma), t_V(\delta)\},$$

$$sup\{1 - f_V(\epsilon) : \epsilon \in \gamma * \delta\} \leq max\{1 - f_V(\gamma), 1 - f_V(\delta)\},$$

and

$$sup\{w_V^t(\epsilon) : \epsilon \in \gamma * \delta\} \leq max\{w_V^t(\gamma), w_V^t(\delta)\},$$

$$sup\{2\pi - w_V^f(\epsilon) : \epsilon \in \gamma * \delta\} \leq max\{2\pi - w_V^f(\gamma), 2\pi - w_V^f(\delta)\}.$$

Assume that $\gamma, d \in S$. There exist $\eta, \rho \in S$ such that $d \in \gamma * \eta$ and $d \in \rho * \gamma$. Then,

$$max\{\widehat{t_V}(\gamma)\widehat{t_V}(d)\} \geq \widehat{t_V}(\eta),$$

$$max\{\widehat{f_V}(\gamma), \widehat{f_V}(d)\} \geq \widehat{f_V}(\eta)$$

and

$$max\{\widehat{t_V}(\gamma), \widehat{t_V}(d)\} \geq \widehat{t_V}(\rho),$$

$$max\{\widehat{f_V}(\gamma), \widehat{f_V}(d)\} \geq \widehat{f_V}(\rho).$$

Notation 3.8 means that the conditions $3-6$ of description of anti-vague subhypergroup are yielded for both t_V , $1-f_V$ and w_V^t , $2\pi - w_V^f$.

$(\Longleftarrow)$: Suppose that t_V and $1-f_V$ are anti-vague subhypergroups (H_v-subgroups) of S and w_V^t , $2\pi - w_V^f$ are $\pi-$anti-vague subhypergroups (H_v-subgroups) of S. We want to demonstrate that requirements of anti-vague complex subhypergroups are provided. For every $\gamma, \delta \in S$, we have

$$sup\{t_V(\epsilon) : \epsilon \in \gamma * \delta\} \leq max\{t_V(\gamma), t_V(\delta)\},$$

$$sup\{1-f_V(\epsilon) : \epsilon \in \gamma * \delta\} \leq max\{1-f_V(\gamma), 1-f_V(\delta)\}$$

and

$$sup\{w_V^t(\epsilon) : \epsilon \in \gamma * \delta\} \leq max\{w_V^t(\gamma) w_V^t(\delta)\},$$

$$sup\{2\pi - w_V^f(\epsilon) : \epsilon \in \gamma * \delta\} \leq max\{2\pi - w_V^f(\gamma), 2\pi - w_V^f(\delta)\}.$$

Notation 3.8 means that

$$sup\{\widehat{t_V}(\epsilon) : \epsilon \in \gamma * \delta\} \leq max\{\widehat{t_V}(\gamma), \widehat{t_V}(\delta)\},$$

$$sup\{\widehat{f_V}(\epsilon) : \epsilon \in \gamma * \delta\} \leq max\{\widehat{f_V}(\gamma), \widehat{f_V}(\delta)\}.$$

Suppose that $\gamma, d \in S$. There exist $\eta, \rho \in S$ such that $d \in \gamma * \eta$ and $d \in \rho * \gamma$

$$max\{t_V(\gamma), t_V(d)\} \geq t_V(\eta) \text{ and } max\{w_V^t(\gamma), w_V^t(d)\} \geq w_V^t(\eta),$$

$$max\{1-f_V(\gamma), 1-f_V(d)\} \geq 1-f_V(\eta) \text{ and } max\{2\pi - w_V^f(\gamma), 2\pi - w_V^f(d)\} \geq 2\pi - w_V^f(\eta),$$

and

$$max\{t_V(\gamma), t_V(d)\} \geq t_V(\rho) \text{ and } max\{w_V^t(\gamma), w_V^t(d)\} \geq w_V^t(\rho),$$

$$max\{1-f_V(\gamma), 1-f_V(d)\} \geq 1-f_V(\rho) \text{ and } max\{2\pi - w_V^f(\gamma), 2\pi - w_V^f(d)\} \geq 2\pi - w_V^f(\rho).$$

Notation 3.8 means that the conditions $3-6$ of description of anti-vague complex subhypergroup are yielded for V. □

Lemma 4.7 *Suppose that $(S, *)$ is a hypergroup (H_v-group), V is (homogeneous) an anti-vague complex subhypergroup (or H_v-subgroup) of S. For every $g_1, g_2, \ldots, g_n \in S$,*

$$max\{\widehat{t_V}(g_1), \widehat{t_V}(g_2), \ldots, \widehat{t_V}(g_n)\} \geq sup\{\widehat{t_V}(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\}$$

and

$$max\{\widehat{f_V}(g_1), \widehat{f_V}(g_2), \ldots, \widehat{f_V}(g_n)\} \geq sup\{\widehat{f_V}(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\}.$$

Proof Let $g_1, g_2, \ldots, g_n \in S$ and $\widehat{t}_V(s) = t_V(s)e^{i\alpha w_V^t(s)}, \widehat{f}_V(s) = 1 - f_V(s)e^{i\alpha(2\pi - w_V^f(s))}$. To demonstrate this lemma, it suffices to indicate that

$$max\{t_V(g_1), t_V(g_2), \ldots, t_V(g_n)\} \geq sup\{t_V(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\},$$

$$max\{w_V^t(g_1), w_V^t(g_2), \ldots, w_V^t(g_n)\} \geq sup\{w_V^t(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\}$$

and

$$max\{1 - f_V(g_1), 1 - f_V(g_2), \ldots, 1 - f_V(g_n)\} \geq sup\{1 - f_V(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\},$$

$$max\{2\pi - w_V^f(g_1), 2\pi - w_V^f(g_2), \ldots, 2\pi - w_V^f(g_n)\} \geq sup\{2\pi - w_V^f(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\}.$$

Because V is homogeneous, it is adequate to indicate

$$max\{t_V(g_1), t_V(g_2), \ldots, t_V(g_n)\} \geq sup\{t_V(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\},$$

$$max\{1 - f_V(g_1), 1 - f_V(g_2), \ldots, 1 - f_V(g_n)\} \geq sup\{1 - f_V(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\}.$$

Theorem 4.6 claims that t_V and $1 - f_V$ are anti-vague subhypergroups (H_v-subgroups) of S. Using Lemma 2.13, the proof is completed. □

Definition 4.8 *Let $V = \{(s, \widehat{t}_V(s) = t_V(s)e^{i\alpha w_V^t(s)}, \widehat{f}_V(s) = 1 - f_V(s)e^{i\alpha(2\pi - w_V^f(s))}) : s \in S\}$ be a (homogeneous) vague complex subset of a non-empty set S. Lower level subset $\overline{V}_{(r,t)}$ of S is described as $\overline{V}_{(r,t)} = \{s \in S : \widehat{t}_V(s) \leq r$ and $\widehat{f}_V(s) \leq t\}$, where $r = me^{i\varphi}, t = ne^{i\psi}$ such that $m, n \in [0,1]$ and $\varphi, \psi \in [0, 2\pi]$.*

Remark 4.9 *Let $V = \{(s, \widehat{t}_V(s) = t_V(s)e^{i\alpha w_V^t(s)}, \widehat{f}_V(s) = 1 - f_V(s)e^{i\alpha(2\pi - w_V^f(s))}) : s \in S\}$ be a (homogeneous) vague complex subset of a non-empty set S. Then followings are true:*

i. If $r_1 \leq r_2$, then $\overline{\widehat{t}_{r_1}} \subseteq \overline{\widehat{t}_{r_2}}$,

ii. $\overline{\widehat{t}}_{1e^{2\pi i}} = S$,

iii. If $t_1 \leq t_2$, then $\overline{\widehat{f}}_{t_1} \subseteq \overline{\widehat{f}}_{t_2}$,

iv. $\overline{\widehat{f}}_{1e^{2\pi i}} = S$.

Theorem 4.10 *Assume that $(S, *)$ is a hypergroup (H_v-group), V is a (homogeneous) vague complex subset of S. Then V is an anti-vague complex subhypergroup (H_v-subgroup) of S if and only if for every $r = ke^{i\psi}$, $t = me^{i\varphi}$ such that $k, m \in [0,1]$ and $\psi, \varphi \in [0, 2\pi]$, $\overline{V}_{(r,t)} \neq \varnothing$ is a subhypergroup (H_v-subgroup) of S.*

Proof The proof is similar to that of Theorem 3.17. □

Corollary 4.11 *Assume that* $(S,*)$ *is a hypergroup* (H_v*−group), V is an (homogeneous) anti-vague complex subhypergroup* (H_v*−subgroup) of S with*

$$V=\{(s,\widehat{t}_V(s)=t_V(s)e^{i\alpha w_V^t(s)},\widehat{f}_V(s)=1-f_V(s)e^{i\alpha(2\pi-w_V^f(s))}):s\in S\}.$$

If $0e^{0i}\leq r_1=s_1e^{i\theta_1}<r_2=s_2e^{i\theta_2}\leq 1e^{2\pi i}$, *then* $\overline{\overline{\widehat{t}}}_{r_1}=\overline{\overline{\widehat{t}}}_{r_2}$ *if and only if there is no* $\sigma\in S$ *such that* $r_1<\widehat{t}_V(\sigma)\leq r_2$ *and similarly if* $0e^{0i}\leq m_1=n_1e^{i\varphi_1}<m_2=n_2e^{i\varphi_2}\leq 1e^{2\pi i}$, *then* $\overline{\overline{\widehat{f}}}_{m_1}=\overline{\overline{\widehat{f}}}_{m_2}$ *if and only if there is no* $\sigma\in S$ *such that* $m_1<\widehat{f}_V(\sigma)\leq m_2$.

Proof Suppose that $0e^{0i}\leq r_1=m_1e^{i\varphi_1}<r_2=m_2e^{i\varphi_2}\leq 1e^{2\pi i}$ such that $\overline{\overline{\widehat{t}}}_{r_1}=\overline{\overline{\widehat{t}}}_{r_2}$. Assume that there exists $\sigma\in S$ such that $r_1<\widehat{t}_V(\sigma)\leq r_2$. Then, $\sigma\in\overline{\overline{\widehat{t}}}_{r_2}=\overline{\overline{\widehat{t}}}_{r_1}$. This refers $\widehat{t}_V(\sigma)\leq r_1$ and it is a contradiction. Similarly, suppose that $0e^{0i}\leq m_1=n_1e^{i\varphi_1}<m_2=n_2e^{i\varphi_2}\leq 1e^{2\pi i}$, such that $\overline{\overline{\widehat{f}}}_{m_1}=\overline{\overline{\widehat{f}}}_{m_2}$. Assume that there exists $\sigma\in S$ such that $m_1<\widehat{f}_V(\sigma)\leq m_2$. Then, $\sigma\in\overline{\overline{\widehat{f}}}_{m_2}=\overline{\overline{\widehat{f}}}_{m_1}$. This refers that $\widehat{f}_V(\sigma)\leq m_1$ and it is a contradiction.

Because of $0e^{0i}\leq r_1=m_1e^{i\varphi_1}<r_2=m_2e^{i\varphi_2}\leq 1e^{2\pi i}$, it follows by Remark 4.9 $\overline{\overline{\widehat{t}}}_{r_1}\subseteq\overline{\overline{\widehat{t}}}_{r_2}$. In order to indicate $\overline{\overline{\widehat{t}}}_{r_2}\subseteq\overline{\overline{\widehat{t}}}_{r_1}$, suppose that $\sigma\in\overline{\overline{\widehat{t}}}_{r_2}$. Then, $\widehat{t}_V(\sigma)\leq r_2$. Because there is no $\sigma\in S$ such that $r_1<\widehat{t}_V(\sigma)\leq r_2$, it follows $\widehat{t}_V(\sigma)\leq r_1$. Hence, $\sigma\in\overline{\overline{\widehat{t}}}_{r_1}$ and $\overline{\overline{\widehat{t}}}_{r_2}\subseteq\overline{\overline{\widehat{t}}}_{r_1}$. Similarly, since $0e^{0i}\leq m_1=n_1e^{i\varphi_1}<m_2=n_2e^{i\varphi_2}\leq 1e^{2\pi i}$, it follows by Remark 4.9, $\overline{\overline{\widehat{f}}}_{m_1}\subseteq\overline{\overline{\widehat{f}}}_{m_2}$. In order to show $\overline{\overline{\widehat{f}}}_{m_2}\subseteq\overline{\overline{\widehat{f}}}_{m_1}$, suppose that $\sigma\in\overline{\overline{\widehat{f}}}_{m_2}$. Then, $\widehat{f}_V(\sigma)\leq m_2$. Because there is no $\sigma\in S$ such that $m_1<\widehat{f}_V(\sigma)\leq m_2$, it follows that $\widehat{f}_V(\sigma)\leq m_1$. Hence, $\sigma\in\overline{\overline{\widehat{f}}}_{m_1}$ and $\overline{\overline{\widehat{f}}}_{m_2}\subseteq\overline{\overline{\widehat{f}}}_{m_1}$. □

Corollary 4.12 *Assume that* $(S,*)$ *is a hypergroup* (H_v*−group), V is an (homogeneous) anti-vague complex subhypergroup* (H_v*−subgroup) of S with*

$$V=\{(s,\widehat{t}_V(s)=t_V(s)e^{i\alpha w_V^t(x)},\widehat{f}_V(s)=1-f_V(s)e^{i\alpha(2\pi-w_V^f(s))}):s\in S\}.$$

If the range of $\widehat{t}_V$ *is finite set* $\{r_1,r_2,\ldots,r_n\}$ *and* $\widehat{f}_V$ *is finite set* $\{s_1,s_2,\ldots,s_n\}$, *then the sets* $\{\widehat{t}_{r_i}:i=1,2,\ldots,n\}$ *and* $\{\widehat{f}_{s_i}:i=1,2,\ldots,n\}$ *comprises whole the lower level subhypergroups* (H_v*−subgroups) of S. Besides of this, if* $r_1\leq r_2\leq\ldots\leq r_n$, *then whole the lower level subhypergroups of S create the chain* $\overline{\overline{\widehat{t}}}_{r_1}\subseteq\overline{\overline{\widehat{t}}}_{r_2}\subseteq\ldots\subseteq\overline{\overline{\widehat{t}}}_{r_n}$. *Similarly, if* $s_1\leq s_2\leq\ldots\leq s_n$, *then whole the lower level subhypergroups of S create the chain* $\overline{\overline{\widehat{f}}}_{s_1}\subseteq\overline{\overline{\widehat{f}}}_{s_2}\subseteq\ldots\subseteq\overline{\overline{\widehat{f}}}_{s_n}$.

Proof Assume that $\bar{\bar{t}}_m, \bar{\bar{f}}_m \neq \varnothing$ are lower level subhypergroups (H_v-subgroups) of S such that $\bar{\bar{t}}_m \neq \bar{\bar{t}}_{r_i}$ and $\bar{\bar{f}}_m \neq \bar{\bar{f}}_{s_i}$ for all $1 \leq i \leq n$. Suppose that r_q and s_q are closest complex numbers to m. There are two cases : $m < r_q$ and $m < s_q$, $m > r_q$ and $m > s_q$. We think the first case, the second case is like that of the first case. Because the ranges of $\widehat{t}_V$ and $\widehat{f}_V$ are finite sets $\{r_1, r_2, \dots, r_n\}$ and $\{s_1, s_2, \dots, s_n\}$ respectively, it follows that there is no $\epsilon \in S$ such that $m < \widehat{t}_V(\epsilon) \leq r_q$ and $m < \widehat{f}_V(\epsilon) \leq s_q$. Using by Corollary 4.11, we have a contradiction. □

Proposition 4.13 *Assume that* $(S, *)$ *is the biset hypergroup like* $\gamma * \delta = \{\gamma, \delta\}$ *for every* $\gamma, \delta \in S$ *and* V *is any homogeneous vague complex subset of* S*. Then* V *is an anti-vague complex subhypergroup of* S*.*

Proof The proof is similar to that of Proposition 3.20. □

Proposition 4.14 *Assume that* $(S, *)$ *is the total hypergroup like* $\gamma * \delta = S$ *for every* $\gamma, \delta \in S$ *and* V *is any homogeneous vague complex subset of* S*. Then* V *is an anti-vague complex subhypergroup of* S *if and only if* $\widehat{t}_V$ *and* $\widehat{f}_V$ *are stable complex functions.*

Proof The proof is similar to that of Proposition 3.21. □

Proposition 4.15 *Assume that* $(S, *)$ *is a hypergroup (*H_v-*group),* V *is a (homogeneous) vague complex fuzzy subset of* S*. Then* V *is an anti-vague complex subhypergroup (*H_v-*subgroup) of* S *if and only if for every* $r = ke^{i\psi}$ *and* $t = me^{i\varphi}$ *such that* $k, m \in [0, 1]$ *and* $\psi, \varphi \in [0, 2\pi]$*, the following properties are provided:*

i. $\overline{V}_{(r,t)} * \overline{V}_{(r,t)} \subseteq \overline{V}_{(r,t)}$,

ii. $\upsilon * (S - \overline{V}_{(r,t)}) - (S - \overline{V}_{(r,t)}) \subseteq \upsilon * \overline{V}_{(r,t)}$, *for every* $\upsilon \in \overline{V}_{(r,t)}$,

iii. $(S - \overline{V}_{(r,t)}) * \upsilon - (S - \overline{V}_{(r,t)}) \subseteq \overline{V}_{(r,t)} * \upsilon$, *for every* $\upsilon \in \overline{V}_{(r,t)}$.

Proof Suppose that V is an anti-vague complex subhypergroup (H_v-subgroup) of S. Then by Theorem 4.10, $\overline{V}_{(r,t)}$ is a subhypergroup (H_v-subgroup) of S as $\upsilon * \overline{V}_{(r,t)} = \overline{V}_{(r,t)}$ for every $\upsilon \in \overline{V}_{(r,t)}$. Hence, $\overline{V}_{(r,t)} * \overline{V}_{(r,t)} \subseteq \overline{V}_{(r,t)}$. We need to indicate $\upsilon * (S - \overline{V}_{(r,t)}) - (S - \overline{V}_{(r,t)}) \subseteq \upsilon * \overline{V}_{(r,t)}$. Suppose that $\eta \in \upsilon * (S - \overline{V}_{(r,t)}) - (S - \overline{V}_{(r,t)})$. We have η is not an element in $(S - \overline{V}_{(r,t)})$. This refers $\eta \in \overline{V}_{(r,t)} = \upsilon * \overline{V}_{(r,t)}$. Similarly, condition iii may be proved.

Conversely, assume that the conditions i, ii are yielded. Using Theorem 4.10, it is adequate to demonstrate $\overline{V}_{(r,t)}$ is a subhypergroup (H_v-subgroup) of S like $\upsilon * \overline{V}_{(r,t)} = \overline{V}_{(r,t)} * \upsilon = \overline{V}_{(r,t)}$

for every $\upsilon \in \overline{V}_{(r,t)}$. Suppose that there exists $\gamma \in \overline{V}_{(r,t)}$ such that γ is not an element in $\upsilon * \overline{V}_{(r,t)}$. Using reproduction properties of $(S,*)$, there exists $\delta \in S$ such that $\gamma \in \upsilon * \delta$. Two situations are here for δ:

Case 1: $\delta \in \overline{V}_{(r,t)}$. We have $\gamma \in \upsilon * \delta \subseteq \upsilon * \overline{V}_{(r,t)}$. This is a contradiction.

Case 2 : $\delta \notin \overline{V}_{(r,t)}$. We have $\delta \in (S - \overline{V}_{(r,t)})$. $\gamma \in \upsilon * \delta$ means $\gamma \in \upsilon * (S - \overline{V}_{(r,t)})$. Because $\gamma \in \overline{V}_{(r,t)}$, it follows that γ is not in $(S - \overline{V}_{(r,t)})$. Hence, using by assumption $\gamma \in \upsilon * (S - \overline{V}_{(r,t)}) - (S - \overline{V}_{(r,t)}) \subseteq \upsilon * \overline{V}_{(r,t)}$. This is a contradiction. Similarly, using condition iii, we can prove $\overline{V}_{(r,t)} * \upsilon = \overline{V}_{(r,t)}$. □

Proposition 4.16 *Assume that $(S,*)$ is a hypergroup (H_v− group), V is an (homogeneous) anti-vague complex subhypergroup (H_v− subgroup) of S with*

$$V = \{(s, \widehat{t_V}(s) = t_V(s)e^{i\alpha w_V^t(s)}, \widehat{f_V}(s) = 1 - f_V(s)e^{i\alpha(2\pi - w_V^f(s))}) : s \in S\},$$

Describe $\overline{\overline{\widehat{t}}}$ and $\overline{\overline{\widehat{f}}}$ as follows:

$$\overline{\overline{\widehat{t}}} = \{s \in S : \widehat{t_V}(s) = 0e^{0i}\} \text{ and } \overline{\overline{\widehat{f}}} = \{s \in S : \widehat{f_V}(s) = 0e^{0i}\},$$

Then, $\overline{\overline{\widehat{t}}}$ and $\overline{\overline{\widehat{f}}}$ are empty or subhypergroups (H_v− subgroups) of S.

Proof We need to demonstrate that $\tau * \overline{\overline{\widehat{t}}} = \overline{\overline{\widehat{t}}} = \overline{\overline{\widehat{t}}} * \tau$ for every $\tau \in \overline{\overline{\widehat{t}}}$. Let $d \in \overline{\overline{\widehat{t}}}$ and $\eta \in \tau * d$. Then, $\widehat{t_V}(\eta) \leq \max\{\widehat{t_V}(\tau), \widehat{t_V}(d)\} = 0e^{0i}$ implies that $\widehat{t_V}(\eta) = 0e^{0i}$. Hence, $\eta \in \tau * d \subseteq \overline{\overline{\widehat{t}}}$. For every $\tau, d \in \overline{\overline{\widehat{t}}}$, there exists $\rho \in S$ such that $d \in \tau * \rho$ and $\widehat{t_V}(\rho) \leq \max\{\widehat{t_V}(\tau), \widehat{t_V}(d)\} = 0e^{0i}$. This refers that $\widehat{t_V}(\rho) = 0e^{0i}$ and $\rho \in \overline{\overline{\widehat{t}}}$.

Similarly, we want to indicate that $\upsilon * \overline{\overline{\widehat{f}}} = \overline{\overline{\widehat{f}}} = \overline{\overline{\widehat{f}}} * \upsilon$ for every $\upsilon \in \overline{\overline{\widehat{f}}}$. Let $d \in \overline{\overline{\widehat{f}}}$ and $\eta \in \upsilon * d$. Then, $\widehat{f_V}(\eta) \leq \max\{\widehat{f_V}(\upsilon), \widehat{f_V}(d)\} = 0e^{0i}$ implies that $\widehat{f_V}(\eta) = 0e^{0i}$. Hence, $\eta \in \upsilon * d \subseteq \overline{\overline{\widehat{f}}}$. For every $\upsilon, d \in \overline{\overline{\widehat{f}}}$, there exists $\rho \in S$ such that $d \in \upsilon * \rho$ and $\widehat{f_V}(\rho) \leq \max\{\widehat{f_V}(\upsilon), \widehat{f_V}(d)\} = 0e^{0i}$. This refers that $\widehat{f_V}(\rho) = 0e^{0i}$ and $\rho \in \overline{\overline{\widehat{f}}}$. □

Proposition 4.17 *Assume that $(S,*)$ is a hypergroup (H_v− group), V is an (homogeneous) anti-vague complex subhypergroup (H_v− subgroup) of S with*

$$V = \{(s, \widehat{t_V}(s) = t_V(s)e^{i\alpha w_V^t(s)}, \widehat{f_V}(s) = 1 - f_V(s)e^{i\alpha(2\pi - w_V^f(s))}) : s \in S\}$$

Describe $\overline{supp}(\widehat{t})$ *and* $\overline{supp}(\widehat{f})$ *as:*

$$\overline{supp}(\widehat{t}) = \{s \in S : \widehat{t}_V(s) < 1e^{2\pi i}\} \text{ and } \overline{supp}(\widehat{f}) = \{s \in S : \widehat{f}_V(s) < 1e^{2\pi i}\}$$

Then, $\overline{supp}(\widehat{t})$ *and* $\overline{supp}(\widehat{f})$ *are empty or subhypergroups (H_v− subgroups) of* S*.*

Proof We need to demonstrate that $\tau * \overline{supp}(\widehat{t}) = \overline{supp}(\widehat{t}) = \overline{supp}(\widehat{t}) * \tau$ for every $\tau \in \overline{supp}(\widehat{t})$. Let $d \in \overline{supp}(\widehat{t})$ and $\eta \in \tau * d$. Then, $\widehat{t}_V(\eta) \le \max\{\widehat{t}_V(\tau), \widehat{t}_V(d)\} < 1e^{2\pi i}$ implies that $\widehat{t}_V(\eta) < 1e^{2\pi i}$. Hence, $\eta \in \tau * d \subseteq \overline{supp}(\widehat{t})$. For every $\tau, d \in \overline{supp}(\widehat{t})$, there exists $\rho \in S$ such that $d \in \tau * \rho$ and $\widehat{t}_V(\rho) \le \max\{\widehat{t}_V(\tau), \widehat{t}_V(d)\} < 1e^{2\pi i}$. This refers that $\widehat{t}_V(\rho) < 1e^{2\pi i}$ and $\rho \in \overline{supp}(\widehat{t})$.

Similarly, we want to demonstrate that $\upsilon * \overline{supp}(\widehat{f}) = \overline{supp}(\widehat{f}) = \overline{supp}(\widehat{f}) * \upsilon$ for every $\upsilon \in \overline{supp}(\widehat{f})$. Let $d \in \overline{supp}(\widehat{f})$ and $\eta \in \upsilon * d$. Then, $\widehat{f}_V(\eta) \le \max\{\widehat{f}_V(\upsilon), \widehat{f}_V(d)\} < 1e^{2\pi i}$ implies that $\widehat{f}_V(\eta) < 1e^{2\pi i}$. Hence, $\eta \in \upsilon * d \subseteq \overline{supp}(\widehat{f})$. For every $\upsilon, d \in \overline{supp}(\widehat{f})$, there exists $\rho \in S$ such that $d \in \upsilon * \rho$ and $\widehat{f}_V(\rho) \le \max\{\widehat{f}_V(\upsilon), \widehat{f}_V(d) < 1e^{2\pi i}$. This refers that $\widehat{f}_V(\rho) < 1e^{2\pi i}$ and $\rho \in \overline{supp}(\widehat{f})$.

□

Theorem 4.18 *Assume that* $(S, *)$ *is a hypergroup (H_v− group),* V *is a homogeneous vague complex subset of* S*. Then* V *is a vague complex subhypergroup (H_v− subgroup) of* S *if and only if* V^C *is an anti-vague complex subhypergroup (H_v− subgroup) of* S*.*

Proof $(\Longrightarrow)$: Let V be a vague complex subhypergroup (H_v-subgroup) of S. Using Theorem 3.12, we have that $t_V, 1 - f_V$ are vague subhypergroups (H_v−subgroups) of S and $w_V^t, 2\pi - w_V^f$ are π−vague subhypergroups (H_v−subgroups) of S. Using Theorem 2.14, $t_{V^C}, 1 - f_{V^C}$ are anti-vague subhypergroups (H_v−subgroups) of S and $w_{V^C}^t, 2\pi - w_{V^C}^f$ are π−anti-vague subhypergroups (or H_v−subgroups) of S. From Theorem 4.6, V^C is an anti-vague complex subhypergroup (H_v−subgroup) of S.

$(\Longleftarrow)$: The proof is similar to the previous part. □

Corollary 4.19 *Assume that* $(S, *)$ *is a hypergroup (H_v− group),* V *is a homogeneous vague complex subset of* S*. Then* V *is a vague complex and anti-vague complex subhypergroup (H_v− subgroup) of* S *if and only if* V^C *is a vague complex and anti-vague complex subhypergroup (H_v− subgroup) of* S*.*

Proof The proof is clear from Theorem 4.18. □

Example 4.20 *Assume that* $(S, *)$ *is the biset hypergroup and* V *is any homogeneous vague* *complex subset of* S. *Using Proposition 3.20 and 4.13,* V *and* V^C *are vague complex and anti-vague complex subhypergroups of* S.

Example 4.21 *Assume that* $(S,*)$ *is any hypergroup* (H_v-*group) with the vague complex subset* V *of* S, *which is described:*

$$V = \{(s, \widehat{t_V}(s) = t_V(s)e^{i\alpha w_V^t(s)}, \widehat{f_V}(s) = 1 - f_V(s)e^{i\alpha(2\pi - w_V^f(s))}) : s \in S\},$$

where $t_V(s)$, $f_V(s) \in [0,1]$ *such that* $0 \le t_V(s) + f_V(s) \le 1$ *and* $w_V^t(s), w_V^f(s) \in [0,1]$ *and* $\alpha \in [0, 2\pi]$ *are fixed real numbers. Therefore,* V *and* V^C *are homogeneous vague complex and anti-vague complex subhypergroups* (H_v-*subgroups) of* S.

5. Conclusion

This study has examined the concepts of vague complex and anti-vague complex subhypergroups (H_v-subgroups) using the framework of vague complex sets and hyperstructures. With this work, subhyperstructures were studied from a broader perspective with the help of vague complex sets. Various examples were provided to illustrate these concepts and their properties. We have also explored the relationships between vague and anti-vague subypergroups, establishing key distinctions and connections.

The findings of this research contribute to a deeper understanding of the structure and behavior of H_v-subgroups under vague and anti-vague conditions. However, several open problems remain for further investigation. One such problem is the study of neutrosophic complex H_v-subgroups, which could provide a more generalized and nuanced framework for analyzing uncertainties and indeterminacies in mathematical structures. Researchers are encouraged to explore these areas to broaden the scope and applicability of hyperstructure theory.

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Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Sanem Yavuz]: Collected the data, contributed to completing the research and solving the problem, wrote the manuscript (%40).

Author [Serkan Onar]: Contributed to research method or evaluation of data, contributed to completing the research and solving the problem (%35).

Author [Bayram Ali Ersoy]: Thought and designed the research/problem, contributed to research method or evaluation of data (%25).

Conflicts of Interest

The authors declare no conflict of interest.

References

[1] Al-Tahan M., Davvaz B., *Complex fuzzy* H_v-*subgroups of an* H_v-*group*, Iranian Journal of Fuzzy Systems, 17(3), 177-186, 2020.

[2] Corsini P., Leorenau V., Applications of Hyperstructures Theory, Advances in Mathematics, Kluwer Academic Publisher, 2003.

[3] Davvaz B., Polygroup Theory and Related Systems, World Scientific Publishing Company Pte. Ltd., 2013.

[4] Davvaz B., *Fuzzy* H_v-*groups*, Fuzzy Sets and Systems, 101(1), 191-195, 1999.

[5] Davvaz B., Cristea I., Fuzzy Algebraic Hyperstructures: An Introduction, Studies in Fuzziness and Soft Computing, 321, Springer, 2015.

[6] Gau W.L., Buehrer D.J., *Vague sets*, IEEE Transaction on Systems, 23(2), 610-614, 1993.

[7] Husban R.A., Salleh A.R., *Complex vague relation*, AIP Conference Proceedings, 1691(1), 040010, 2015

[8] Marty F., *Sur une generalization de la notion de group*, In 8th Congress Math. Scandenaves, 45-49, 1934.

[9] Rossenfeld A., *Fuzzy groups*, Journal of Mathematical Analysis and Applications, 35(3), 512-517, 1971.

[10] Ramot D., Friedman M., Langholz G., Kandel A., *Complex fuzzy sets*, IEEE Transaction of Fuzzy Systems, 11(4), 450-461, 2003.

[11] Ramot D., Milo R., Friedman M., Kandel A., *Complex fuzzy logic*, IEEE Transaction of Fuzzy Systems, 10(2), 171-186, 2002.

[12] Singh P.K., *Complex vague set based concept lattice*, Chaos, Solitons and Fractal, 96, 145-153, 2017.

[13] Vougiouklis T., *The fundamental relation in hyperrings. The general hyperfield*, Proceedings of the Fourth International Congress on Algebraic Hyperstructures and Applications, Greece, 203-211, 1991.

[14] Vougiouklis T., Hyperstructures and Their Representations, Aviani Editor, Hadronic Press, 1994.

[15] Vougiouklis T., *A new class of hyperstructures*, Journal of Combinatorial Information Sciences, 20, 229-235, 1995.

[16] Zadeh L.A., *Fuzzy sets*, Information and Control, 8(3), 338-353, 1965.