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On Reverse and Generalized of Bellman Type Inequality

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Bellman Tipi Eşitsizliğin Tersi ve Genelleştirilmesi Üzerine

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Öz

Bu çalışmanın amacı, önceki çalışmalar incelendikten sonra dışbükey ve integrallenebilir ve diferansiyellenebilir fonksiyonlar için ters Bellman tipi eşitsizliği elde etmektir. Bununla birlikte Bellman tipi eşitsizliğin genelleştirilmesinin ispatı yapılmıştır. Elde edilen ana sonuçlar yardımıyla dışbükey ve integrallenebilir, diferansiyellenebilir fonksiyonlar üzerine bazı özel durumlar elde edildi. Bu sonuçlar kullanılarak analitik olarak hesaplanması zor integraller için bir alt sınır elde dilmiştir.

Anahtar Kelimeler: *Konvekslik; Eşitsizlikler; İntegral Operatörü; Bellman.*

Abstract

The aim of this work, to obtain reverse Bellman type inequality for convex and integrable-differentiable functions after examining the previous studies on the Bellman inequality and giving the results that are a source of inspiration for us. With that we proved the generalized of Bellman type inequality. With the help of obtained main results, we have obtained some special cases on convex and integrable-differentiable functions. For convexity we have also included some examples to make these exercises more understandable. On the other hand, we have expressed with examples to obtain a lower bound for hard integrals.

Keywords: *Convexity; Inequalities; Integral operator; Belmann.*

1. Introduction

Calculus, using different notions of derivatives and integrals of arbitrary order, has become in recent years one of the centers of attention of mathematical researchers, both pure and applied. By other hand, one of the most developed mathematical areas in the last 30 years is that of integral inequalities, associated with different functional notions: convex, q-calculus, synchronous functions within the framework of Riemann, fractional and generalized integral operators (Alp 2021, Delgado et al. 2021, Roberts et al. 1974). Moreover, many research papers have studied the properties of convex functions that make this concept interesting in mathematical analysis. In recent years, important generalizations have been made in the context of convexity: quasi-convex, pseudo-convex, invex and preinvex, strongly convex, approximately convex, MT-convex, (α;m)-convex, and strongly (s;m)-convex (Cristescu and Lupsa 2002, Vivas-Cortez et al. 2020, Youness 1999).

Recently, some studies have been done on the Bellman inequality, which has an important place in the literature on inequalities. Pecaric and Mitronic (Mitrinovic and Pecaric, 1988, Pecaric 1982) obtained Bellman type generalization of Steffensen's inequalities. In 2014, Mirzapour (Mirzapour and Moslehian 2014) proved some results on Bellman operator inequality. Moreover in 2013 Morassaei (Morassaei et al. 2013) proved the Bellman operator inequality. In 2014 Iddrisu (Iddrisu et al. 2014) proved Bellman type inequality. On the other hand, in 2020 Sababheh (Sababheh et al. 2020) obtained reversing Bellman operator inequality. The inspiration for this work is the inequality achieved by Iddrisu. The purpose of this paper, to obtain inverse inequality of (4) and prove generalized of (4) with some results. By using this result we obtain upper and lower bounds for integrals.

2. Preliminaries

We begin with convex functions.

Definition 1. (Convex Function). Let real function f be defined on some non-empty interval I of real line $\mathbb{R}$. The function f said to be convex on I, if the inequality

$$f(tb+(1-t)a) \leq tf(b)+(1-t)f(a),$$

$a,b \in I$ and $t \in [0,1]$.

The following inequality was discovered in 1918 by Steffensen (Steffensen 1947):

Assume that two integrable functions $h(x)$ and $\Im(x)$are defined on [a,b], that $h(x)$ never increases and that $0 \leq \Im(x) \leq 1$ in (a,b).

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Then

$$\int_{b-\lambda}^{b} h(x)dx \le \int_{a}^{b} h(x)\phi(x)dx$$

$$\le \int_{a}^{a+\lambda} h(x)dx \qquad (1)$$

where $\lambda = \int_a^b \phi(x)dx$.

Bellman in (Bellman 1959) prove the following inequality:

$$\left(\int_a^b \Psi(s)\Lambda(s)ds\right)^p \le \int_a^{a+c} \Psi^p(s)ds, \qquad (2)$$

where $c = a + \left(\int_a^b \Lambda(s)ds\right)^p$.

But, Godunova and Levin in (Godunova and Levin 1968) noted Inequality (2) that the generalisation of Inequality (1) was incorrect. Pecaric (Pecaric 1982) corrected the Bellman generalisation with a narrow subclass. The corrected result is

$$\left(\int_0^1 \Psi(s)\Lambda(s)ds\right)^p \le \int_0^{\lambda} \Lambda^p(s)ds, \qquad (3)$$

where

$$\lambda = \left(\int_0^1 \Psi(s)ds\right)^p, \Lambda: [0,1] \to \mathbb{R}$$

is a nonnegative and nonincreasing function, $\Psi: [0,1] \to \mathbb{R}$ is an integrable function with $0 \le \Psi(x) \le 1$ and $p \ge 1$, for the proof; see (Pecaric 1982) and the references therein.

Iddrisu et al. (Iddrisu et al. 2014) prove the following refinement of Bellman type inequality:

Theorem 1. Let the function $\Psi: [0,1] \to \mathbb{R}$ be continuous such that $0 \le \Im(x) \le 1$. If $\Im: [0,1] \to \mathbb{R}$ is a convex, differentiable function with $\Im(0) = 0$, then

$$\Im\left(\int_0^1 \Psi(s)ds\right) \le \int_0^1 \Psi(s)\Im'(s)ds \qquad (4)$$

for all $s \in [0,1]$.

The purpose of this paper, to obtain reversi of (4) and prove generalized of (4) with some results. By using obtained our results we obtain lower bounds for integrals.

3. Reverse Bellman Type Inequality

In this section, we prove reverse of Inequality (4) and by following theorems:

Theorem 2 (Reverse Bellman Type Inequality). Assume $\Psi, \Im: [0,1] \to \mathbb{R}$ are integrable-differentiable functions and $\Im$ is convex. For $0 \le \Psi(t) \le 1$ and $\Im(0) = 0$, then the following inequality holds:

$$\int_0^1 \Psi(x)\Im'\left(\frac{x}{2}\right)dx + 2\Im\left(\frac{1}{2}\right) - \Im(1)$$

$$\le \Im\left(\int_0^1 \Psi(x)dx\right). \qquad (5)$$

(5) changes direction when $\Im$ is concave.

Proof. Assume $0 \le \Psi(t) \le 1$ and

$$0 \le \int_0^x \Psi(t)dt,$$

$$\frac{x}{2} \le \frac{1}{2}\left(\int_0^x \Psi(t)dt + x\right).$$

Now choosing $F(x) = \frac{1}{2}\left(\int_0^x \Psi(t)dt + x\right)$, then we have

$$G(x) = \Im(F(x)) = \Im\left\{\frac{1}{2}\left(\int_0^x \Psi(t)dt + x\right)\right\}$$

$\Im'$ is nondecreasing and this yields

$$G'(x) = F'(x)\Im'(F(x))$$

$$= \frac{\Psi(x)+1}{2}\Im'\left\{\frac{1}{2}\left(\int_0^x \Psi(t)dt + x\right)\right\}$$

$$\ge \frac{\Psi(x)+1}{2}\Im'\left(\frac{x}{2}\right).$$

By integrating the last inequality on $[0,1]$ and for $\Im(0) = 0$, we get

$$\int_0^x G'(x)dx = G(1) - G(0)$$

$$= \Im\left\{\frac{1}{2}\left(\int_0^1 \Psi(t)dt + 1\right)\right\}$$

$$\ge \int_0^1 \frac{[\Psi(x)+1]}{2}\Im'\left(\frac{x}{2}\right)dx.$$

Due to convexity of $\Im$, we obtain

$$\frac{1}{2}\left\{\Im\left(\int_0^1 \Psi(t)dt\right) + \Im(1)\right\}$$

$$\ge \Im\left\{\frac{1}{2}\left(\int_0^1 \Psi(t)dt + 1\right)\right\}$$

$$\geq \int_0^1 \frac{[\Psi(x)+1]}{2} \Im'\left(\frac{x}{2}\right) \mathrm{d}x$$

$$= \frac{1}{2}\int_0^1 \Psi(x)\Im'\left(\frac{x}{2}\right) \mathrm{d}x + \frac{1}{2}\int_0^1 \Im'\left(\frac{x}{2}\right) \mathrm{d}x.$$

So, we have

$$\Im\left(\int_0^1 \Psi(t)\mathrm{d}t\right) + \Im(1)$$

$$\geq \int_0^1 \Psi(x)\Im'\left(\frac{x}{2}\right)\mathrm{d}x + 2\Im\left(\frac{1}{2}\right),$$

i.e.

$$\Im\left(\int_0^1 \Psi(t)\mathrm{d}t\right) \tag{6}$$

$$\geq \int_0^1 \Psi(x)\Im'\left(\frac{x}{2}\right)\mathrm{d}x + 2\Im\left(\frac{1}{2}\right) - \Im(1).$$

Thus, the proof is completed. □

Corollary 1. In (5), assume $\Im(\mathrm{t}) = t^p$ *for* $p \geq 1$, then we get

$$\frac{p}{2^{p-1}}\int_0^1 x^{p-1}\Psi(x)\mathrm{d}x + \frac{1-2^{p-1}}{2^{p-1}}$$

$$\leq \left(\int_0^1 \Psi(x)\mathrm{d}x\right)^p.$$

4. Generalized Results

In this section we generalize Theorem 1.and 2.

Theorem 3 (Generalized Bellman Type Inequality). Assume $\Psi, \Lambda, \Im : [a,b] \to \mathbb{R}$ are integrable-differentiable functions and $\Im(\mathrm{x})$ is a convex. For $\Psi(x) \leq M$, $\Lambda'(t) \geq 0$, and $\Im(0) = 0$ the following inequality holds

$$\Im\left(\int_a^b \Lambda'(t)\Psi(t)\mathrm{d}t\right)$$

$$\leq \int_a^b \Lambda'(t)\Psi(t)\Im'\{M[\Lambda(t)-\Lambda(a)]\}\mathrm{d}t. \tag{7}$$

where $M \in \mathbb{R}$.

Proof. Assume $m \leq \Psi(x) \leq M$ and $\Lambda'(t) \geq 0$,

$$\Lambda'(t)\Psi(t) \leq \Lambda'(t)M$$

$$\int_a^x \Lambda'(t)\Psi(t)\mathrm{d}t \leq M\int_a^x \Lambda'(t)\mathrm{d}t$$

$$= M[\Lambda(x)-\Lambda(a)].$$

Now choosing

$$\mathrm{E}(x) = \int_a^x \Lambda'(t)\Psi(t)\mathrm{d}t,$$

then, we have

$$\mathrm{H}(x) = \Im(\mathrm{E}(x)) = \Im\left(\int_a^x \Lambda'(t)\Psi(t)\mathrm{d}t\right).$$

$\Im'$ is nondecreasing due to convexity, so this yields

$$\mathrm{H}'(x) = \mathrm{E}'(x)\Im'(\mathrm{E}(x))$$

$$= \Lambda'(x)\Psi(x)\Im'\left(\int_a^x \Lambda'(t)\Psi(t)\mathrm{d}t\right)$$

$$\leq \Lambda'(x)\Psi(x)\Im'\{M[\Lambda(x)-\Lambda(a)]\}.$$

By integrating the last inequality on $[a,b]$, we have

$$\int_a^b \mathrm{H}'(x)\mathrm{d}x$$

$$\leq \int_a^b \Lambda'(x)\Psi(x)\Im'\{M[\Lambda(x)-\Lambda(a)]\}\mathrm{d}x.$$

Since $\Im(0) = 0$ and

$$\int_a^b \mathrm{H}'(x)\mathrm{d}x = \mathrm{H}(b) - \mathrm{H}(a)$$

$$= \Im\left(\int_a^b \Lambda'(t)\Psi(t)\mathrm{d}t\right),$$

and the desired result is achieved as follows

$$\Im\left(\int_a^b \Lambda'(t)\Psi(t)\mathrm{d}t\right)$$

$$\leq \int_a^b \Lambda'(x)\Psi(x)\Im'\{M[\Lambda(x)-\Lambda(a)]\}\mathrm{d}x.$$

Corollary 2. Considering the conditions in Theorem 3, assume $\Lambda(t) = t$ in Inequality (7), then we obtain

$$\Im\left(\int_a^b \Psi(t)\mathrm{d}t\right) \leq \int_a^b \Psi(t)\Im'\{M[t-a]\}\mathrm{d}t. \tag{8}$$

Remark 1. In (8), assume $a = 0$, $b = 1$ and $m = 0 \leq \Psi(x) \leq 1 = M$, then we reach (4).

Corollary 3. In (7), assume $\Im(\mathrm{t}) = t^p$ for $p \geq 1$, then we get

$$\left(\int_a^b \Lambda'(x)\Psi(x)\mathrm{d}x\right)^p \tag{9}$$

$$\leq pM^{p-1}\int_a^b \Lambda'(x)\Psi(x)[\Lambda(x)-\Lambda(a)]^{p-1}\mathrm{d}x.$$

Corollary 4. In (9), assume $\Lambda(t)=t,$ then we get

$$\left(\int_a^b \Psi(x)\mathrm{d}x\right)^p$$

$$\leq pM^{p-1}\int_a^b (x-a)^{p-1}\Psi(x)\mathrm{d}x. \tag{10}$$

Example 1. In (10), assume $p=2$, then we get

$$\left(\int_a^b \Psi(x)\mathrm{d}x\right)^2 \leq 2M\int_a^b (x-a)\Psi(x)\mathrm{d}x.$$

Theorem 4 (Generalizied Reverse Bellman Type Inequality)**.** Assume Ψ, Λ, $\Im$: $[\mathrm{a},\mathrm{b}]\to\mathbb{R}$ are integrable-differentiable functions and $\Im(\mathrm{x})$ is a convex. For $\mathrm{m}\leq\Psi(\mathrm{x})$, $\Lambda'(\mathrm{x})\geq 0$, and $\Im$ (0) = 0 the following inequality holds

$$\int_a^b \Lambda'(x)\Psi(x)\Im'\left(\frac{(m+1)\Lambda(x)-m\Lambda(a)}{2}\right)\mathrm{d}x \tag{11}$$
$$+\frac{2}{(m+1)}\left[\Im\left(\frac{(m+1)\Lambda(b)-m\Lambda(a)}{2}\right)+m\Im\left(\frac{\Lambda(a)}{2}\right)\right]$$
$$-\Im\big(\Lambda(b)\big)$$
$$\leq \Im\left(\int_a^b \Lambda'(x)\Psi(x)\mathrm{d}x\right),$$

where $m\in\mathbb{R}$.

Proof. Assume $\mathrm{m}\leq\Psi(\mathrm{x})$ and $\Lambda'(\mathrm{t})\geq 0$, then we write

$$\mathrm{m}\Lambda'(\mathrm{t})\leq\Lambda'(\mathrm{t})\Psi(\mathrm{t}),$$

$$m[\Lambda(x)-\Lambda(a)]\leq\int_a^x \Lambda'(t)\Psi(t)\mathrm{d}t,$$

$$\frac{m[\Lambda(x)-\Lambda(a)]+\Lambda(x)}{2}$$
$$\leq\frac{1}{2}\left(\int_a^x \Lambda'(t)\Psi(t)\mathrm{d}t+\Lambda(x)\right).$$

Now, choosing $\mathrm{K}(x)=\frac{1}{2}\left(\int_a^x \Lambda'(t)\Psi(t)\mathrm{d}t+\Lambda(x)\right)$, then we have

$$\mathrm{T}(x)=\Im\big(\mathrm{K}(x)\big)$$
$$=\Im\left\{\frac{1}{2}\left(\int_a^x \Lambda'(t)\Psi(t)\mathrm{d}t+\Lambda(x)\right)\right\}.$$

$\Im'$ is nondecreasing and this yields

$$\mathrm{T}'(x)=\mathrm{K}'(x)\Im'\big(\mathrm{K}(x)\big)$$
$$=\frac{1}{2}\big(\Lambda'(x)\Psi(x)+\Lambda'(x)\big)$$
$$\times\Im'\left\{\frac{1}{2}\left(\int_a^x \Lambda'(t)\Psi(t)\mathrm{d}t+\Lambda(x)\right)\right\}$$
$$\geq\frac{1}{2}\big(\Lambda'(x)\Psi(x)+\Lambda'(x)\big)$$
$$\times\Im'\left\{\frac{m[\Lambda(x)-\Lambda(a)]+\Lambda(x)}{2}\right\}.$$

If the last inequality integrate on $[a,b]$, the following inequality is obtained

$$\int_a^b \mathrm{G}'(x)\mathrm{d}x=\mathrm{G}(b)-\mathrm{G}(a)$$
$$=\Im\left\{\frac{1}{2}\left(\int_a^b \Lambda'(t)\Psi(t)\mathrm{d}t+\Lambda(b)\right)\right\}-\Im\left(\frac{\Lambda(a)}{2}\right)$$
$$\geq\frac{1}{2}\int_a^b\{\Lambda'(x)[\Psi(x)+1]$$
$$\times\Im'\left(\frac{m[\Lambda(x)-\Lambda(a)]+\Lambda(x)}{2}\right)\Bigg\}dx,$$

i.e.

$$\Im\left\{\frac{1}{2}\left(\int_a^b \Lambda'(t)\Psi(t)\mathrm{d}t+\Lambda(b)\right)\right\}$$
$$\geq\frac{1}{2}\int_a^b\{\Lambda'(x)[\Psi(x)+1]$$
$$\times\Im'\left(\frac{m[\Lambda(x)-\Lambda(a)]+\Lambda(x)}{2}\right)\Bigg\}dx+\Im\left(\frac{\Lambda(a)}{2}\right).$$

Due to convexity of $\Im$, we obtain

$$\frac{1}{2}\Im\left(\int_a^b \Lambda'(t)\Psi(t)\mathrm{d}t\right)+\frac{1}{2}\Im\big(\Lambda(b)\big)$$

$$\geq \Im\left\{\frac{1}{2}\left(\int_a^b \Lambda'(t)\Psi(t)\mathrm{d}t + \Lambda(b)\right)\right\}$$

$$\geq \frac{1}{2}\int_a^b \{\Lambda'(x)[\Psi(x)+1]$$

$$\times \Im'\left(\frac{m[\Lambda(x)-\Lambda(a)]+\Lambda(x)}{2}\right)\right\} dx + \Im\left(\frac{\Lambda(a)}{2}\right).$$

So, the following inequality is obtained

$$\Im\left(\int_a^b \Lambda'(t)\Psi(t)\mathrm{d}t\right) \tag{12}$$

$$\geq \int_a^b \{\Lambda'(x)[\Psi(x)+1]$$

$$\times \Im'\left(\frac{m[\Lambda(x)-\Lambda(a)]+\Lambda(x)}{2}\right)\right\} dx$$

$$+2\Im\left(\frac{\Lambda(a)}{2}\right) - \Im(\Lambda(b)).$$

By editing the integral in right side of (12), we get

$$\int_a^b \{\Lambda'(x)[\Psi(x)+1] \tag{13}$$

$$\times \Im'\left(\frac{\Lambda(x)[m+1]-m\Lambda(a)}{2}\right)\right\} dx$$

$$= \int_a^b \Lambda'(x)\Psi(x)\Im'\left(\frac{\Lambda(x)[m+1]-m\Lambda(a)}{2}\right) dx$$

$$+ \int_a^b \Lambda'(x)\Im'\left(\frac{\Lambda(x)[m+1]-m\Lambda(a)}{2}\right) dx$$

$$= \int_a^b \Lambda'(x)\Psi(x)\Im'\left(\frac{\Lambda(x)[m+1]-m\Lambda(a)}{2}\right) dx$$

$$+\frac{2}{m+1}.\frac{m+1}{2}\int_a^b \Lambda'(x)\Im'\left(\frac{\Lambda(x)[m+1]-m\Lambda(a)}{2}\right) dx$$

$$= \int_a^b \Lambda'(x)\Psi(x)\Im'\left(\frac{\Lambda(x)[m+1]-m\Lambda(a)}{2}\right) dx$$

$$+\frac{2}{m+1}\Im\left(\frac{[m+1]\Lambda(b)-m\Lambda(a)}{2}\right)$$

$$-\frac{2}{m+1}\Im\left(\frac{[m+1]\Lambda(a)-m\Lambda(a)}{2}\right)$$

$$= \int_a^b \Lambda'(x)\Psi(x)\Im'\left(\frac{\Lambda(x)[m+1]-m\Lambda(a)}{2}\right) dx$$

$$+2\frac{\Im\left(\frac{(m+1)\Lambda(b)-m\Lambda(a)}{2}\right) - \Im\left(\frac{\Lambda(a)}{2}\right)}{m+1}.$$

If (13) is substituted in (12), we get

$$\Im\left(\int_a^b \Lambda'(t)\Psi(t)\mathrm{d}t\right)$$

$$\geq \int_a^b \{\Lambda'(x)[\Psi(x)+1]$$

$$\times \Im'\left(\frac{m[\Lambda(x)-\Lambda(a)]+\Lambda(x)}{2}\right)\right\} dx$$

$$+2\Im\left(\frac{\Lambda(a)}{2}\right) - \Im(\Lambda(b))$$

$$= \int_a^b \Lambda'(x)\Psi(x)\Im'\left(\frac{\Lambda(x)[m+1]-m\Lambda(a)}{2}\right) dx$$

$$+2\frac{\Im\left(\frac{(m+1)\Lambda(b)-m\Lambda(a)}{2}\right) - \Im\left(\frac{\Lambda(a)}{2}\right)}{m+1}$$

$$+2\Im\left(\frac{\Lambda(a)}{2}\right) - \Im(\Lambda(b))$$

$$= \int_a^b \Lambda'(x)\Psi(x)\Im'\left(\frac{\Lambda(x)[m+1]-m\Lambda(a)}{2}\right) dx$$

$$+\frac{2}{m+1}\Im\left(\frac{(m+1)\Lambda(b)-m\Lambda(a)}{2}\right)$$

$$+\frac{2m}{m+1}\Im\left(\frac{\Lambda(a)}{2}\right) - \Im(\Lambda(b)).$$

Thus, the desired result is achieved and the proof is completed.

Remark 2. In (11), assume $a = 0, b = 1, \Lambda(x) = x$ and $m = 0 \leq \Psi(x) \leq 1 = M$ we reach (5).

Corollary 5. Considering the conditions in Theorem 4, in (11), assume $\Lambda(t) = t$ we obtain

$$\int_a^b \Psi(x)\Im'\left(\frac{(m+1)x-ma}{2}\right)\mathrm{d}x$$

$$+\frac{2}{(m+1)}\left[\Im\left(\frac{(m+1)b-ma}{2}\right) + m\Im\left(\frac{a}{2}\right)\right] - \Im(b)$$

$$\leq \Im\left(\int_a^b \Psi(x)\mathrm{d}x\right),$$

where $m \in \mathbb{R}$.

Corollary 6. In (11), assume $\Im(t) = t^p$, for $p \geq 1$ then we get

$$\frac{p}{2^{p-1}} \int_a^b \Lambda'(x)\Psi(x)\big((m+1)\Lambda(x) - m\Lambda(a)\big)^{p-1} \mathrm{d}x + \frac{2}{(m+1)}\left[\left(\frac{(m+1)\Lambda(b)-m\Lambda(a)}{2}\right)^p + m\left(\frac{\Lambda(a)}{2}\right)^p\right] - \big(\Lambda(b)\big)^p \leq \left(\int_a^b \Lambda'(x)\Psi(x)\mathrm{d}x\right)^p, \quad (14)$$

where $m \in \mathbb{R}$.

Corollary 7. In (14), assume $\Lambda(t) = t$ then we get

$$\frac{p}{2^{p-1}} \int_a^b \Psi(x)\big((m+1)x - ma\big)^{p-1} \mathrm{d}x + \frac{\big((m+1)b-ma\big)^p + ma^p - 2^{p-1}(m+1)b^p}{2^{p-1}(m+1)} \leq \left(\int_a^b \Psi(x)\mathrm{d}x\right)^p. \quad (15)$$

Corollary 8. In (15), assume $p = 2$ then we get

$$\int_a^b \Psi(x)\big((m+1)x - ma\big)\mathrm{d}x + \frac{\big((m+1)b-ma\big)^2 + ma^2 - 2(m+1)b^2}{2(m+1)} \leq \left(\int_a^b \Psi(x)\mathrm{d}x\right)^2. \quad (16)$$

Finally, we give some examples to calculate some bound value for hard integrals using the obtained results.

Example 2. By using (16) we obtain a lower bound for $\int_a^b e^{x^2}dx$ as follows

$$\int_a^b xe^{x^2}\mathrm{d}x - \frac{b^2}{2} \leq \left(\int_a^b e^{x^2}dx\right)^2,$$

$$\sqrt{\frac{e^{b^2} - e^{a^2} - b^2}{2}} \leq \int_a^b e^{x^2}dx.$$

Example 3. By using (16) we obtain a lower bound for $\int_a^b sin(x^2)dx$ as follows

$$\int_a^b xsin(x^2)\mathrm{d}x - \frac{b^2}{2} \leq \left(\int_a^b sin(x^2)\mathrm{d}x\right)^2,$$

$$\sqrt{\frac{sin(a^2) - sin(b^2) - b^2}{2}} \leq \int_a^b sin(x^2)dx.$$

5. Conclusion

In this research, we have proved generalized of Bellman type inequality and obtained Reverse Bellman type inequality with some more results for convex functions. We also get some bounds for hard integrals with examples. Our results can be applied to other analyses especially q-calculus. Similar inequalities in convexity types (for example, quasi-convex, pseudo-convex, invex, preinvex, strongly convex, approximately convex, MT-convex, (α;m)-convex, strongly (s;m)-convex) can be investigated. Moreover, similar results on the Time Scale can be explored.

Declaration of Ethical Standards
The authors declare that they comply with all ethical standards.

Credit Authorship Contribution Statement
Author-1: Conceptualization, Methodology/Study design, Software, Validation, Formal analysis, Investigation, Resources, Data curation, Writing–original draft, Writing–review and editing, Visualization, Supervision

Declaration of Competing Interest
The authors have no conflicts of interest to declare regarding the content of this article.

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All data generated or analyzed during this study are included in this published article.

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