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Research Article

Bayesian inference and optimal plan for the family of inverted exponentiated distributions under doubly censored data

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Abstract

In this paper, we consider inference upon unknown parameters of the family of inverted exponentiated distributions when it is known that data are doubly censored. Maximum likelihood and Bayes estimates under different loss functions are derived for estimating the parameters. We use Metropolis-Hastings algorithm to draw Markov chain Monte Carlo samples, which are used to compute the Bayes estimates and construct the Bayesian credible intervals. Further, we present point and interval predictions of the censored data using the Bayesian approach. The performance of proposed methods of estimation and prediction are investigated using simulation studies, and two illustrative examples are discussed in support of the suggested methods. Finally, we propose the optimal plans under double censoring scheme.

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1. Introduction

In life testing studies like mortality analysis and bio-assay experiments subjects either leave the study before its completion or there is no follow-up after certain time period. In such situations, complete failure times data of all subjects may not be recorded. This leads to the concept of censoring. In survival and reliability analyses, data are always subject to censoring. Left and right censoring schemes are two important methods which are commonly utilized in practice. In right censoring, the recorded failure times of subjects are smaller than the observed censoring time. In case of left censoring, an experimenter have only partial information on failure time that it is more than considered left censored time. One of the important censoring schemes is observed in the context when failure times are left and right censored as well. This is the case of a double censoring. Such

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schemes have attracted attention of several researchers, Balakrishnan [2], Kotb and Ragab [18], and Fernández [10, 11] present many useful applications of double censoring in life test studies. As such studies are conducted under several constraints in the form of time and budget limitations so inference on unknown parameters are mostly derived based on censored data only. In fact, truncation of some initial and final data in a study may happen due to various reasons. It is of practical importance to obtain inference for quantity of interest under these limited observations. Here, we present analysis under type-II double censoring where certain observations may not be recorded in the beginning and towards end of a test. Double censoring has wide practical applications. Peer et al. [30] reported that such censoring is quite useful in the study of age-dependent growth rate of primary breast cancer and useful estimates under different age groups can be derived based on this method of censoring. Extensive studies have been discussed for double censoring under various approaches, see, for example, [6], [12], [36].

Several probability distributions with different shape characteristics and probabilistic properties have been introduced and studied in the literature. Among others, two important families are referred as exponentiated distributions and inverted exponentiated distributions. Initially, Ghitany et al. [13] reported several properties of inverted family of exponentiated distributions. In this model, shape parameter takes care of geometric shape of density function. However, associated scale parameter takes care of height of the distribution curve. This family possesses flexible probability models which are useful in fitting many positively skewed real data arising from various studies. Below we observe that inverted exponentiated exponential, inverted exponentiated Rayleigh, and inverted exponentiated Pareto distributions belong to this family. Many authors have analyzed this family under different context. Abouammoh and Alshingiti [1], Rastogi and Tripathi [32], and Maurya et al. [25] presented many applications of this family in lifetime analysis. One may further refer to [7], [19], [38] for important results on considered family of distributions.

The cumulative distribution function (CDF) is given as

$$F_Y(y) = (1 - e^{-\lambda Q(y)})^{\alpha}; \quad y > 0, \lambda, \alpha > 0,$$

where α and λ denote shape and scale parameters. Also $Q(y)$ is increasing function and with $lim_{y\to 0} Q(y) = 0$ and $\lim_{y\to\infty} Q(y) = \infty$. Let $X = 1/Y$, and then the corresponding CDF of a transformed random variable is given by

$$F_X(x) = 1 - (1 - e^{-\lambda Q(1/x)})^{\alpha}; \quad x > 0, \lambda, \alpha > 0$$

which is as given in [13].

The probability density function (PDF) is

$$f_X(x) = \alpha\lambda \frac{Q^{'}(1/x)}{x^2} e^{-\lambda Q(1/x)} (1 - e^{-\lambda Q(1/x)})^{\alpha - 1}; \quad x > 0, \alpha, \lambda > 0, \tag{1.1}$$

from which we observe that the following probability models belong to this family:

- $Q(1/x) = 1/x$: inverted exponentiated exponential distribution
- $Q(1/x) = 1/x^2$: inverted Burr X distribution
- $Q(1/x) = ln(1 + 1/x)$: inverted exponentiated Pareto distribution

In this article, estimation for the family of inverted exponentiated distributions is considered under classical and Bayesian approaches. We assume gamma priors for unknown parameters. Then, different estimates are obtained using squared error, linear-exponential (LINEX), and general entropy loss functions. Deriving estimates for censored observations under some prior information is of high significant importance in many inference problems. Such issues have received considerable attention among practitioners. Our objective is to find estimates for missing samples based on observed data. Prediction for censored data under different censoring schemes is studied by many researchers. Kotb and Raqab [18] obtained prediction estimates for missing observations under double censored data by considering modified Weibull distribution. Kayal et al. [17] presented prediction results for inverted exponentiated Rayleigh distribution under hybrid censoring. Wang et al. [39] investigated inference and prediction problems under progressively Type-II censored data when the lifetime follows the unit-generalized Rayleigh distribution. We further mention that, in life test studies, finding optimum censoring schemes is an important practical issue to assess reliability of a product. It has received wide attention at recent past. See, for example, [5, 20, 23, 28, 31, 37]. It is mentioned that finding adequate optimal criteria often helps in determination of better censoring schemes. Mostly variations of information measures are taken into consideration to define such criteria. Idea is to select the scheme that offers reasonable information about unknown quantities. We refer to [20] for useful discussion in this aspect.

In Section 2, data description is discussed. Further maximum likelihood estimators (MLEs) of parameters are obtained. Section 3 discusses observed Fisher information matrix. Subsequently approximate confidence intervals are constructed based on doubly censored samples. Bayes estimates, along with Bayesian credible intervals, are derived in Section 4. The prediction of censored data is considered in Section 5 from Bayes viewpoint. In Section 6, we conduct simulation study and evaluate performance of proposed estimation and prediction methods. Two numerical data sets are used to illustrate application viewpoint. In Section 7, we present Bayesian optimal design under double censoring. Finally, some concluding remarks are given in Section 8.

2. Model description and maximum likelihood estimates

A random sample of size n is taken from $f_X(x;\theta)$ model. Suppose that $x_r, x_{r+1} \dots x_s$ are ordered observations, and also $r-1$ smallest data and $n-s$ largest data are censored. Here $1 \le r < s \le n$. The likelihood of $\theta = (\alpha, \lambda)$ based on observed data is then

$$L(\theta, \boldsymbol{x}) = \frac{n! \prod_{i=r}^{s} f_X(x_i;\theta)}{(r-1)!(n-s)!} \Big((F_X(x_r))^{r-1} (1 - F_X(x_r))^{n-s} \Big),$$

where $\boldsymbol{x} = (x_r, x_{r+1}, \dots, x_s)'$. Thus, the likelihood of the considered distribution is

$$L(\alpha, \lambda; \boldsymbol{x}) = \frac{n! (\alpha\lambda)^{s-r+1}}{(r-1)!(n-s)! \prod_{i=r}^{s} x_i^2} \Big(\prod_{i=r}^{s} Q'(1/x_i) e^{-\lambda Q(1/x_i)} (1 - e^{-\lambda Q(1/x_i)})^{\alpha-1} \Big)$$
$$[1 - (1 - e^{-\lambda Q(1/x_r)})^{\alpha}]^{r-1} (1 - e^{-\lambda Q(1/x_s)})^{\alpha(n-s)}.$$

The log-likelihood is given as follows:

$$\ln L(\alpha, \lambda; \boldsymbol{x}) \propto (s-r+1) \ln \alpha + (s-r+1) \ln \lambda - \lambda \sum_{i=r}^{s} Q(1/x_i) + (\alpha - 1) \cdot$$
$$\sum_{i=r}^{s} \ln(\phi(\lambda, x_i) + (r-1) \ln[1 - (\phi(\lambda, x_r))^{\alpha}] + \alpha(n-s) \ln(\phi(\lambda, x_s)),$$

where

$$\phi(\lambda, x_i) = 1 - e^{-\lambda Q(1/x_i)}, \quad r \le i \le s.$$

Likelihood equations are obtained as

$$\frac{\partial \ln L}{\partial \alpha} = \frac{(s-r+1)}{\alpha} + \sum_{i=r}^{s} \ln(\phi(\lambda, x_i)) - \frac{(r-1)(\phi(\lambda, x_r))^{\alpha} \ln(\phi(\lambda, x_r))}{1 - (\phi(\lambda, x_r))^{\alpha}} + (n-s) \ln(\phi(\lambda, x_s)), \tag{2.1}$$

and

$$\begin{aligned}\frac{\partial \ln L}{\partial \lambda} =&\frac{(s-r+1)}{\lambda} - \sum_{i=r}^{s} Q(1/x_i) + (\alpha-1)\sum_{i=r}^{s} \frac{Q(1/x_i)(1-\phi(\lambda,x_i))}{\phi(\lambda,x_i)} \\ &- \frac{\alpha(r-1)Q(1/x_r)(1-\phi(\lambda,x_r))(\phi(\lambda,x_r))^{\alpha-1})}{1-(\phi(\lambda,x_r))^{\alpha}} + \frac{\alpha(n-s)Q(1/x_s)(1-\phi(\lambda,x_s))}{\phi(\lambda,x_s)}.\end{aligned} \tag{2.2}$$

The required MLEs of α and λ are computed by solving (2.1) and (2.2) simultaneously. We use *R* program with the *nleqslv* library for the numerical computations. We mention that *nleqslv* function solves a system of nonlinear equations with either Broyden or Newton-Raphson methods. The Broyden method often yields super linear convergence while the Newton-Raphson method usually shows quadratic convergence. As errors in the approximation is squared at each iteration, so this leads to a rapid convergence for Newton-Raphson method. In the Newton-Raphson method, selecting an initial guess near the true value is crucial for convergence, and some priori knowledge may be used if available. However, there is no strict rule for choosing the initial guess. In our case, we generate random samples by assigning different parameter values to (α, λ). Accordingly we select initial guess to be near true parameter values. We refer to [15] for a discussion on this topic.

Remark 2.1. It should be emphasized that it is difficult to prove the existence and uniqueness of the MLEs due to the intricate forms of the likelihood equations. Two equations in two variables, however, ought to be solved with modern computing capability, see also [40].

3. Fisher information matrix

In this section, we obtain observed and expected Fisher information matrices. We mention that to construct approximate confidence intervals we have used observed Fisher information matrix. We further note that the expected Fisher information matrix is used in finding optimal designs of life tests, see [23].

Here we derive the Fisher information matrix of model parameters. The second derivatives of log-likelihood are

$$I_{11} = \frac{\partial^2 \ln L}{\partial \alpha^2} = \frac{-(s-r+1)}{\alpha^2} - (r-1)(\phi(\lambda,x_r))^{\alpha}(\ln(\phi(\lambda,x_r)))^2(1-(\phi(\lambda,x_r))^{\alpha})^{-2},$$

$$\begin{aligned}I_{12} = \frac{\partial^2 \ln L}{\partial \alpha \partial \lambda} =&\sum_{i=r}^{s} \frac{Q(1/x_i)(1-\phi(\lambda,x_i))}{\phi(\lambda,x_i)} - (r-1)Q(1/x_r)(1-\phi(\lambda,x_r))(\phi(\lambda,x_r))^{\alpha-1} \\ &\frac{\Big(1-(\phi(\lambda,x_r))^{\alpha} + \alpha\ln\phi(\lambda,x_r)\Big)}{(1-(\phi(\lambda,x_r))^{\alpha})^2} + \frac{(n-s)Q(1/x_s)(1-\phi(\lambda,x_s))}{\phi(\lambda,x_s)} \\ =&\frac{\partial^2 \ln L}{\partial \lambda \partial \alpha} = I_{21},\end{aligned}$$

and

$$\begin{aligned}
I_{22} =& \frac{-(s-r+1)}{\lambda^2} - (\alpha-1)\sum_{i=r}^{s}\left(\frac{(Q(1/x_i))^2(1-\phi(\lambda,x_i))}{(\phi(\lambda,x_i))^2}\right) \\
& - \alpha(r-1)(Q(1/x_r))^2(\phi(\lambda,x_r))^{\alpha-2}(1-\phi(\lambda,x_r)) \\
& \left[\left((\alpha-1)(1-\phi(\lambda,x_r))(1-(\phi(\lambda,x_r))^{\alpha})\right) - \left(\phi(\lambda,x_r)(1-(\phi(\lambda,x_r))^{\alpha})\right)\right. \\
& \left. + \alpha(\phi(\lambda,x_r))^{\alpha}(1-\phi(\lambda,x_r))\right](1-(\phi(\lambda,x_r))^{\alpha})^{-2} \\
& - \alpha(n-s)(Q(1/x_s))^2(1-\phi(\lambda,x_s))(\phi(\lambda,x_s))^{-2}.
\end{aligned}$$

In order to obtain the expected value of I_{ij}, $i,j=1,2$, we need the PDF of the k-th, $k=1,2,\ldots,n$, order statistic from the inverted exponentiated family, which is given as

$$\begin{aligned}
f_{X_k}(x_k) =& \frac{n!}{(n-k)!(k-1)!}\frac{\alpha\lambda Q'(1/x_k)}{x_k^2}(\phi(\lambda,x_k))^{\alpha(n-k)+\alpha-1} \\
& (1-\phi(\lambda,x_k))(1-(\phi(\lambda,x_k))^{\alpha})^{k-1}, \quad x_k>0,\ \alpha,\lambda>0.
\end{aligned}$$

Therefore, the Fisher information matrix is

$$I(\theta) = \begin{bmatrix} -E(I_{11}) & -E(I_{12}) \\ -E(I_{21}) & -E(I_{22}) \end{bmatrix},$$

where

$$-E[I_{11}] = \frac{s-r+1}{\alpha^2} + \frac{n!\alpha}{(n-r)!(r-2)!}\int_0^1 t^{\alpha(n-r)+2\alpha-1}(1-t^{\alpha})^{r-3}(\ln t)^2 dt,$$

$$\begin{aligned}
& -E[I_{12}] \\
=& \sum_{i=r}^{s}\frac{n!\alpha}{(n-i)!(i-1)!\lambda}\int_0^1 t^{\alpha(n-i)+\alpha-2}(1-t)(1-t^{\alpha})^{i-1}(\ln(1-t))dt \\
& - \frac{n!\alpha}{(n-r)!(r-2)!\lambda}\int_0^1 t^{\alpha(n-r)+2\alpha-2}(1-t)(1-t^{\alpha})^{r-3}\ln(1-t)(1-t^{\alpha}+\alpha\ln t)dt \\
& + \frac{n!\alpha}{(n-s-1)!(s-1)!\lambda}\int_0^1 t^{\alpha(n-s)+\alpha-2}(1-t)(1-t^{\alpha})^{s-1}\ln(1-t)dt \\
=& -E[I_{21}],
\end{aligned}$$

and

$$\begin{aligned}
-E[I_{22}] =& \frac{(s-r+1)}{\lambda^2} \\
& + \frac{\alpha(\alpha-1)}{\lambda^2}\sum_{i=r}^{s}\frac{n!}{(n-i)!(i-1)!}\int_0^1 t^{\alpha(n-i)+\alpha-3}(1-t)(1-t^{\alpha})^{i-1}(\ln(1-t))^2 dt \\
& + \frac{n!\alpha^2}{(n-r)!(r-2)!\lambda^2}\int_0^1 t^{\alpha(n-r)+2\alpha-3}(1-t)(1-t^{\alpha})^{r-3}(\ln(1-t))^2 A(t)dt \\
& - \frac{n!\alpha^2}{(n-s-1)!(s-1)!\lambda^2}\int_0^1 t^{\alpha(n-s)+\alpha-3}(1-t)(1-t^{\alpha})^{s-1}(\ln(1-t))^2 dt.
\end{aligned}$$

Here, we define

$$A(t) = \alpha(1-t) - (1-t^{\alpha}).$$

Under some mild regularity conditions, the MLE $\hat{\theta}$ is asymptotically normally distributed with mean θ and variance-covariance matrix $I^{-1}(\theta)$. We will use the Fisher information matrix to obtain the optimal design of life tests in Section 7.

Efron and Hinkley [9] pointed out that there are two approximations of $I(\theta)$. One is called expected Fisher information matrix which is $I(\theta)$ evaluated at the MLE $\hat{\theta}$, that is, $I(\hat{\theta}) = I(\theta)|_{\theta=\hat{\theta}}$. Another approximation is called the observed Fisher information matrix, that is,

$$\widehat{I(\theta)} = -\begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix}\Bigg|_{\theta=\hat{\theta}}.$$

The observed Fisher information matrix can be used to construct the approximate confidence intervals for α and λ. Thus, the $100(1-\nu)\%$, $0<\nu<1$, approximate confidence intervals for α and λ are $\hat{\alpha} \pm z_{\nu/2}\sqrt{I^{11}}$ and $\hat{\lambda} \pm z_{\nu/2}\sqrt{I^{22}}$, respectively, where $z_{\nu/2}$ is the $\nu/2$ upper quantile of standard normal distribution and I^{ij} is the (i,j)-th element of the inverse of the observed Fisher information matrix.

4. Bayesian estimate

Now, we discuss Bayes estimates of parameters α and λ under doubly censored data. Note that the parameters of considered family take positive values. So, the priors of α and λ can be taken as gamma distribution with the density as

$$\pi_0(a,b;\alpha) = \frac{b^a}{\Gamma(a)} e^{-b\alpha} \alpha^{(a-1)}, \quad \alpha > 0,\ a,b>0,$$

and

$$\pi_1(c,d;\lambda) = \frac{d^c}{\Gamma(c)} e^{-d\lambda} \lambda^{(c-1)}, \quad \lambda > 0,\ c,d>0,$$

where a, b, c, and d are hyper-parameters. In practice, these quantities are chosen to gain some information on parameters α and λ, respectively. Selection of gamma priors in such case is quite natural and common. This is because such class of priors is flexible and specify improper prior also for specific values of a, b, c, and d. Assume that α and λ are independent. The joint prior is written as

$$\pi(a,b,c,d;\alpha,\lambda) = \frac{b^a}{\Gamma(a)} \frac{d^c}{\Gamma(c)} e^{-(b\alpha+d\lambda)} \alpha^{(a-1)} \lambda^{(c-1)}, \quad \alpha,\lambda > 0.$$

Then, the posterior distribution of α and λ is given by

$$\pi(\alpha,\lambda|\text{data}) \propto \prod_{i=r}^{s} \frac{Q'(1/x_i)}{x_i^2} \alpha^{(a+s-r)} \lambda^{(c+s-r)} e^{-b\alpha} e^{-\lambda(\sum_{i=r}^{s} Q(1/x_i)+d)}$$
$$\prod_{i=r}^{s} \phi(\lambda,x_i)^{(\alpha-1)} [1-\phi(\lambda,x_r)^{\alpha}]^{(r-1)} \phi(\lambda,x_s)^{\alpha(n-s)}.$$

4.1. Loss functions

Bayes estimation of θ depends on loss function which is chosen. We apply squared error, linear-exponential (LINEX), and general entropy loss functions in this regard.

4.1.1. Squared error loss function. This is one of the popular loss functions. It is a symmetric function and defined as

$$L_1(\theta,\delta) = (\theta-\delta)^2.$$

Bayes estimate of $T(\alpha,\lambda)$ is the posterior mean given by

$$\hat{T}_{BS}(\alpha,\lambda) = E(T(\alpha,\lambda) \mid \text{data}) = \int_0^\infty \int_0^\infty T(\alpha,\lambda)\pi(\alpha,\lambda|\text{data})\, d\alpha\, d\lambda.$$

As this loss function is symmetric in this sense that it provides equal penalty to both over and under estimation.

4.1.2. LINEX loss function. This loss function is also highly popular in Bayes computations. It was initially discussed in [41]. This function is asymmetric in nature. It is given by

$$L_2(\theta,\delta)=e^{k(\delta-\theta)}-k(\delta-\theta)-1,$$

where k controls the degree of asymmetry. The Bayes estimate is now given as

$$\hat{T}_{BL}(\alpha,\lambda)=-\frac{1}{k}\ln[E(e^{-kT(\alpha,\lambda)})|\text{data}]=\frac{-1}{k}\ln\left[\int_0^\infty\int_0^\infty e^{-kT(\alpha,\lambda)}\pi(\alpha,\lambda|\text{data})d\alpha d\lambda\right].$$

4.1.3. General entropy loss function. In Bayesian set up, LINEX loss function is very useful for estimating location parameters (see [4], [29]). Basu and Ebrahimi [4] studied a new loss function referred as general entropy loss function, which is defined as

$$L_3(\theta,\delta)=\left(\frac{\delta}{\theta}\right)^m-m\ln\left(\frac{\delta}{\theta}\right)-1.$$

This is also an asymmetric loss function. The Bayes estimate of $T(\alpha,\lambda)$ is

$$\begin{aligned}\hat{T}_{BE}(\alpha,\lambda) &= \Big(E((T(\alpha,\lambda))^{-m}|\text{data})\Big)^{-1/m}\\ &= \left[\int_0^\infty\int_0^\infty (T(\alpha,\lambda))^{-m}\pi(\alpha,\lambda|\text{data})d\alpha d\lambda\right]^{-1/m}.\end{aligned}$$

It can be noted that the integral of the above equation is not in nice form. Therefore, we use Markov chain Monte Carlo (MCMC) technique to compute the Bayes estimate.

4.2. MCMC approach

This technique is useful in simulating samples from posterior densities. These samples are further used to obtain required estimates under different loss functions. We can also construct credible intervals based on such samples. Robert and Casella [33] discussed many applications of this method. It is illustrated that such sampling is useful in cases posterior distributions are not in known forms. Therefore, we use Metropolis-Hastings algorithm to obtain required estimates of parameters. Random numbers from these distributions can be generated by using bivariate normal distribution BVN(μ,Σ) as a proposal density. Algorithm 1 shows the required steps for the sample generation from $\pi(\alpha,\lambda|\text{data})$ under the MCMC technique.

Algorithm 1. Algorithm to generate posterior sample

1: Assume an initial guess of $(\alpha,\lambda)=(\alpha_{(0)},\lambda_{(0)})$.
2: Set $l=1$.
3: Generate a proposal α^* and λ^* from the BVN(μ,Σ) where $\mu=(\alpha_{(l)},\lambda_{(l)})$ and $\Sigma=\hat{\Sigma}$ which is the variance-covariance matrix of the MLEs.
4: Evaluate the acceptance probability

$$\rho=\min\left\{1,\frac{\pi(\alpha^*,\lambda^*|\text{data})}{\pi(\alpha_{(l-1)},\lambda_{(l-1)}|\text{data})}\right\}$$

5: Generate u from $U(0,1)$ distribution.
6: If $u\le\rho$, accept the proposal and set $\alpha_{(l)}=\alpha^*$ and $\lambda_{(l)}=\lambda^*$, else set $\alpha_{(l)}=\alpha_{(l-1)}$ and $\lambda_{(l)}=\lambda_{(l-1)}$.
7: Set $l=l+1$ and repeat Steps 3 to 6, D times and obtain the posterior samples of (α_1,λ_1), $(\alpha_2,\lambda_2),\ldots,(\alpha_D,\lambda_D)$.

Therefore, the Bayes estimates of any arbitrary function of $\theta=(\alpha,\lambda)$ are as follows:

- under squared error loss function, $\hat{\theta} = \frac{1}{D-D_0}\sum_{l=D_0+1}^{D}\theta_l$,
- under LINEX loss function, $\hat{\theta} = -\frac{1}{k}\ln\left(\frac{1}{D-D_0}\sum_{l=D_0+1}^{D}e^{-k\theta^l}\right)$,
- under general entropy loss function, $\hat{\theta} = \left(\frac{1}{D-D_0}\sum_{l=D_0+1}^{D}\theta_l^{-m}\right)^{-1/m}$,

where the initial D_0 is a burn-in sample to be discarded due to the convergence of posterior samples.

It is important to compute $100(1-\nu)\%$ credible intervals of parameters to summarize the inferences. We obtain these estimates again by using MCMC samples. Bayesian credible intervals are constructed also. To obtain the required intervals of α and λ, first order α_j and λ_j as $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(D-D_0)}$ and $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(D-D_0)}$. Then, the $100(1-\nu)\%$ credible intervals are given by

$$\left(\alpha^{([(D-D_0)(\nu/2)])}, \alpha^{([(D-D_0)(1-\nu/2)])}\right), \quad \text{and} \quad \left(\lambda^{([(D-D_0)(\nu/2)])}, \lambda^{([(D-D_0)(1-\nu/2)])}\right).$$

5. Bayesian prediction

Prediction of the censored observations under some given information is an important practical problem in many studies. Several examples arise in agricultural, clinical, financial, and industrial experiments where prediction of unobserved data is required. It has attracted attention of several researchers. Dey et al. [8] obtained such inferences for generalized inverted exponential distribution under progressive censoring. Shafay et al. [35] obtained similar results under jointly type-II censored samples by considering two exponential populations. Kotb and Raqab [18] discussed prediction inference under doubly censored data by considering a modified version of Weibull distribution. Sometimes we may have prior information on parameters of interest. In such a situation, Bayesian prediction is a natural choice. Long [24], Maurya et al. [26], and Kayal et al. [16] discussed many applications of prediction inferences in lifetime analysis. Here our objective is to obtain predictive estimate and interval of l-th order statistic, X_l, $l = 1, 2, \ldots, r-1$ for left side prediction and $l = s+1, s+2, \ldots, n$ for right side prediction. Next, we discuss the procedure to evaluate the predictive estimates for both the cases and subsequently prediction intervals obtained as well.

5.1. Point prediction

The predictive conditional density function of x_l given $\boldsymbol{x} = (x_r, x_{r+1}\ldots, x_s)$ for considered family of distributions under the doubly censored sample is given by,

$$f(x_l|\boldsymbol{x}) = \begin{cases} \frac{(r-1)!\alpha\lambda Q'(1/x_l)}{(l-1)!(r-l-1)!x_l^2}\frac{[1-(\phi(\lambda,x_l))^{\alpha}]^{(l-1)}}{[1-(\phi(\lambda,x_r))^{\alpha}]^{(r-1)}} \\ \qquad (\phi(\lambda,x_l))^{\alpha-1}(1-\phi(\lambda,x_l))[(\phi(\lambda,x_l))^{\alpha}-(\phi(\lambda,x_r))^{\alpha}]^{r-l-1}, \\ \qquad\qquad l = 1,2,\ldots,r-1 \\ \frac{(n-s)!\alpha\lambda Q'(1/x_l)}{(l-s-1)!(n-l)!x_l^2}\frac{(\phi(\lambda,x_l))^{\alpha(n-l)}}{(\phi(\lambda,x_s))^{\alpha(n-s)}} \\ \qquad (\phi(\lambda,x_l))^{\alpha-1}(1-\phi(\lambda,x_l))[(\phi(\lambda,x_s))^{\alpha}-(\phi(\lambda,x_l))^{\alpha}]^{l-s-1}, \\ \qquad\qquad l = s+1, s+2,\ldots,n. \end{cases} \tag{5.1}$$

Note that Equation (5.1) is based on the ordered joint density function and the conditional density function. The posterior predictive density of x_l under prior $\pi(\alpha,\lambda)$ is,

$$f^*(x_l|\boldsymbol{x}) = \int_0^{\infty}\int_0^{\infty} f(x_l|\boldsymbol{x})\pi(\alpha,\lambda|\boldsymbol{x})d\alpha d\lambda.$$

To obtain predictive estimate of x_l, we consider the loss functions as defined in Section 4.1. Under squared error loss function, for left as well as right predictive estimates are given by

$$\begin{aligned}\hat{x}_l &= \int_0^\infty x_l f^*(x_l|\boldsymbol{x})dx_l \\ &= \int_0^\infty \int_0^\infty I_{1l}(\alpha,\lambda)\pi(\alpha,\lambda|\boldsymbol{x})d\alpha d\lambda,\end{aligned} \tag{5.2}$$

where

$$I_{1l}(\alpha,\lambda)=\begin{cases}\frac{(r-1)!\alpha\lambda}{(l-1)!(r-l-1)!}\int_0^{x_r}\frac{Q'(1/x_l)}{x_l}\frac{[1-(\phi(\lambda,x_l))^\alpha]^{(l-1)}}{[1-(\phi(\lambda,x_r))^\alpha]^{(r-1)}}(\phi(\lambda,x_l))^{\alpha-1}(1-\phi(\lambda,x_l)) \\ \qquad [(\phi(\lambda,x_l))^\alpha-(\phi(\lambda,x_r))^\alpha]^{r-l-1}dx_l, \quad l=1,2,\ldots,r-1 \\ \\ \frac{(n-s)!\alpha\lambda}{(l-s-1)!(n-l)!}\int_{x_s}^\infty\frac{Q'(1/x_l)}{x_l}\frac{(\phi(\lambda,x_l))^{\alpha(n-l)}}{(\phi(\lambda,x_s))^{\alpha(n-s)}}(\phi(\lambda,x_l))^{\alpha-1}(1-\phi(\lambda,x_l)) \\ \qquad [(\phi(\lambda,x_s))^\alpha-(\phi(\lambda,x_l))^\alpha]^{l-s-1}dx_l, \quad l=s+1,s+2,\ldots,n.\end{cases}$$

Under LINEX loss function, for left as well as right predictive values is given by

$$\begin{aligned}\hat{x}_l &= \frac{-1}{k}\ln\Big[\int_0^\infty e^{-kx_l}f^*(x_l|\boldsymbol{x})dx_l\Big] \\ &= \frac{-1}{k}\ln\Big[\int_0^\infty\int_0^\infty I_{2l}(\alpha,\lambda)\pi(\alpha,\lambda|\boldsymbol{x})d\alpha d\lambda\Big],\end{aligned} \tag{5.3}$$

where

$$I_{2l}(\alpha,\lambda)=\begin{cases}\frac{(r-1)!\alpha\lambda}{(l-1)!(r-l-1)!}\int_0^{x_r}e^{-kx_l}\frac{Q'(1/x_l)}{x_l^2}\frac{[1-(\phi(\lambda,x_l))^\alpha]^{l-1}}{[1-(\phi(\lambda,x_r))^\alpha]^{r-1}}(\phi(\lambda,x_l))^{\alpha-1}(1-\phi(\lambda,x_l)) \\ \qquad [(\phi(\lambda,x_l))^\alpha-(\phi(\lambda,x_r))^\alpha]^{r-l-1}dx_l, \quad l=1,2,\ldots,r-1 \\ \\ \frac{(n-s)!\alpha\lambda}{(l-s-1)!(n-l)!}\int_{x_s}^\infty e^{-kx_l}\frac{Q'(1/x_l)}{x_l^2}\frac{(\phi(\lambda,x_l))^{\alpha(n-l)}}{(\phi(\lambda,x_s))^{\alpha(n-s)}}(\phi(\lambda,x_l))^{\alpha-1}(1-\phi(\lambda,x_l)) \\ \qquad [(\phi(\lambda,x_s))^\alpha-(\phi(\lambda,x_l))^\alpha]^{l-s-1}dx_l, \quad l=s+1,s+2,\ldots,n.\end{cases}$$

Under general entropy loss function, for left as well as right predictive values are given by

$$\begin{aligned}\hat{x}_l &= \Big[\int_0^\infty x_l^{-m}f^*(x_l|\boldsymbol{x})dx\Big]^{-1/m} \\ &= \Big[\int_0^\infty\int_0^\infty I_{3l}(\alpha,\lambda)\pi(\alpha,\lambda|\boldsymbol{x})d\alpha d\lambda\Big]^{-1/m},\end{aligned} \tag{5.4}$$

where

$$I_{3l}(\alpha,\lambda)=\begin{cases}\frac{(r-1)!\alpha\lambda}{(l-1)!(r-l-1)!}\int_0^{x_r}x_l^{-m}\frac{Q'(1/x_l)}{x_l^2}\frac{[1-(\phi(\lambda,x_l))^\alpha]^{l-1}}{[1-(\phi(\lambda,x_r))^\alpha]^{r-1}}(\phi(\lambda,x_l))^{\alpha-1}(1-\phi(\lambda,x_l)) \\ \qquad [(\phi(\lambda,x_l))^\alpha-(\phi(\lambda,x_r))^\alpha]^{r-l-1}dx_l, \quad l=1,2,\ldots,r-1 \\ \\ \frac{(n-s)!\alpha\lambda}{(l-s-1)!(n-l)!}\int_{x_s}^\infty x_l^{-m}\frac{Q'(1/x_l)}{x_l^2}\frac{(\phi(\lambda,x_l))^{\alpha(n-l)}}{(\phi(\lambda,x_s))^{\alpha(n-s)}}(\phi(\lambda,x_l))^{\alpha-1}(1-\phi(\lambda,x_l)) \\ \qquad [(\phi(\lambda,x_s))^\alpha-(\phi(\lambda,x_l))^\alpha]^{l-s-1}dx_l, \quad l=s+1,s+2,\ldots,n.\end{cases}$$

It can be seen that equations (5.2), (5.3), and (5.4) are not in nice forms. Therefore, in order to predict x_l, we apply the Metropolis-Hastings algorithm. The Bayes predictive estimates of x_l under the squared error, LINEX, and general entropy loss functions turn out to be

- under squared error loss function, $\hat{x}_l = \frac{1}{D-D_0}\sum_{q=D_0+1}^{D} I_{1l}(\alpha_q,\lambda_q)$,
- under LINEX loss function, $\hat{x}_l = -\frac{1}{k}\ln\left(\frac{1}{D-D_0}\sum_{q=D_0+1}^{D} I_{2l}(\alpha_q,\lambda_q)\right)$,
- under general entropy loss function, $\hat{x}_l = \left(\frac{1}{D-D_0}\sum_{l=D_0+1}^{D} I_{3l}(\alpha_q,\lambda_q)^m\right)^{-1/m}$.

5.2. Interval prediction

Here, we evaluate Bayesian interval prediction based on the predictive survival function. The posterior predictive survival function is

$$S^*(t|\mathbf{X}) = \int_0^\infty \int_0^\infty S(t|\mathbf{X})\pi(\alpha,\lambda|\mathbf{X})d\alpha d\lambda,$$

where $S(t|\mathbf{X})$ denotes the prior predictive survival function and it is given by

$$S(t|\mathbf{X}) = \begin{cases} \frac{P(x_l<t|\mathbf{X})}{P(x_l<x_r|\mathbf{X})} = \frac{\int_0^t f_X(x_l|\mathbf{X})dx_l}{\int_0^{x_r} f_X(x_l|\mathbf{X})dx_l}, & l = 1,2,\ldots,r-1, \\ \\ \frac{P(x_l>t|\mathbf{X})}{P(x_l>x_s|\mathbf{X})} = \frac{\int_t^n f_X(x_l|\mathbf{X})dx_l}{\int_{x_s}^n f_X(x_l|\mathbf{X})dx_l}, & l = s+1, s+2,\ldots,n. \end{cases}$$

Now a $100(1-\nu)\%$ equal tail prediction interval (L,U) of x_l can be obtained from following equations

$$S^*(L|\mathbf{X}) = 1-\nu/2 \quad \text{and} \quad S^*(U|\mathbf{X}) = \nu/2.$$

In a similar manner, the corresponding predictive interval of x_l is obtained.

6. Simulation study and applications

In this section, we perform some numerical studies and evaluate behavior of proposed estimation and prediction estimates. We consider $Q(x) = 1/x$ and show the results for the inverted exponentiated exponential (IEE) distribution. We, firstly, conduct extensive Monte Carlo simulation experiments in order to investigate the performance of the proposed estimations. Then, we analyzed two real data sets in reference to the studied problem.

6.1. Monte Carlo Simulation study

We study performances of various estimators through Monte Carlo simulation. Point estimators of α and λ are evaluated based on average estimates (AEs) and mean squared errors (MSEs). Note that Bayes estimates are obtained under squared error (SE), LINEX, and general entropy (GE) loss functions. Interval estimators are compared against average lengths (ALs) and coverage probabilities (CPs). For the Bayes estimate, informative prior (IP) and non-informative prior (NIP) are considered for finding estimates. Under IP setup, hyper-parameters are chosen in such a way that the prior mean is close to the true parameter. For the simulation, We take two values of n as 50 and 80. Corresponding to the values of n, we take a different set of values of r and s. The AEs and MSEs are computed for various combinations of (n,r,s). For the simulation, parameters are assigned as $(\alpha,\lambda) = (1.2, 0.8)$ and $(0.80, 1.40)$ in an arbitrary manner. The hyper-parameters are assigned as closed to zero value in the case of NIP. Under IP case, these are assigned as $(a,b,c,d) = (2.16, 1.80, 1.92, 2.40)$ and $(1.44, 1.80, 3.36, 2.40)$. The results for point estimates are tabulated in Tables 1 and 3, whereas for interval estimation, it is tabulated in Table 2 and 4 based on 10,000 simulation runs. The nominal confidence level is 95%. One may refer to [3] and [7] for some useful discussion on performance of various interval estimates in such situations. In addition, predictive estimates for censored observation based on different loss functions and two sets of prior are computed in Table 5. We draw the following conclusion from the simulation results:

- As the effective sample size increases (s increases or r decreases), the MLEs and the Bayes estimate under different loss functions are improved in terms of the lesser value of MSEs.

Table 1. AEs and MSEs of all estimates for $\alpha = 1.2$ and $\lambda = 0.8$.

			MLE	Bayesian (NIP)					Bayesian (IP)				
				SE	LINEX		GE		SE	LINEX		GE	
(n,r,s)	θ				$k=-0.5$	$k=0.5$	$m=-1.5$	$m=-0.5$		$k=-0.5$	$k=0.5$	$m=-1.5$	$m=-0.5$
(50,11,30)	α	AE	1.3420	1.3177	1.2854	1.3527	1.2946	1.3408	1.2791	1.2545	1.3055	1.2607	1.2976
		MSE	0.1631	0.1362	0.1175	0.1596	0.1264	0.1472	0.0807	0.0711	0.0927	0.0754	0.0867
	λ	AE	0.8673	0.8448	0.8350	0.8548	0.8331	0.8563	0.8320	0.8239	0.8402	0.8222	0.8416
		MSE	0.0480	0.0406	0.0386	0.0428	0.0392	0.0421	0.0272	0.0259	0.0286	0.0263	0.0282
(50,11,35)	α	AE	1.3463	1.3198	1.2781	1.3660	1.2899	1.3500	1.2729	1.2428	1.3058	1.2502	1.2959
		MSE	0.1693	0.1389	0.1156	0.1698	0.1264	0.1534	0.0753	0.0649	0.0892	0.0694	0.0823
	λ	AE	0.8631	0.8366	0.8254	0.8479	0.8231	0.8497	0.8231	0.8142	0.8321	0.8123	0.8337
		MSE	0.0476	0.0393	0.0373	0.0416	0.0380	0.0409	0.0249	0.0237	0.0262	0.0241	0.0258
(50,21,45)	α	AE	1.3220	1.3020	1.2771	1.3285	1.2840	1.3200	1.2737	1.2538	1.2948	1.2588	1.2888
		MSE	0.1344	0.1165	0.1033	0.1325	0.1096	0.1241	0.0756	0.068	0.0847	0.0715	0.0802
	λ	AE	0.8608	0.8425	0.8338	0.8513	0.8322	0.8526	0.8321	0.8247	0.8396	0.8232	0.8408
		MSE	0.0427	0.0374	0.0357	0.0393	0.0363	0.0388	0.0266	0.0254	0.0278	0.0258	0.0275
(50,11,40)	α	AE	1.2970	1.2731	1.2216	1.3315	1.2346	1.3119	1.2307	1.1954	1.2699	1.2029	1.2587
		MSE	0.1459	0.1201	0.0984	0.1525	0.1083	0.1348	0.0607	0.0530	0.0729	0.0566	0.0665
	λ	AE	0.8413	0.8106	0.7978	0.8237	0.7946	0.8262	0.8029	0.7933	0.8128	0.7908	0.8148
		MSE	0.0443	0.0365	0.0348	0.0385	0.0358	0.0376	0.0222	0.0214	0.0232	0.0219	0.0227
(50,16,45)	α	AE	1.3009	1.2796	1.2528	1.3084	1.2598	1.2995	1.2537	1.2328	1.2758	1.2376	1.2698
		MSE	0.1045	0.0945	0.0830	0.1091	0.0883	0.1016	0.0593	0.0533	0.0669	0.0559	0.0632
	λ	AE	0.8619	0.8389	0.8280	0.8500	0.8258	0.8516	0.8284	0.8194	0.8376	0.8175	0.8391
		MSE	0.0477	0.0413	0.0391	0.0438	0.0399	0.0430	0.0281	0.0268	0.0297	0.0272	0.0292
(50,11,45)	α	AE	1.2319	1.2102	1.1835	1.2391	1.1891	1.2315	1.1988	1.1783	1.2207	1.1822	1.2156
		MSE	0.0639	0.0605	0.0550	0.0686	0.0578	0.0642	0.0387	0.0363	0.0424	0.0376	0.0404
	λ	AE	0.8331	0.8057	0.7921	0.8196	0.7887	0.8222	0.8027	0.7919	0.8137	0.7892	0.8159
		MSE	0.0448	0.0382	0.0364	0.0404	0.0375	0.0394	0.0253	0.0243	0.0266	0.0249	0.0261
(80,11,60)	α	AE	1.2955	1.2873	1.2669	1.3089	1.2723	1.3023	1.2657	1.2487	1.2835	1.2529	1.2785
		MSE	0.1040	0.0967	0.0870	0.1083	0.0915	0.1023	0.0674	0.0615	0.0745	0.0642	0.0711
	λ	AE	0.8393	0.8287	0.8232	0.8343	0.8221	0.8352	0.8221	0.8173	0.8270	0.8163	0.8279
		MSE	0.0247	0.0230	0.0223	0.0238	0.0225	0.0236	0.0176	0.0171	0.0182	0.0172	0.0180
(80,11,65)	α	AE	1.2817	1.2742	1.2573	1.2920	1.2617	1.2869	1.2583	1.2438	1.2733	1.2473	1.2693
		MSE	0.0812	0.0781	0.0712	0.0862	0.0744	0.0821	0.0570	0.0526	0.0623	0.0546	0.0598
	λ	AE	0.8383	0.8290	0.8239	0.8341	0.8229	0.8350	0.8235	0.8190	0.8281	0.8181	0.8290
		MSE	0.0236	0.0224	0.0218	0.0232	0.0219	0.0230	0.0177	0.0172	0.0182	0.0173	0.0181
(80,21,75)	α	AE	1.2658	1.2559	1.2420	1.2703	1.2453	1.2665	1.2445	1.2324	1.2571	1.2352	1.2539
		MSE	0.0660	0.0638	0.0592	0.0692	0.0613	0.0665	0.0485	0.0454	0.0521	0.0468	0.0504
	λ	AE	0.8363	0.8255	0.8198	0.8312	0.8186	0.8323	0.8208	0.8157	0.8260	0.8146	0.8270
		MSE	0.0251	0.0239	0.0231	0.0247	0.0234	0.0244	0.0189	0.0184	0.0195	0.0185	0.0194
(80,11,70)	α	AE	1.2727	1.2652	1.2510	1.2801	1.2545	1.2760	1.2529	1.2404	1.2659	1.2433	1.2625
		MSE	0.0718	0.0693	0.0640	0.0754	0.0665	0.0724	0.0528	0.0492	0.0570	0.0509	0.0550
	λ	AE	0.8345	0.8259	0.8213	0.8307	0.8203	0.8315	0.8216	0.8174	0.8259	0.8165	0.8267
		MSE	0.0219	0.0209	0.0204	0.0216	0.0205	0.0214	0.0169	0.0165	0.0174	0.0166	0.0173
(80,16,75)	α	AE	1.2668	1.2581	1.2452	1.2716	1.2483	1.2680	1.2476	1.2362	1.2594	1.2387	1.2565
		MSE	0.0637	0.0615	0.0572	0.0665	0.0592	0.0641	0.0477	0.0447	0.0512	0.0460	0.0495
	λ	AE	0.8356	0.8263	0.8213	0.8313	0.8203	0.8322	0.8221	0.8176	0.8267	0.8167	0.8275
		MSE	0.0232	0.0221	0.0215	0.0228	0.0217	0.0226	0.0180	0.0175	0.0185	0.0176	0.0184
(80,11,75)	α	AE	1.2603	1.2531	1.2410	1.2655	1.2438	1.2623	1.2441	1.2334	1.2552	1.2358	1.2525
		MSE	0.0552	0.0541	0.0505	0.0582	0.0522	0.0563	0.0432	0.0406	0.0462	0.0418	0.0448
	λ	AE	0.8310	0.8233	0.8189	0.8277	0.8180	0.8285	0.8198	0.8158	0.8239	0.8150	0.8246
		MSE	0.0195	0.0187	0.0183	0.0193	0.0184	0.0191	0.0157	0.0153	0.0161	0.0154	0.0160

- In the case of point estimation, the MSEs under Bayesian approach are less than those under maximum likelihood method in most cases. It is also observed that the MSEs under NIP are higher than that of IP setup.
- We note that MLEs of both the parameters compete well with Bayes estimates obtained under noninformative prior. In this case, MSE values for MLEs get closer to the corresponding NIP estimates with an increase in effective sample size. Furthermore, performance of respective Bayes estimates obtained under informative prior is quite good compared to both MLEs and NIP estimates. Under LINEX loss,

Table 2. ALs and CPs of all estimates for $\alpha = 1.2$ and $\lambda = 0.8$.

		ACI		BCI			
				NIP		IP	
(n, r, s)	Criterion	α	λ	α	λ	α	λ
(50,11,30)	AL	2.8700	1.2590	1.4715	0.7958	1.4064	0.7520
	CP	0.9530	0.9696	0.9797	0.9795	0.9797	0.9815
(50,11,35)	AL	2.2252	1.0399	1.3424	0.7481	1.3011	0.7202
	CP	0.9631	0.9770	0.9761	0.9728	0.9766	0.9795
(50,21,45)	AL	2.0145	1.3435	1.1327	0.8329	1.0827	0.7921
	CP	0.9548	0.9644	0.9719	0.9696	0.9719	0.9734
(50,11,40)	AL	1.7294	0.8907	1.2045	0.7095	1.1764	0.6898
	CP	0.9689	0.9775	0.9613	0.9559	0.9771	0.9698
(50,16,45)	AL	1.6791	1.0078	1.1209	0.7486	1.0884	0.7252
	CP	0.9649	0.9737	0.9755	0.9651	0.9759	0.9741
(50,11,45)	AL	1.3660	0.7826	1.0827	0.6733	1.0604	0.6578
	CP	0.9658	0.9717	0.9501	0.9505	0.9620	0.9651
(80,11,60)	AL	1.1481	0.5966	0.9948	0.5441	0.9781	0.5345
	CP	0.9663	0.962	0.9525	0.9527	0.9595	0.9604
(80,11,65)	AL	1.0313	0.5685	0.9215	0.5270	0.9068	0.5182
	CP	0.9680	0.9545	0.9543	0.9465	0.9602	0.9557
(80,21,75)	AL	0.9918	0.6284	0.8465	0.5591	0.8329	0.5493
	CP	0.9665	0.9721	0.9525	0.9518	0.9595	0.9595
(80,11,70)	AL	0.9428	0.5434	0.8579	0.5091	0.8459	0.5016
	CP	0.9644	0.9516	0.9491	0.9449	0.9573	0.9543
(80,16,75)	AL	0.9199	0.5682	0.8231	0.5248	0.8114	0.5164
	CP	0.9716	0.9548	0.9537	0.9449	0.9579	0.9523
(80,11,75)	AL	0.8633	0.5219	0.7977	0.4936	0.7876	0.4869
	CP	0.9577	0.9541	0.9471	0.9491	0.9521	0.9561

it is seen that assignment $k = -0.5$ yields marginally better estimates compared to the other choices of k. This holds for both NIP and IP prior cases. Similar observations hold for estimates obtained under GE loss function. Overall, Bayes estimates evaluated under assignments $k = -0.5$ and $m = -1.5$ provide really good estimates compared to other tabulated estimates.

- In the case of interval estimation, the ALs reduce as sample sizes increase. All the CPs are close to the nominal level of 0.95.
- We also observe that the ALs of the Bayesian credible intervals (BCIs) are smaller than those of approximate confidence intervals (ACIs). Also, ALs have larger values in the case of NIP than those of IP distributions.
- From Table 5, it can be seen that the point prediction under squared error, LINEX, and general entropy are close to the true value, and the Bayesian prediction interval (BPI) contains the true value.
- We observe that increased sample sizes lead to improved estimates for unknown parameters. This holds for both point and interval estimation.

6.2. Application

In this section, we analyze two real datasets for illustrative purposes.

6.2.1. Application I. The dataset I is taken from [22] and shown in Table 6. It represents the survival time (in days) of the 58 patients with Hypernephroma.

Considered data is fitted by the IEE distribution which is a member of inverted exponentiated family. The value of Kolmogorov-Smirnov's (K-S) test is 0.0904 with p-value

Table 3. AEs and MSEs of all estimates for $\alpha = 0.8$ and $\lambda = 1.4$.

			MLE	Bayesian (NIP)					Bayesian (IP)				
				SE	LINEX		GE		SE	LINEX		GE	
(n, r, s)	θ				$k = -0.5$	$k = 0.5$	$m = -1.5$	$m = -0.5$		$k = -0.5$	$k = 0.5$	$m = -1.5$	$m = -0.5$
(50,11,30)	α	AE	0.9508	0.9076	0.8854	0.9313	0.8849	0.9304	0.8672	0.8517	0.8837	0.8504	0.8841
		MSE	0.1388	0.0878	0.0771	0.1004	0.0798	0.0967	0.0452	0.0403	0.0511	0.0413	0.0497
	λ	AE	1.5916	1.5127	1.4614	1.5664	1.4779	1.5463	1.4679	1.432	1.5054	1.4432	1.4921
		MSE	0.2907	0.2085	0.1828	0.241	0.1988	0.22	0.0995	0.0887	0.1138	0.0948	0.1052
(50,11,35)	α	AE	0.9151	0.8876	0.8715	0.9046	0.8708	0.9045	0.8577	0.8458	0.8702	0.8447	0.8708
		MSE	0.0990	0.0714	0.0644	0.0795	0.0662	0.0772	0.0404	0.0368	0.0445	0.0376	0.0435
	λ	AE	1.5738	1.5168	1.4719	1.5637	1.4867	1.5460	1.4763	1.4434	1.5106	1.4539	1.4983
		MSE	0.2538	0.1983	0.1755	0.2267	0.1894	0.2087	0.1034	0.0926	0.1173	0.0987	0.1090
(50,21,45)	α	AE	0.8840	0.8593	0.8475	0.8716	0.8464	0.8721	0.8380	0.8294	0.8470	0.8282	0.8479
		MSE	0.0600	0.0485	0.0446	0.0530	0.0456	0.0518	0.0270	0.0252	0.0292	0.0255	0.0287
	λ	AE	1.6072	1.5373	1.4760	1.6023	1.4966	1.5766	1.4855	1.4431	1.5306	1.4568	1.5138
		MSE	0.3598	0.2661	0.2285	0.3143	0.2520	0.2828	0.1242	0.1084	0.1455	0.1173	0.1326
(50,11,40)	α	AE	0.8943	0.8756	0.8635	0.8884	0.8627	0.8886	0.8530	0.8436	0.8628	0.8426	0.8635
		MSE	0.0695	0.0568	0.0520	0.0621	0.0532	0.0607	0.0349	0.0323	0.0378	0.0328	0.0371
	λ	AE	1.5642	1.5203	1.4802	1.5621	1.4938	1.5462	1.483	1.4526	1.5145	1.4625	1.5031
		MSE	0.2254	0.1886	0.1677	0.2141	0.1802	0.1982	0.1053	0.0946	0.1187	0.1006	0.1108
(50,16,45)	α	AE	0.8796	0.8611	0.8507	0.8720	0.8498	0.8725	0.8423	0.8342	0.8507	0.8332	0.8514
		MSE	0.0572	0.0483	0.0448	0.0523	0.0456	0.0513	0.0298	0.0279	0.0320	0.0283	0.0315
	λ	AE	1.5687	1.5207	1.4745	1.5692	1.4901	1.5506	1.4809	1.4467	1.5167	1.4578	1.5037
		MSE	0.2532	0.2090	0.1840	0.2401	0.1992	0.2203	0.1112	0.0992	0.1268	0.1060	0.1175
(50,11,45)	α	AE	0.8687	0.8548	0.8456	0.8643	0.8447	0.8649	0.8407	0.8332	0.8484	0.8322	0.8491
		MSE	0.0482	0.0426	0.0397	0.0458	0.0404	0.0450	0.0287	0.0270	0.0306	0.0274	0.0302
	λ	AE	1.5296	1.4948	1.4595	1.5316	1.4712	1.5180	1.4679	1.4403	1.4966	1.4491	1.4864
		MSE	0.1821	0.1605	0.1445	0.1801	0.1542	0.1678	0.0961	0.0874	0.1069	0.0923	0.1005
(80,11,60)	α	AE	0.8489	0.8421	0.8350	0.8495	0.8341	0.8502	0.8331	0.8269	0.8394	0.8260	0.8401
		MSE	0.0325	0.0309	0.0291	0.0329	0.0295	0.0324	0.0227	0.0215	0.0240	0.0218	0.0238
	λ	AE	1.4778	1.4584	1.4374	1.4799	1.444	1.4726	1.4453	1.4275	1.4635	1.433	1.4574
		MSE	0.0979	0.0924	0.0864	0.0997	0.0899	0.0953	0.0655	0.0616	0.0702	0.0638	0.0674
(80,11,65)	α	AE	0.8429	0.8363	0.8302	0.8425	0.8294	0.8432	0.8295	0.8242	0.8349	0.8234	0.8356
		MSE	0.0293	0.0282	0.0268	0.0297	0.0272	0.0294	0.0217	0.0208	0.0228	0.0210	0.0226
	λ	AE	1.4694	1.4516	1.4322	1.4715	1.4383	1.4648	1.4409	1.4242	1.4580	1.4294	1.4524
		MSE	0.0904	0.0862	0.0811	0.0924	0.0842	0.0886	0.0629	0.0594	0.0671	0.0614	0.0646
(80,21,75)	α	AE	0.843	0.8349	0.8297	0.8402	0.8289	0.8409	0.8287	0.8242	0.8334	0.8234	0.8340
		MSE	0.0244	0.0234	0.0224	0.0245	0.0226	0.0243	0.0180	0.0173	0.0188	0.0174	0.0186
	λ	AE	1.4896	1.4683	1.4446	1.4926	1.4522	1.4841	1.4542	1.4342	1.4746	1.4405	1.4677
		MSE	0.1112	0.1049	0.0972	0.1141	0.1017	0.1085	0.0734	0.0685	0.0795	0.0713	0.0760
(80,11,70)	α	AE	0.8382	0.8322	0.8270	0.8374	0.8262	0.8381	0.8267	0.8221	0.8314	0.8214	0.8321
		MSE	0.0239	0.0233	0.0223	0.0244	0.0225	0.0241	0.0185	0.0178	0.0193	0.0179	0.0192
	λ	AE	1.4689	1.4528	1.4346	1.4715	1.4403	1.4652	1.4432	1.4273	1.4595	1.4322	1.4541
		MSE	0.0838	0.08030	0.0756	0.0859	0.0783	0.0825	0.0601	0.0568	0.0640	0.0586	0.0617
(80,16,75)	α	AE	0.8361	0.8293	0.8245	0.8342	0.8238	0.8349	0.8244	0.8201	0.8287	0.8194	0.8294
		MSE	0.0218	0.0212	0.0203	0.0221	0.0205	0.0219	0.0169	0.0163	0.0175	0.0164	0.0174
	λ	AE	1.4703	1.4526	1.4327	1.4731	1.4389	1.4661	1.4425	1.4254	1.4601	1.4307	1.4543
		MSE	0.0900	0.0862	0.0809	0.0926	0.0840	0.0887	0.0633	0.0597	0.0676	0.0618	0.0651
(80,11,75)	α	AE	0.8361	0.8303	0.8258	0.8349	0.8251	0.8355	0.8257	0.8217	0.8299	0.8210	0.8305
		MSE	0.0209	0.0204	0.0197	0.0213	0.0198	0.0211	0.0167	0.0161	0.0173	0.0162	0.0172
	λ	AE	1.4687	1.4537	1.4365	1.4714	1.4419	1.4655	1.4448	1.4296	1.4603	1.4343	1.4552
		MSE	0.0784	0.0754	0.0711	0.0805	0.0736	0.0775	0.0574	0.0544	0.0611	0.0561	0.0590

0.7302. Since the p-value is relatively high, which shows that the data are reasonably fitted by IEE distribution. Further, we also show the fitting using some graphical technique such as empirical CDF versus theoretical CDF plot, Probability-Probability (P-P) plot and Quantile-Quantile (Q-Q) plot in Figure 1. These plots also suggest that the data are reasonably fitted by IEE distribution.

Next, we generate two sets of doubly censored data by choosing specific values of r and s, as shown in Table 7.

The convergence of MCMC samples under given data sets is illustrated in Figure 2 via trace and density plots of posterior samples of α and λ. Figure 2 indicates that

Table 4. ALs and CPs of all estimates for $\alpha = 0.8$ and $\lambda = 1.4$.

		ACI		BCI			
				NIP		IP	
(n, r, s)	Criterion	α	λ	α	λ	α	λ
(50,11,30)	AL	1.4869	2.026	0.9499	1.5342	0.9199	1.4482
	CP	0.9510	0.9640	0.9560	0.9720	0.9800	0.9880
(50,11,35)	AL	1.0991	1.7496	0.8330	1.4541	0.8100	1.3864
	CP	0.9582	0.9631	0.9500	0.9552	0.9674	0.9766
(50,21,45)	AL	0.9265	2.0831	0.7254	1.6587	0.6965	1.5696
	CP	0.9518	0.9507	0.9651	0.9550	0.9786	0.9831
(50,11,40)	AL	0.8802	1.5867	0.7425	1.3858	0.7244	1.3310
	CP	0.9685	0.9532	0.9512	0.9518	0.9683	0.9730
(50,16,45)	AL	0.8094	1.7000	0.6931	1.4753	0.6737	1.4110
	CP	0.9599	0.9548	0.9555	0.9539	0.9665	0.9725
(50,11,45)	AL	0.7409	1.4641	0.6634	1.3215	0.6480	1.2707
	CP	0.9642	0.9462	0.9516	0.9492	0.9600	0.9656
(80,11,60)	AL	0.6476	1.1222	0.6016	1.0540	0.5905	1.0246
	CP	0.9627	0.9512	0.9608	0.9550	0.9654	0.9636
(80,11,65)	AL	0.5949	1.0792	0.5586	1.0192	0.5498	0.9940
	CP	0.9546	0.9478	0.9478	0.9528	0.9545	0.9620
(80,21,75)	AL	0.5481	1.1737	0.5215	1.1150	0.5120	1.0840
	CP	0.9581	0.9464	0.9532	0.953	0.9609	0.9638
(80,11,70)	AL	0.5509	1.0471	0.5224	0.9936	0.5147	0.9706
	CP	0.9534	0.9492	0.9510	0.9525	0.9548	0.9618
(80,16,75)	AL	0.5280	1.0857	0.5042	1.0354	0.4961	1.0081
	CP	0.9548	0.9487	0.9530	0.9543	0.9595	0.9613
(80,11,75)	AL	0.5154	1.0188	0.4915	0.9698	0.4849	0.9481
	CP	0.9532	0.9519	0.9501	0.9545	0.9537	0.9608

MCMC chains provide acceptable behavior for α and β. The plot of density converges well under varying chains. This indicates that samples may have been generated from assumed posterior distribution. Next, we obtain estimates of α and λ under doubly censored data. When there is no information on parameter, Bayes estimates are evaluated under NIP. The results are given in Table 8.

The results listed in Table 8 suggest that all the point estimate are closed to each other, and Bayes estimates under LINEX loss function is under-estimated in the positive loss parameter and over-estimated in the negative loss parameter. Also, the same results are obtained in the case of the general entropy loss function. Further, Bayes intervals are better than approximate intervals under the criterion of interval length.

6.2.2. Application II. This data represent failure times of air conditioning systems of planes, see [34]. The data are listed in Table 9. We divide data by 10 for ease of computation purpose. Similarly, as in Dataset I, we also draw the plots for Dataset II.

Again we check goodness-of-fit for this data by IEE distribution using K-S test. It is found that the K-S value is 0.2109 and p-value is 0.1388. We observe that p-value is greater than pre-specified significance level 0.05. This reasonably indicates that the considered data set is appropriately fitted by IEE model. Further, we also show the fitting using some graphical techniques in Figure 3, which also suggests that the data fitted good by the IEE distribution.

Here, we generate two different sets of doubly censored data by choosing specific values of r and s as shown in Table 10.

Table 5. Bayesian point and interval predictive estimations under different prior settings.

			Point prediction			Interval prediction	
(n, r, s)	l	True value	SE	LINEX (-0.002)	GE (-0.700)	BPI	Length
Prior I: $a = 1.44$, $b = 1.8$, $c = 2.88$, $d = 2.40$							
(30,5,26)	1	0.2053	0.3663	0.3663	0.3608	(0.2828 0.4585)	0.1757
	2	0.3619	0.4850	0.4850	0.4808	(0.3997 0.5697)	0.1700
	3	0.3972	0.5871	0.5871	0.5842	(0.5212 0.6486)	0.1274
	4	0.6444	0.6822	0.6822	0.6806	(0.6415 0.7151)	0.0736
	27	27.1443	24.9352	25.4548	24.1317	(19.9143 31.7562)	11.8419
	28	48.3692	44.6443	47.5444	40.8333	(25.9927 72.1402)	46.1475
	29	104.7182	116.2216	109.5389	102.9703	(68.5045 134.1018)	65.6035
	30	165.7776	175.7860	174.2773	142.8336	(129.2558 196.9236)	67.6678
(35,6,30)	1	0.2293	0.3089	0.3088	0.3044	(0.2291 0.3829)	0.1538
	2	0.2397	0.4015	0.4015	0.3981	(0.3296 0.4730)	0.1434
	3	0.3167	0.4809	0.4809	0.4783	(0.4141 0.5439)	0.1298
	4	0.4182	0.5519	0.5519	0.5500	(0.5011 0.5993)	0.0982
	5	0.4581	0.6220	0.6220	0.6210	(0.5919 0.6469)	0.0550
	31	13.1001	15.2949	16.3833	14.9864	(12.9653 18.2609)	5.2956
	32	24.1963	23.9111	23.6154	22.5257	(15.3677 35.0016)	19.6339
	33	44.2238	43.5758	47.3863	38.8193	(20.7207 74.4157)	53.6950
	34	105.4059	119.2495	107.2124	101.0801	(61.1404 131.1298)	69.9894
	35	141.8698	169.0679	164.7949	135.4872	(122.1321 191.3960)	69.2639
Prior II: $a = 2.16$, $b = 1.80$, $c = 3.60$, $d = 2.40$							
(30,5,26)	1	0.3257	0.3290	0.3290	0.3258	(0.2683 0.3838)	0.1155
	2	0.3695	0.4108	0.4108	0.4087	(0.3571 0.4584)	0.1013
	3	0.5593	0.4759	0.4759	0.4746	(0.4342 0.5097)	0.0755
	4	0.5759	0.5316	0.5316	0.5309	(0.5102 0.5501)	0.0399
	27	6.1942	6.2258	6.1611	6.1039	(5.0189 7.2616)	2.2427
	28	6.5969	9.2037	36.4867	8.9900	(7.2277 12.7268)	5.4991
	29	10.4379	17.0450	7.5380	15.1259	(8.2520 31.6645)	23.4125
	30	17.1398	52.1332	58.7644	41.4539	(15.8194 99.6399)	83.8205
(35,6,30)	1	0.3852	0.2970	0.2970	0.2929	(0.2275 0.3613)	0.1338
	2	0.4389	0.3864	0.3864	0.3833	(0.3234 0.4508)	0.1274
	3	0.4418	0.4549	0.4549	0.4524	(0.3914 0.5085)	0.1171
	4	0.4511	0.5226	0.5226	0.5209	(0.4748 0.5602)	0.0854
	5	0.5187	0.5883	0.5883	0.5874	(0.5643 0.6104)	0.0461
	31	14.7049	14.0760	15.8457	13.9040	(12.3943 16.2255)	3.8312
	32	18.2224	20.0438	19.0002	19.2751	(14.7475 27.1166)	12.3691
	33	24.4327	34.4450	35.6048	31.4276	(18.3428 58.7377)	40.3949
	34	35.2306	68.0574	77.6848	58.8785	(29.2314 120.6116)	91.3802
	35	70.4484	148.8468	158.1031	122.1909	(91.8836 190.0679)	98.1843

Table 6. Real dataset I from [22].

77	18	8	68	35	8	26	84	17	52	26	108	18	72	38	99	9	56
36	108	10	36	6	9	12	5	104	6	115	9	21	14	52	9	48	15
5	28	25	25	40	16	8	70	6	8	12	20	8	99	12	181	20	14
26	16	30	20														

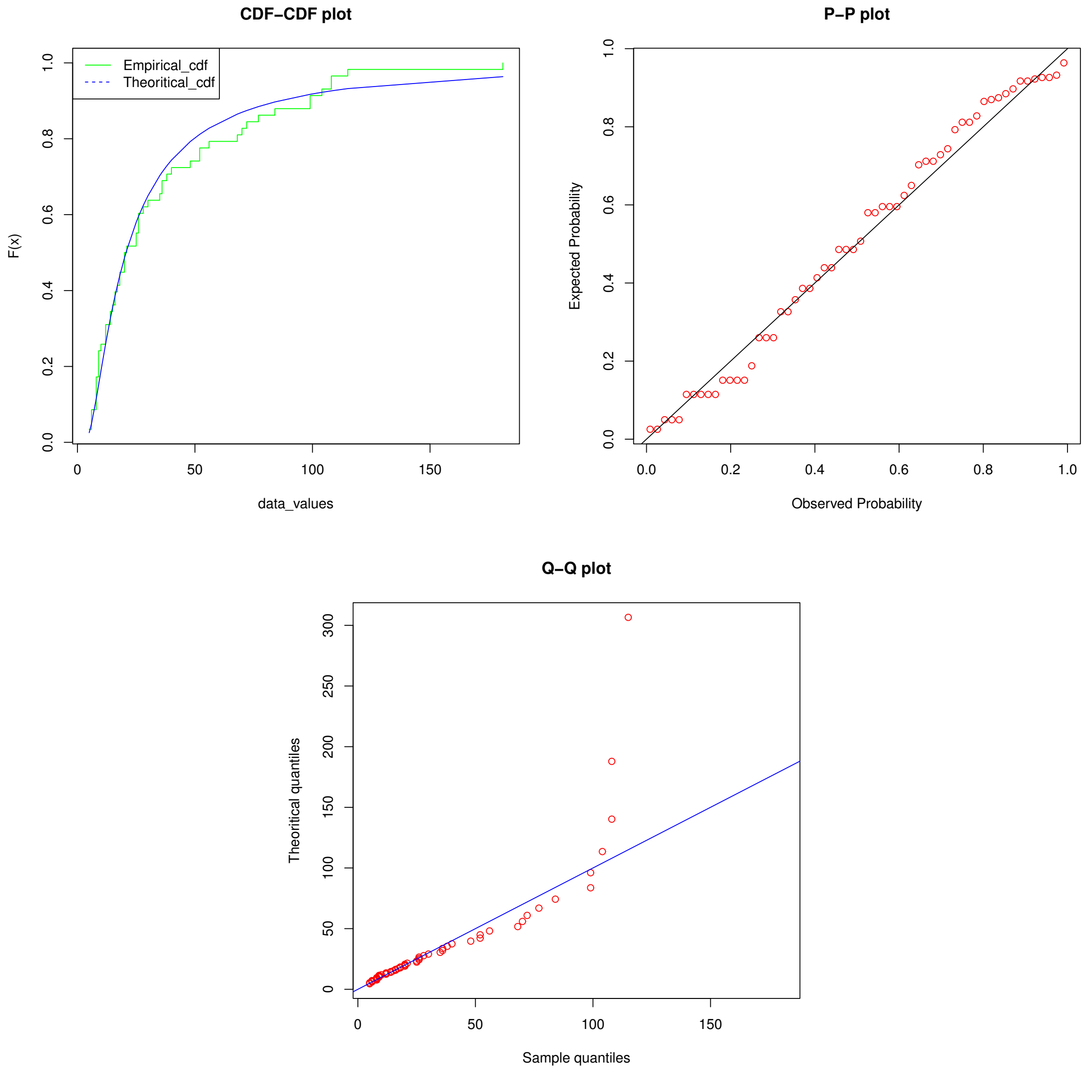


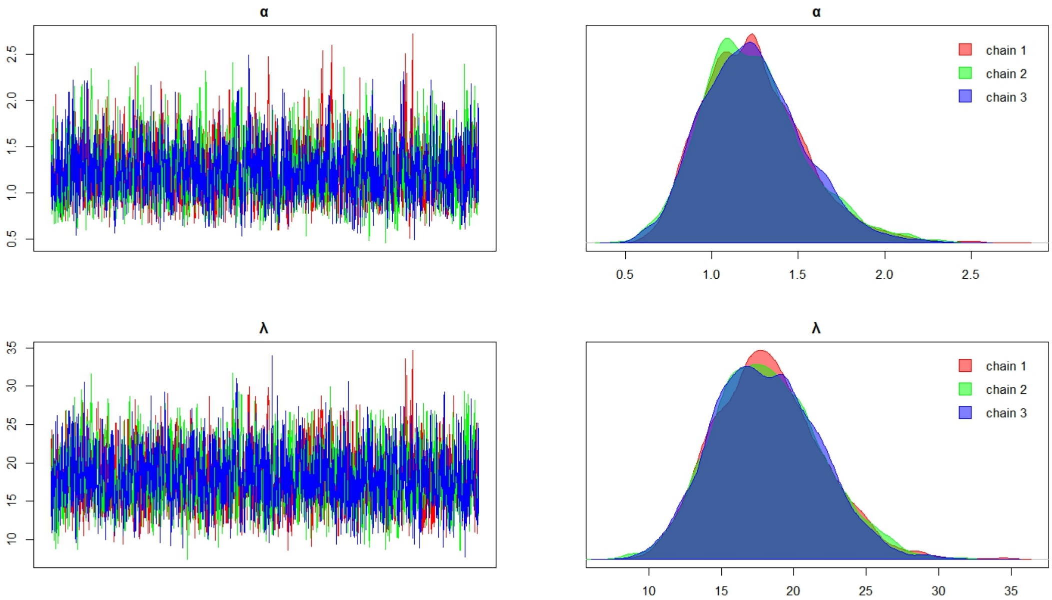
Figure 1. Empirical and theoretical distribution plots, PP plot, and QQ plot for the IEE distribution from dataset I.

Table 7. Doubly censored data from dataset I.

Data 1: $(n, r, s) = (58, 6, 45)$																	
8	8	8	8	8	9	9	9	9	10	12	12	12	14	14	15	16	16
17	18	18	20	20	20	21	25	25	26	26	26	28	30	35	36	36	38
40	48	52	52														
Data 2: $(n, r, s) = (58, 11, 45)$																	
9	9	9	9	10	12	12	12	14	14	15	16	16	17	18	18	20	20
20	21	25	25	26	26	26	28	30	35	36	36	38	40	48	52	52	

Figure 4 indicates that MCMC chains show acceptable pattern for α and β under given data. The plot of density converges well under different chains which indicates that samples may come from assumed posterior distribution. We next obtain estimates of α and λ under doubly censored data. We evaluate Bayes estimates under NIP. We report the results in Table 11.

(a) Trace and density plot of both parameters α (up) and λ (down) for data 1 of dataset I.



(b) Trace and density plot of both parameters α (up) and λ (down) for data 2 of real dataset I.

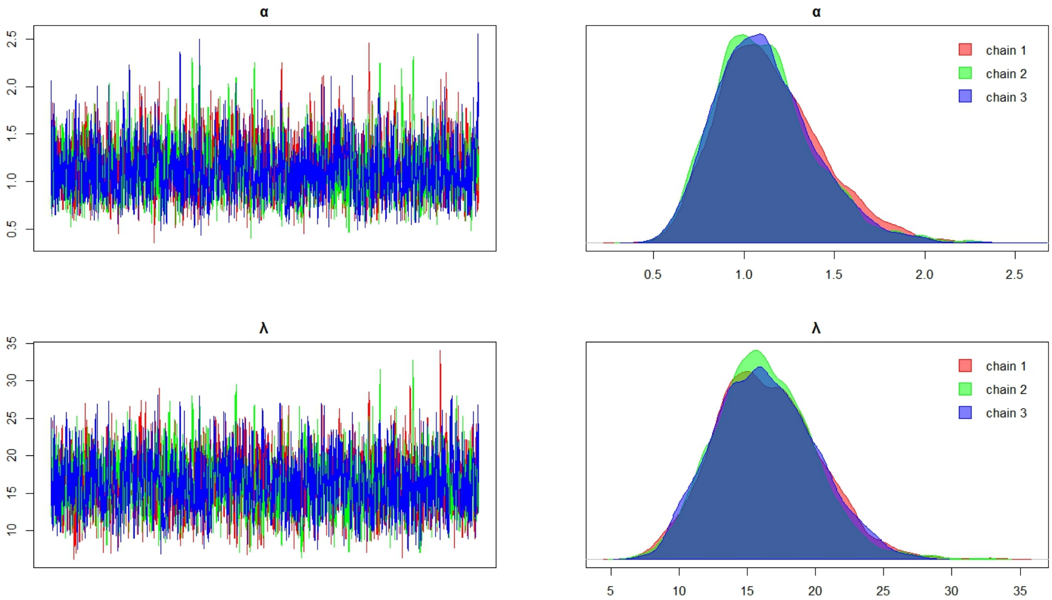


Figure 2. Trace and density plot for dataset I.

Furthermore, we obtain the point and interval predictive values of the censored observations. From Table 12, we observe that the predictive estimates of the first four censored data are close to the true value. The same results are obtained for the last four observations as well. We also observe that the difference between the true value and the predicted value is less than the nearer value from the censored data. Similarly, we can obtain the predictive value of censored observation for dataset II. To save space, the results are not listed here.

Table 8. Point and interval estimates for dataset I.

Data 1			α	λ
MLE			1.2438	18.4017
Bayes estimate	SE		1.2482	18.3802
	LINEX	$k = 0.5$	1.2263	15.5488
		$k = -0.5$	1.2716	22.8323
	GE	$m = -0.5$	1.2306	18.1971
		$m = -1.5$	1.2659	18.5616
ACI			(0.6674 1.8202)[1.1528]	(11.3100 25.4934)[14.1834]
BCI			(0.6903 1.8261)[1.1358]	(11.4434 25.5885)[14.1451]
Data 2			α	λ
MLE			1.1323	16.5044
Bayes estimate	SE		1.1290	16.3202
	LINEX	$k = 0.5$	1.1080	13.4789
		$k = -0.5$	1.1523	23.3725
	GE	$m = -0.5$	1.1106	16.1039
		$m = -1.5$	1.1479	16.5355
ACI			(0.5804 1.6841)[1.1037]	(9.1912 23.8175)[14.6263]
BCI			(0.6086 1.6802)[1.0716]	(9.2161 23.5245)[14.3087]

Table 9. Dataset II from [34]

1	3	5	7	11	11	11	12	14	14	14	16	16	20	21	23	42	47
52	62	71	71	87	90	95	120	120	225	246	261						

Table 10. Doubly censored data from dataset II

Data 1: $(n, r, s) = (30, 7, 23)$																	
1.1	1.2	1.4	1.4	1.4	1.6	1.6	2.0	2.1	2.3	4.2	4.7	5.2	6.2	7.1	7.1	8.7	

Data 2: $(n, r, s) = (30, 4, 26)$																	
0.7	1.1	1.1	1.1	1.2	1.4	1.4	1.4	1.6	1.6	2.0	2.1	2.3	4.2	4.7	5.2	6.2	7.1
7.1	8.7	9.0	9.5	12.0	12.0												

7. Bayesian optimal design

We have discussed till now estimations of parameters under double censoring scheme when the lifetime of an item follows IEE distribution. We evaluated estimates when censoring scheme, *i.e.*, (n, r, s), is known in advance. The design parameters (n, r, s) can be selected in many ways for a given sample size n_0. Due to cost factors, finding optimal schemes for various practical scenarios may be difficult. In fact, deriving the optimal plans for life tests has received wide attention under different censoring schemes. Pradhan and Kundu [31], Pareek et al. [28], Kundu [20], Bhattacharya et al. [5], Lodhi et al. [23] presented many important results in this direction. Optimal criterion selection is another important factor in such inferences. Here we adopt Bayesian approach for obtaining such plans under double censoring. The posterior variance is taken up as the optimal criterion measure for the censoring scheme (n, r, s).

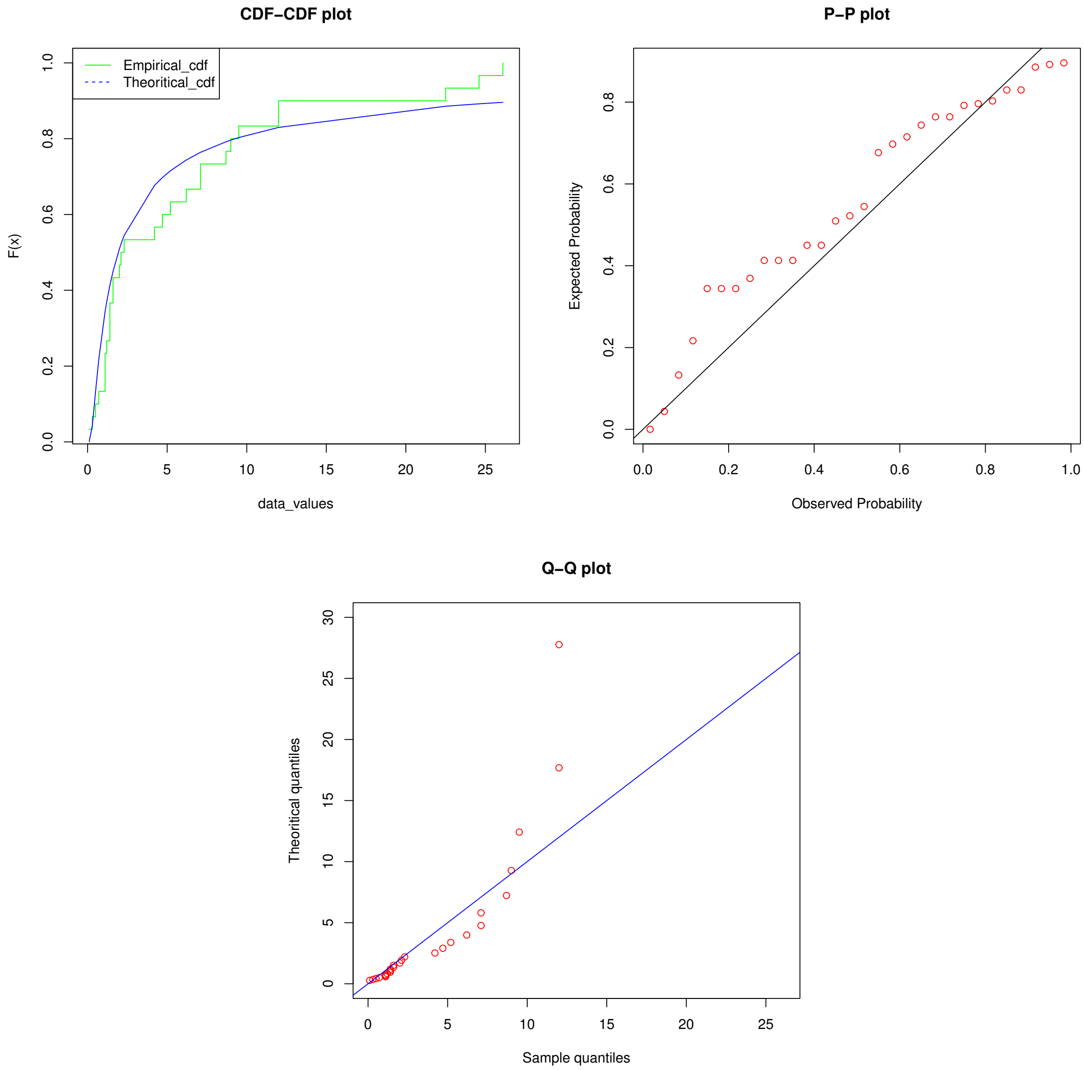


Figure 3. Empirical distribution and fitted IEE distribution plot, PP plot, and QQ plot for the IEE distribution from real dataset II.

7.1. Objective function

Variance of logarithm of the p-th ($p \in (0,1)$) quantile of inverted exponentiated family is given by (see also, [21])

$$\phi(n,r,s) = E_{\text{data}:n,r,s}\big[Var_{\text{posterior}}\big(\ln x_p|\text{data}\big)\big], \tag{7.1}$$

where x_p is the p-th quantile of the distribution given by

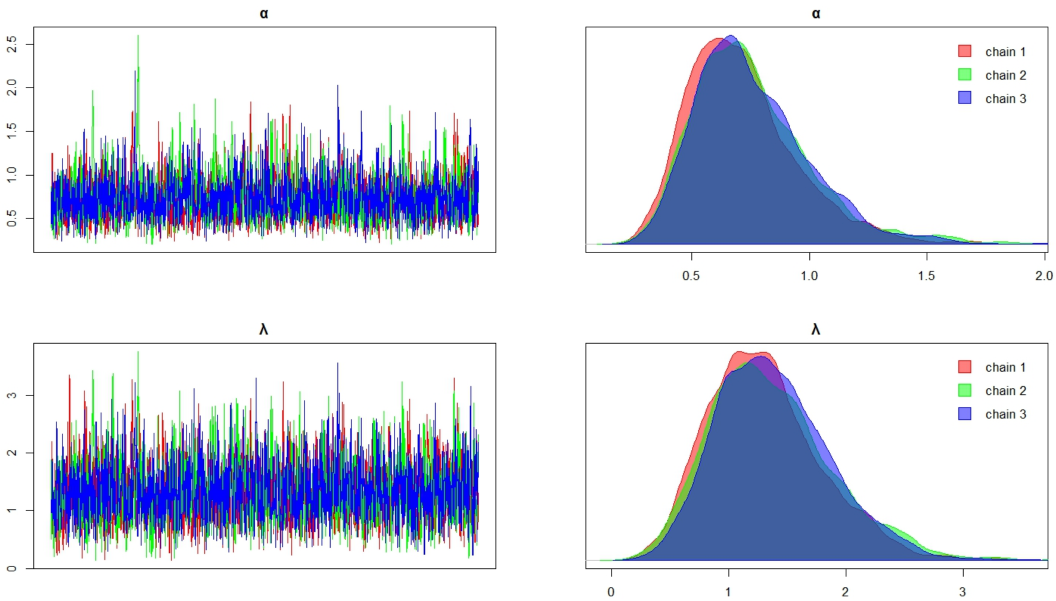
$$x_p = Q^{-1}\left[-\frac{\lambda}{\ln\left[1-(1-p)^{(1/\alpha)}\right]}\right]$$

and $Var_{\text{posterior}}(\cdot|\text{data})$ denotes the variance of posterior distribution and E_{data} is the expectation under likelihood.

Optimal design criterion problem can be considered in sequel to evaluate n, r, and s so that (7.1) is minimum. We have $Var_{\text{posterior}}\big(\ln x_p|\text{data}\big)$ given as

$$Var_{\text{posterior}}\big(\ln x_p|\text{data}\big) = \delta^t Var_{\text{posterior}}(\theta|\text{data})\delta,$$

(a) Trace and density plot of both parameters α (up) and λ (down) for data 1 of real dataset II.



(b) Trace and density plot of both parameters α (up) and λ (down) for data 2 of real dataset I.

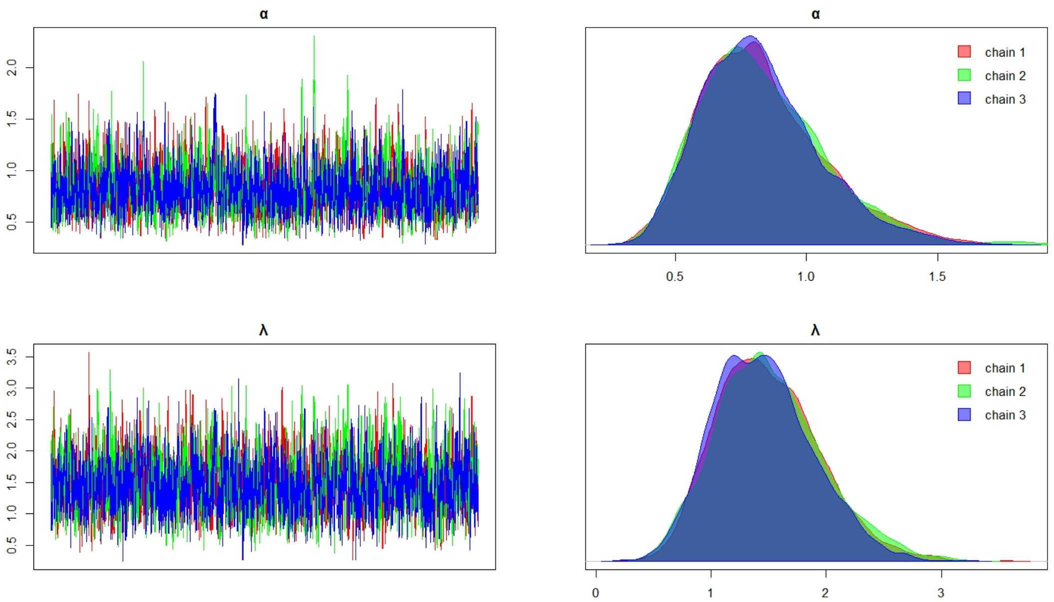


Figure 4. Trace and density plot for dataset II.

where

$$\delta^t = \begin{pmatrix} \frac{\partial \ln x_p}{\partial \alpha} & \frac{\partial \ln x_p}{\partial \lambda} \end{pmatrix}.$$

Now, (7.1) can be expressed as

$$\phi(n, r, s) \approx E_{\text{data}:n,r,s}[\delta^t Var_{\text{posterior}}(\theta|\text{data})\delta]. \tag{7.2}$$

Table 11. Point and interval estimates for dataset II

Data 1			α	λ
MLE			0.7472	1.3905
Bayes estimate	SE		0.7346	1.3730
	LINEX	$k = 0.5$	0.7220	1.3173
		$k = -0.5$	0.7479	1.4331
	GE	$m = -0.5$	0.7176	1.3299
		$m = -1.5$	0.7517	1.4145
ACI			(0.3025 1.1919)[0.8894]	(0.4498 2.3313)[1.8815]
BCI			(0.3415 1.1795)[0.8380]	(0.4816 2.2995)[1.8179]
Data 2			α	λ
MLE			0.9108	1.5985
Bayes estimate	SE		0.8955	1.5649
	LINEX	$k = 0.5$	0.8816	1.5186
		$k = -0.5$	0.9100	1.6137
	GE	$m = -0.5$	0.8800	1.5339
		$m = -1.5$	0.9110	1.5950
ACI			(0.4314 1.3902)[0.9588]	(0.7182 2.4788)[1.7606]
BCI			(0.4784 1.3594)[0.881]	(0.7239 2.4040)[1.6801]

Table 12. Prediction for dataset I from Table 6.

			Point prediction			Interval prediction	
(n, r, s)	l	True value	SE	LINEX (-0.002)	GE (-0.700)	BPI	Length
(58,5,54)	1	5	3.5319	3.5325	3.5037	(3.1785 3.8451)	0.6630
	2	5	4.3589	4.3594	4.3414	(4.0342 4.6105)	0.5763
	3	6	4.9780	4.9784	4.9672	(4.7627 5.1560)	0.3933
	4	6	5.5127	5.5129	5.5074	(5.3949 5.6039)	0.2090
	55	108	136.8131	139.2143	135.1780	(124.7190 151.9873)	27.2683
	56	108	194.5060	205.9849	188.6324	(160.2806 237.9956)	77.7150
	57	115	285.0472	310.6712	268.2246	(239.4847 326.3485)	86.8638
	58	181	313.9642	323.5371	261.0000	(259.0013 337.2843)	78.2830

Under large sample size, the posterior distribution is approximated as a multivariate normal distribution (see also, [27]). Thus, $Var_{\text{posterior}}(\theta|\text{data})$ becomes

$$Var_{\text{posterior}}(\theta|\text{data}) \approx \left[S^{-1}(\theta) + I(\hat{\theta})\right]^{-1}, \tag{7.3}$$

where $I(\hat{\theta})$ is the Fisher information evaluated at $\hat{\theta}$ and $S(\theta)$ is the prior variance-covariance matrix which is obtained as

$$S(\theta) = \begin{bmatrix} Var(\alpha) & Cov(\alpha, \lambda) \\ Cov(\lambda, \alpha) & Var(\lambda) \end{bmatrix} = \begin{bmatrix} \frac{a_1}{b_1^2} & 0 \\ 0 & \frac{a_2}{b_2^2} \end{bmatrix}.$$

If we substitute the approximation of $Var_{\text{posterior}}(\theta|\text{data})$ from (7.3) into (7.2), we get

$$\phi(n, r, s) \approx E_{\text{data}:n,r,s}\left[\delta^t \left(S^{-1}(\theta) + I(\hat{\theta})\right)^{-1} \delta\right]. \tag{7.4}$$

Following the concept of [27], we say that, for large sample size, $\hat{\theta}|\theta$ converges to θ in probability. Therefore, $\delta^t\big(S^{-1}(\theta)+I(\hat{\theta})\big)^{-1}\delta$ in (7.4) can be approximated as $\delta^t\big(S^{-1}(\theta)+I(\theta)\big)^{-1}\delta$. Further, the objective function (7.4) depends on p and so using the discussion of [14], the optimization criterion is re-expressed as

$$\begin{aligned}\phi(n,r,s) =&E_{\text{data}:n,r,s}\int_0^1 Var_{\text{posterior}}\big(\ln x_p|\text{data}\big)dW(p)\\ \approx&E_{\text{data}:n,r,s}\int_0^1 \big[\delta^t\big(S^{-1}(\theta)+I(\theta)\big)^{-1}\delta\big]dW(p),\end{aligned} \tag{7.5}$$

where $0\leq W(p)\leq 1$ is a non-negative weight function defined on $[0,1]$. We take (7.5) as the optimization function. Accordingly, a plan (n^*,r^*,s^*) is said to be better than plan (n,r,s) if (n^*,r^*,s^*) leads to smaller value of $\phi(n,r,s)$ than (n,r,s).

7.2. Sensitivity analysis and numerical illustration

The procedure of evaluating optimal plans depends on assignment of hyper-parameters. It is important to analyze effect of these values on proposed plans. For illustrative purpose, substitute $Q(1/x) = 1/x$ to make Equation (1.1) as a pdf of the IEE distribution. A numerical study using Algorithm 2 is done to obtain optimal schemes when lifetime of the unit follows IEE distribution with parameters α and λ. We consider $W(p) = 1$ and the hyper-parameters of α and λ is to be considered as below

- Prior Set I :$\{a_1 = 3.0, b_1 = 2.5, a_2 = 3.0, b_2 = 2.0\}$.
- Prior Set II: $\{a_1 = 3.4, b_1 = 2.2, a_2 = 1.5, b_2 = 1.2\}$
- Prior Set III: $\{a_1 = 2.0, b_1 = 1.2, a_2 = 3.6, b_2 = 2.0\}$

Algorithm 2. Algorithm to obtain optimum testing plan under double censoring scheme

1: Start with an initial guess of the sample size, say n_0.
2: For a fixed n_0, sample size n can be considered as $2\leq n\leq n_0$.
3: For a given n, s can be selected as $2\leq s\leq n$.
4: For a given s, r can be selected as $1\leq r\leq (s-1)$.
5: For every possible combination of n, r and s, compute the values of the objective function defined in (7.5).
6: Choose the optimum testing plan (n^*, r^*, s^*) for which value of objective function is minimum.

Next, we use the above algorithm and obtain the different optimum testing plans for different values of n_0 and the results are reported in Table 13. From Table 13, we draw the following conclusions:

- As the sample size, n_0, increases, the value of the objective function decreases.
- We observe that the values of the optimal plan (n^*,r^*,s^*) are different for the same value of n_0 with different prior sets. This implies that the optimal plan is sensitive to the values of the hyper-parameters.
- It can be observed that, for almost all the cases, the optimal solution r^* is equal to 1. This means that no left censoring occurs in a double censoring scheme. It is an interesting result even though we deal with double censoring.

8. Conclusions

We have considered problems of estimation, prediction and optimal plan for a family of inverted exponentiated distributions under double censoring. We applied both classical and Bayesian approaches to obtain inference results. We derived MLEs and approximate

Table 13. Optimum solutions obtained based on prior sets for different values of n_0.

n_0	Prior set	n^*	r^*	s^*	$\phi(n^*, r^*, s^*)$
10	I	9	1	6	0.032854360000
10	II	8	1	5	0.019597410000
10	III	10	2	3	0.020558200000
15	I	13	1	9	0.020153400000
15	II	14	1	10	0.008702918000
15	III	11	1	10	0.000110277200
20	I	20	1	15	0.010754500000
20	II	14	1	10	0.008702918000
20	III	11	1	10	0.000110277200
25	I	23	1	18	0.000359902400
25	II	24	1	18	0.007948447000
25	III	11	1	10	0.000110277200
30	I	23	1	18	0.000359902400
30	II	30	1	23	0.004591003000
30	III	11	1	10	0.000110277200
35	I	23	1	18	0.000359902400
35	II	32	1	25	0.000040833930
35	III	11	1	10	0.000110277200
40	I	38	1	31	0.000243234200
40	II	32	1	25	0.000040833930
40	III	38	1	38	0.000006756209
45	I	45	1	37	0.000039638330
45	II	32	1	25	0.000040833930
45	III	38	1	38	0.000006756209
50	I	45	1	37	0.000039638330
50	II	49	1	39	0.000000082493
50	III	38	1	38	0.000006756209

intervals for a member of this family of densities. Bayes estimates are evaluated under proper and improper priors based on different loss functions. In sequel credible intervals are constructed as well. Our numerical study suggested that MLEs show good behavior in comparison with NIP Bayes estimates. However, estimates obtained under IP outperform all other proposed estimates. We further predicted censored observations using a Bayesian approach. Our finding is that prediction intervals contain censored observations of a given data set. We have also observed that predictive intervals obtained using credible interval have smaller interval lengths when we predict the observations just preceding the censored (for left) observation and just succeeding the censored observation (for right). We have observed that optimal scheme is sensitive to the value of hyper-parameters. Also the objective function decreases as the sample size increases. Though we have considered a SE, LINEX and GE loss functions under the Bayesian setup, yet other loss functions can also be considered. The present work can also be extended to other censoring schemes, such as adaptive censoring and generalized progressive first-failure censoring.

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