Ikonion Journal of Mathematics 6(1) (2024) 9-20

Ikonion Journal of Mathematics

https://dergipark.org.tr/tr/pub/ikjm

Research Article

Open Access

https://doi.org/10.54286/ikjm.1378951

ISSN: 2687-6531

Trans-Sasakian Indefinite Finsler Manifolds

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Keywords:
Trans-Sasakian manifold,
α −Sasakian manifold ,
β −Kenmotsu manifold ,
Pseudo-Finsler metric ,
Indefinite Finsler manifold

Abstract — In this paper we introduce some properties and results for trans-Sasakian structures on indefinite Finsler manifolds and give the examples of such manifolds.These structures are established on the $(M^0)^h$ and $(M^0)^v$ vector subbundles, where M is an $(2n+1)$ dimensional C^∞ manifold, $M^0 = (M^0)^h \oplus (M^0)^v$ is a non-empty open submanifold of TM. F^* is the fundamental Finsler function and $F^{2n+1} = (M, M^0, F^*)$ is an indefinite Finsler manifold. We use the Sasaki Finsler metric $G = G^H + G^V = g_{ij}^{F^*} dx^i \otimes dx^j + g_{ij}^{F^*} \delta y^i \otimes \delta y^i$. Furthermore, we give some formulas for α −Sasakian and β −Kenmotsu Finsler manifolds with pseudo-Finsler metric. Finally, it is shown that the conformally flat trans-Sasakian indefinite Finsler manifolds $((M^0)^h, \phi^H, \xi^H, \eta^H, G^H)$ and $((M^0)^v, \phi^V, \xi^V, \eta^V, G^V)$ are the η − Einstein manifolds if and only if $\alpha.\beta = 0$, where α, β are constant functions defined on $(M^0)^h$ and $(M^0)^v$.

Subject Classification (2020): 53B30, 53B35, 53B40

1. Introduction

Oubina introduced the idea of trans-Sasakian manifold of classification (α,β). Indefinite Sasakian manifold is a notable category of indefinite trans-Sasakian manifold for α=1, β=0. Also, indefinite cosymplectic manifold is the other category of indefinite trans-Sasakian manifold for α=0, β=0. Indefinite Kenmotsu manifold is given with α=0, β=1. M. D. Siddiqi, A. N. Siddiqui and O. Bahadır study the trans-Sasakian manifolds with a quarter-symmetric nonmetric connection [12]. R. Prasad, U. K. Gautam, J. Prakash and A. K. Rai study (ε)−Lorentzian trans-Sasakian manifolds [16].

The papers interested in contact structures with Riemannian metric or pseudo-Riemannian metric but in this paper, we are also related to the contact structures with pseudo-Finsler metric.

After Finsler published his thesis about curves and surfaces, a lot of articles are dedicated to Finsler geometry, see references [4, 5, 10, 13, 14, 15] but the theory of indefinite Finsler manifold has been investigated by few researchers [1, 2, 7, 8, 9]. We also make reference to the reader to the recent monograph for detailed information in this field.

Hence, our aim is to present trans-Sasakian indefinite Finsler manifolds and to obtain the formulas for α −Sasakian and β −Kenmotsu indefinite Finsler manifolds. The paper is organized as follows: after

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Article History: Received: 26.10.2023 — Accepted: 19.02.2024 — Published: 28.04.2024

introduction and background, we give some preliminaries about indefinite Finsler manifolds. Then, we deal with the trans-Sasakian indefinite Finsler manifolds, α −Sasakian and β −Kenmotsu indefinite Finsler manifolds. Finally, it is shown that the conformally flat trans-Sasakian indefinite Finsler manifolds $((M^0)^h, \phi^H, \xi^H, \eta^H, G^H)$ and $((M^0)^v, \phi^V, \xi^V, \eta^V, G^V)$ are the $\eta-$Einstein manifolds if and only if $\alpha.\beta = 0$, where α and β are constant functions defined on $(M^0)^h$ and $(M^0)^v$.

2. Preliminaries

2.1. Indefinite Finsler Manifolds

Let M be a real $(2n+1)-$ dimensional smooth manifold and TM be the tangent bundle of M. A coordinate system in M can be stated with $\{(U, \varphi): x^1, \dots, x^{2n+1}\}$, where U is an open subset of M; for any $x \in U, \varphi: U \to \mathbb{R}^{2n+1}$ is a diffeomorphism of U onto $\varphi(U)$, and $\varphi(x) = (x^1, \dots, x^{2n+1})$. On M, denote by π the canonical projection of TM and by T_xM the fibre, at $x \in M$, i.e., $T_xM = \pi^{-1}(x)$. Through the coordinate system $\{(U, \varphi): x^i\}$ in M, we can describe a new coordinate system $\{(U^*, \Phi); x^1, \dots, x^{2n+1}, y^1, \dots, y^{2n+1}\}$ or shortly $\{(U^*, \Phi): x^i, y^i\}$ in TM, where $U^* = \pi^{-1}(U)$ and $\Phi: U^* \to \mathbb{R}^{4n+2}$ is a diffeomorphism of U^* on $\varphi(U) \times \mathbb{R}^{2n+1}$, and $\Phi(y_x) = (x^1, \dots, x^{2n+1}, y^1, \dots, y^{2n+1})$ for any $x \in U$ and $y_x \in T_xM$. Let M^0 be a non-empty open submanifold of TM such that $\pi(M^0) = M$ and $\theta(M) \cap M^0 = \emptyset$, where θ is the zero section of TM. Assume that $M_x^0 = T_xM \cap M^0$ is a positive conic set, for any $k > 0$ and $y \in M_x^0$. we have $ky \in M_x^0$. Obviously, the largest M^0 holding the above circumstances is $TM \setminus \theta(M)$, ordinarily given with the description of a Finsler manifold.

We now consider a smooth function $F: M^0 \to (0, \infty)$ and take $F^* = F^2$. Then suppose that for any coordinate system $\{(U^0, \Phi^0); x^i, y^i\}$ in M^0, the following conditions are fulfilled:

(**$F1$**) F is positively homogenous of degree one regarding $(y^1, \dots, y^{2n+1})$, *i.e.*, we get, for all $k > 0$ and $(x, y) \in \Phi^0(U^0)$,

$$F(x^1, \dots, x^{2n+1}, ky^1, \dots, ky^{2n+1}) = k\, F(x^1, \dots, x^{2n+1}, y^1, \dots, y^{2n+1})$$

($F2$) At any point $(x, y) \in \Phi^0(U^0)$,

$$g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 F^*}{\partial y_i \partial y_j}(x, y), \quad i, j \in \{1,2, \dots, 2n+1\}$$

are the components of a positive definite quadratic form on $\mathbb{R}^{2n+1}$.

We say that the triple $F^{2n+1} = (M, M^0, F)$ is a Finsler manifold, and F is the fundamental function of F^{2n+1}.

Certainly, condition $(F2)$ is not appropriate for some applications of Finsler geometry. To remove this inconvenience we consider a positive integer $0 < q < 2n+1$, and a smooth function $F^*: M^0 \to R$, where $M^0$is as above. Moreover, suppose that for any coordinate system $\{(U^0, \Phi^0); x^i, y^i\}$ in M^0, the following conditions are fulfilled:

($F1^*$) F^* is positively homogenous of degree two regarding $(y^1, \dots, y^{2n+1})$, we get, for all $k > 0$ and $(x, y) \in \Phi^0(U^0)$,

$$F^*(x^1, \dots, x^{2n+1}, ky^1, \dots, ky^{2n+1}) = k^2 F^*(x^1, \dots, x^{2n+1}, y^1, \dots, y^{2n+1})$$

($F2^*$) At all point $(x, y) \in \Phi^0(U^0)$,

$$g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 F^*}{\partial y_i \partial y_j}(x, y), \quad i, j \in \{1,2, \dots, 2n+1\}$$

are the components of a quadratic form on $\mathbb{R}^{2n+1}$ with $(2n+1)-q$ positive eigenvalues and q negative eigenvalues $(0<q<2n+1)$. In this state $F^{2n+1}=(M,M^0,F^*)$ is called indefinite Finsler manifolds with index q. Particularly, if chosing $q=1$, we get Lorentzian indefinite Finsler manifolds [2].

Consider the structure of $F^{2n+1}=(M,M^0,F^*)$ indefinite Finsler manifold with index q. Then the tangent mapping $\pi_*: TM^0 \to TM$ of the submersion $\pi: M^0 \to M$ and define the vector bundle $(TM^0)^V = ker\pi_*$. As locally, $\pi^i(x,y)=x^i$, we obtain $\pi_*^i\left(\frac{\partial}{\partial x^j}\right)=\delta_j^i$ and $\pi_*^i\left(\frac{\partial}{\partial y^j}\right)=0$, on the coordinate neighborhood $U^0 \subset M^0$. Thus, $\left\{\frac{\partial}{\partial y^i}\right\}$ is a basis of $\Gamma\left((TM^0)^V{}_{|_{U^0}}\right)$. We call $(TM^0)^V$ the vertical vector bundle of F^{2n+1}. Locally, on a coordinate neighborhood $U^0 \subset M^0$. we have $X^V = X^i(x,y)\frac{\partial}{\partial y^i}$, where X^i smooth functions on U^0. After we denote by $(T^*M^0)^V$ the dual vector bundle of $(TM^0)^V$. Thus a Finsler 1-form is smooth section of $(T^*M^0)^V$. Assume $\{\delta y^1,\dots,\delta y^{2n+1}\}$ is a dual basis to $\left\{\frac{\partial}{\partial y^1},\dots,\frac{\partial}{\partial y^{2n+1}}\right\}$, i.e., $\delta y^i\left(\frac{\partial}{\partial y^j}\right)=\delta_j^i$. Then each for $w \in (T^*M^0)^V$, $w^V = w^i(x,y)\,\delta y^i$, where $w^i(x,y) = w\left(\frac{\partial}{\partial y^i}\right)$ [1, 2].

The complementary distribution $(TM^0)^H$ to $(TM^0)^V$ in TM^0 is said a horizontal distribution (non- linear connection) on M^0. Thus we can write

$$TM^0 = (TM^0)^H \oplus (TM^0)^V$$

The set of the local vector fields $\left\{\frac{\delta}{\delta x^1},\dots,\frac{\delta}{\delta x^{2n+1}}\right\}$ is a basis in $\Gamma((TM^0)^H)$. Then

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}$$

Let X be a vector field on M^0. Then locally we get

$$X = X^i \frac{\delta}{\delta x^i} + \tilde{X}^i \frac{\partial}{\partial y^i}$$

Clearly, for $\tilde{X}^i(x,y)=0$, we obtain the subbundle of $(M^0)^h \subset M^0$ and for $X^i(x,y)=0$, we obtain the subbundle of $(M^0)^v \subset M^0$. Suppose $\{dx^1,\dots,dx^{2n+1}\}$ is a dual basis to $\left\{\frac{\delta}{\delta x^1},\dots,\frac{\delta}{\delta x^{2n+1}}\right\}$, i.e., $dx^i\left(\frac{\delta}{\delta x^j}\right) = \delta_j^i$. Then each $w \in \Gamma(T^*M^0)^H$ is locally written as $w^H = \widetilde{w}_i(x,y)dx^i$, where $\widetilde{w}_i = w_i - N_i^j w_j$. Thus we can write

$$\delta y^i = dy^i + N_j^i(x,y)dx^j$$

Consider a w, 1-form, then

$$w = \widetilde{w}_i(x,y)dx^i + w_i(x,y)\ \delta y^i.$$

Also, $w^H(X^V)=0, w^V(X^H)=0$, where $w = w^H + w^V$ [2].

Definition 2.1. A Finsler connection is a linear connection $\nabla = F\Gamma$ with the property that the horizontal linear space $\left(T_{(x,y)}M^0\right)^H$, $(x,y) \in M^0$ of the distribution N is parallel with respect to ∇.

Similarly, a Finsler connection is called linear connection $\nabla = F\Gamma$ with the vertical linear space $\left(T_{(x,y)}M^0\right)^V$, $(x,y) \in M^0$ of the distribution N parallel relative to ∇.

Necessary and sufficient condition for linear connection ∇ on M^0 to be Finsler connection is

$$(\nabla_X^V Y^H) = 0, (\nabla_X^H Y^V) = 0$$

$$\nabla_X Y = \nabla_X^H Y^H + \nabla_X^V Y^V$$

for each $X, Y \in T_{(x,y)}M^0$.

$$\nabla_X w = \nabla_X^H w^H + \nabla_X^V w^V$$

for all $w \in T^*_{(x,y)}M^0$[15].

Let ∇ be a Finsler connection and the curvature of this connection is given with the below equation.

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = R^H(X^H, Y^H)Z^H + R^V(X^V, Y^V)Z^V$$

where $X, Y, Z \in T_{(x,y)}M^0$ [14].

Theorem 2.1. The curvature of a Finsler connection ∇ on $T_{(x,y)}M^0$ is totally stated with the following Finsler tensor fields equations:

$$R^H(X^H, Y^H)Z^H = \nabla_{X^H} \nabla_{Y^H} Z^H - \nabla_{Y^H} \nabla_{X^H} Z^H - \nabla_{[X^H, Y^H]} Z^H$$

$$R^V(X^V, Y^V)Z^V = \nabla_{X^V} \nabla_{Y^V} Z^V - \nabla_{Y^V} \nabla_{X^V} Z^V - \nabla_{[X^V, Y^V]} Z^V$$

[14].

2.2. Almost Contact Pseudo-Metric Finsler Structures

Consider tensor field ϕ, 1-form η and vector field ξ given as below:

$$\phi = \phi^H + \phi^V = \phi_i^j(x,y) \frac{\delta}{\delta x_i} \otimes dx^j + \widetilde{\phi_i^j}(x,y) \frac{\partial}{\partial y^i} \otimes \delta y^j \quad (2.1)$$

$$\eta = \eta^H + \eta^V = \eta_i(x,y) dx^i + \widetilde{\eta_i}(x,y) \delta y^i \quad (2.2)$$

$$\xi = \xi^H + \xi^V = \xi^i(x,y) \frac{\delta}{\delta x_i} + \widetilde{\xi^i}(x,y) \frac{\partial}{\partial y^i} \quad (2.3)$$

Then, we can write the following statements.

$$(\phi^H)^2 X^H = -X^H + \eta^H(X^H)\, \xi^H, (\phi^V)^2 X^V = -X^V + \eta^V(X^V)\, \xi^V \quad (2.4)$$

$$\eta^H(\xi^H) = \eta^V(\xi^V) = 1 \quad (2.5)$$

$$\phi^H(\xi^H) = \phi^V(\xi^V) = 0 \quad (2.6)$$

$$\eta^H \circ \phi^H = \eta^V \circ \phi^V = 0 \quad (2.7)$$

$$rank(\phi^H) = rank(\phi^V) = 2n \quad (2.8)$$

Thus, (ϕ^H, ξ^H, η^H) and (ϕ^V, ξ^V, η^V) are called the almost contact Finsler structures on vector bundles $(M^0)^h$ and $(M^0)^v$,respectively, where $M^0 = (M^0)^h \oplus (M^0)^v$. Also, we call that $\left((M^0)^h, \phi^H, \xi^H, \eta^H\right)$ and $((M^0)^v, \phi^V, \xi^V, \eta^V)$ are almost contact Finsler manifolds [3].

Let $F^{2n+1} = (M, M^0, F^*)$ be an indefinite Finsler manifold. Then, we define

$$g^{F^*}: \Gamma(TM^0)^V \times \Gamma(TM^0)^V \to \mathfrak{F}(M^0),$$

$$g_{ij}^{F^*}(x,y) = g^{F^*}\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right)(x,y).$$

Obviously, g^{F^*} is a symmetric Finsler tensor field. g^{F^*} is called the pseudo-Finsler metric of F^{2n+1}. Thus, g^{F^*} is thought to be a pseudo-Riemannian metric on $(TM^0)^V$.

Similarly, we define the metric for horizontal distrubituon as following:

$$g^{F^*}: \Gamma(TM^0)^H \times \Gamma(TM^0)^H \to \mathfrak{F}(M^0),$$

$$g_{ij}^{F^*}(x,y) = g^{F^*}(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j})(x,y)$$

[1, 2]. A Finsler vector can be described with below statements.

$$g^{F^*}(\mathcal{X},\mathcal{X}) = 0 \text{ and } \mathcal{X} \neq 0 \Rightarrow \text{light-like}$$

$$g^{F^*}(\mathcal{X},\mathcal{X}) > 0 \text{ or } \mathcal{X} = 0 \Rightarrow \text{space-like}$$

$$g^{F^*}(\mathcal{X},\mathcal{X}) < 0 \Rightarrow \text{time-like},$$

where $\mathcal{X} \in T_{(x,y)}M^0$, $(x,y) \in M^0$. The Finsler norm of $\mathcal{X}$ is a nonnegative number and $\|\mathcal{X}\|$ is described with following equation:

$$\|\mathcal{X}\| = \left|g^{F^*}(\mathcal{X},\mathcal{X})\right|^{1/2}.$$

If $g^{F^*}(\mathcal{X},\mathcal{X}) = 1$, $\mathcal{X}$ is called unit space-like Finsler vector or $g^{F^*}(\mathcal{X},\mathcal{X}) = -1$, $\mathcal{X}$ is called unit time-like Finsler vector. $g^{F^*}(\mathcal{X},\mathcal{X}) = \varepsilon$ and ε is said the signature of $\mathcal{X}$ when $\mathcal{X}$ is a unit Finsler vector.

Also,

$$G: \Gamma(TM^0) \times \Gamma(TM^0) \to \mathfrak{F}(M^0)$$

$$G(X,Y) = G^H(X,Y) + G^V(X,Y).$$

is defined. Obviously, G is a symmetric tensor field of type (0,2), non-degenerate and pseudo-Riemannian metric on M^0 with index $2q$. Then, G is called Sasaki Finsler metric on M^0. Then, G can be defined as below.

$$G = G^H + G^V = g_{ij}^{F^*} dx^i \otimes dx^j + g_{ij}^{F^*}\, \delta y^i \otimes \delta y^i$$

[1, 2].

Definition 2.2. Suppose that (ϕ^H,ξ^H,η^H) and (ϕ^V,ξ^V,η^V) are almost contact structures on horizontal and vertical Finsler vector bundles $(M^0)^h$ and $(M^0)^v$. If the G^H and G^V satisfy the following conditions,

$$G^H(\phi X^H,\phi Y^H) = G^H(X^H,Y^H) - \varepsilon\eta^H(X^H)\eta^H(Y^H)$$

$$G^V(\phi X^V,\phi Y^V) = G^V(X^V,Y^V) - \varepsilon\eta^V(X^V)\eta^V(Y^V)$$

$$\eta^H(X^H) = \varepsilon\, G^H(X^H,\xi^H), \eta^V(X^V) = \varepsilon\, G^V(X^V,\xi^V)$$

where $\varepsilon = \pm 1$, then $(\phi^H,\xi^H,\eta^H,G^H)$ and $(\phi^V,\xi^V,\eta^V,G^V)$ are called almost contact pseudo-metric Finsler structures on $(M^0)^h$ and $(M^0)^v$.

Now, we define

$$\Omega\,(X\,,Y) = G(X\,,\phi Y), \qquad \Omega^H(X^H,Y^H) = G^H(X^H,\phi Y^H), \quad \Omega^V(X^V,Y^V) = G^V(X^V,\phi Y^V)$$

and call it the fundamental 2-form [4].

The fundamental 2-form, defined above, satisfies the following equations:

$$\Omega^H(\phi X^H,\, \phi Y^H) = \Omega^H(X^H,Y^H),\;\; \Omega^V(\phi X^V,\, \phi Y^V) = \Omega^V(X^V,Y^V)$$

$$\Omega^H(Y^H,X^H) = -\,\Omega^H(X^H,Y^H)\,,\;\; \Omega^V(Y^V,\, X^V) = -\Omega^V(X^V,Y^V)$$

Proposition 2.1. Let ∇ be a Finsler connection on M^0 and Ω be the fundamental 2-form which satisfies

$$d\eta^V(X^V,Y^V) = \Omega^V(X^V,Y^V), \qquad d\eta^H(X^H,Y^H) = \Omega^H(X^H,Y^H),$$

$$\Omega^H(X^H,Y^H) = (\nabla_X^H\eta^H)(Y^H) - (\nabla_Y^H\eta^H)(X^H) + \eta^H\big(T(X^H\,,Y^H)\big),$$

$$\Omega^V(X^V,Y^V)=(\nabla_X^V\eta^V)(Y^V)-(\nabla_Y^V\eta^V)(X^V)+\eta^V\big(T(X^V\,,Y^V)\big).$$

Then the almost contact pseudo-metric Finsler structure is called almost ε −Sasakian Finsler structure on M^0**.**

$(\phi^H,\xi^H,\eta^H,G^H)$ and $(\phi^V,\xi^V,\eta^V,G^V)$ are called almost ε −Sasakian Finsler structures on $(M^0)^h$ and $(M^0)^v$, respectively [4].

Theorem 2.2. Let Ω be the fundamental 2-form and almost ε −Sasakian Finsler connection ∇ on M^0 is torsion free then

$$\Omega^H(X^H,Y^H)=(\nabla_X^H\eta^H)(Y^H)-(\nabla_Y^H\eta^H)(X^H)$$

$$\Omega^V(X^V,Y^V)=(\nabla_X^V\eta^V)(Y^V)-(\nabla_Y^V\eta^V)(X^V)$$

[4].

Definition 2.3. An almost ε −Sasakian Finsler structure on M^0 is said to be an ε −Sasakian Finsler structure if the 1-form η is a killing vector field, i.e.,

$$(\nabla_X^H\eta^H)(Y^H)+(\nabla_Y^H\eta^H)(X^H)=0,\ (\nabla_X^V\eta^V)(Y^V)+(\nabla_Y^V\eta^V)(X^V)=0$$

$$\Omega^H(X^H,Y^H)=2(\nabla_X^H\eta^H)(Y^H),\ \Omega^V(X^V,Y^V)=2(\nabla_X^V\eta^V)(Y^V)$$

[4].

3. Trans- Sasakian Indefinite Finsler Manifolds

We introduce trans-Sasakian indefinite Finsler manifolds in our main results. Also, we give the special case of these structures α −Sasakian and β −Kenmotsu indefinite Finsler manifolds.
The almost contact pseudo-metric Finsler manifolds $((M^0)^h,\phi^H,\xi^H,\eta^H,G^H)$ and $((M^0)^v,\phi^V,\xi^V,\eta^V,G^V)$ are said to be trans-Sasakian indefinite Finsler manifolds if and only if the following conditions are hold.

$$(\nabla_X^H\phi^H)Y^H=\frac{\alpha}{2}\{G^H(X^H,Y^H)\xi^H-\varepsilon\eta^H(Y^H)X^H\}+\frac{\beta}{2}\{\varepsilon G^H(\phi X^H,Y^H)\xi^H-\eta^H(Y^H)\phi X^H\} \tag{3.1}$$

$$(\nabla_X^V\phi^V)Y^V=\frac{\alpha}{2}\{G^V(X^V,Y^V)\xi^V-\varepsilon\eta^V(Y^V)X^V\}+\frac{\beta}{2}\{\varepsilon G^V(\phi X^V,Y^V)\xi^V-\eta^V(Y^V)\phi X^V\} \tag{3.2}$$

where α and β are smooth functions on $(M^0)^h$ and $(M^0)^v$ then we say such a structure the trans-Sasakian pseudo-metric Finsler structure of type (α,β). If α,β =constant, then the getting α,β =constant from (3.1) and (3.2) we get

$$(\nabla_X^H\xi^H)=-\varepsilon\frac{\alpha}{2}\,\phi X^H+\frac{\beta}{2}(X^H-\eta^H(X^H)\xi^H) \tag{3.3}$$

$$(\nabla_X^V\xi^V)=-\varepsilon\frac{\alpha}{2}\,\phi X^V+\frac{\beta}{2}(X^V-\eta^V(X^V)\xi^V) \tag{3.4}$$

$$(\nabla_X^H\eta^H)(Y^H)=\frac{\alpha}{2}G^H(X^H,\phi Y^H)+\varepsilon\frac{\beta}{2}G^H(\phi X^H,\phi Y^H) \tag{3.5}$$

$$(\nabla_X^V\eta^V)(Y^V)=\frac{\alpha}{2}G^V(X^V,\phi Y^V)+\varepsilon\frac{\beta}{2}G^V(\phi X^V,\phi Y^V) \tag{3.6}$$

Theorem 3.1. In the trans-Sasakian indefinite Finsler manifolds $((M^0)^h,\phi^H,\xi^H,\eta^H,G^H)$ and $((M^0)^v,\phi^V,\xi^V,\eta^V,G^V)$ the following relations hold.

$$R^H(X^H,Y^H)\xi^H = \frac{(\alpha^2-\beta^2)}{4}\{\eta^H(Y^H)X^H - \eta^H(X^H)Y^H\} + \varepsilon\frac{\alpha\beta}{2}\{\eta^H(Y^H)\phi X^H - \eta^H(X^H)\phi Y^H\} \quad (3.7)$$

$$R^V(X^V,Y^V)\xi^V = \frac{(\alpha^2-\beta^2)}{4}\{\eta^V(Y^V)X^V - \eta^V(X^V)Y^V\} + \varepsilon\frac{\alpha\beta}{2}\{\eta^V(Y^V)\phi X^V - \eta^V(X^V)\phi Y^V\} \quad (3.8)$$

$$R^H(\xi^H,X^H)Y^H = \frac{(\alpha^2-\beta^2)}{4}\{\varepsilon G^H(X^H,Y^H)\xi^H - \eta^H(Y^H)X^H\} + \varepsilon\frac{\alpha\beta}{2}\{\eta^H(Y^H)\phi X^H - \varepsilon G^H(\phi X^H,Y^H)\xi^H\} \quad (3.9)$$

$$R^V(\xi^V,X^V)Y^V = \frac{(\alpha^2-\beta^2)}{4}\{\varepsilon G^V(X^V,Y^V)\xi^V - \eta^V(Y^V)X^V\} + \varepsilon\frac{\alpha\beta}{2}\{\eta^V(Y^V)\phi X^V - \varepsilon G^V(\phi X^V,Y^V)\xi^V\} \quad (3.10)$$

$$\eta^H(R^H(X^H,Y^H)Z^H) = \varepsilon\frac{(\alpha^2-\beta^2)}{4}\{G^H(Y^H,Z^H)\eta^H(X^H) - G^H(X^H,Z^H)\eta^H(Y^H)\} + \frac{\alpha\beta}{2}\{\eta^H(X^H)G(\phi Y^H,Z^H) - \eta^H(Y^H)G^H(\phi X^H,Z^H)\} \quad (3.11)$$

$$\eta^V(R^V(X^V,Y^V)Z^V) = \varepsilon\frac{(\alpha^2-\beta^2)}{4}\{G^V(Y^V,Z^V)\eta^V(X^V) - G^V(X^V,Z^V)\eta^V(Y^V)\} + \frac{\alpha\beta}{2}\{\eta^V(X^V)G(\phi Y^V,Z^V) - \eta^V(Y^V)G^V(\phi X^V,Z^V)\} \quad (3.12)$$

$$\eta^H(R^H(X^H,Y^H)\xi^H) = 0, \eta^V(R^V(X^V,Y^V)\xi^V)=0 \quad (3.13)$$

$$S^H(X^H,\xi^H) = n\frac{(\alpha^2-\beta^2)}{2}\eta^H(X^H), \quad S^V(X^V,\xi^V) = n\frac{(\alpha^2-\beta^2)}{2}\eta^V(X^V) \quad (3.14)$$

$$S^H(\xi^H,\xi^H) = n\frac{(\alpha^2-\beta^2)}{2}, \quad S^V(\xi^V,\xi^V) = n\frac{(\alpha^2-\beta^2)}{2} \quad (3.15)$$

$$QX^H = \varepsilon\, n\frac{(\alpha^2-\beta^2)}{2}X^H, \; QX^V = \varepsilon\, n\frac{(\alpha^2-\beta^2)}{2}X^V, \; Q\xi^H = \varepsilon\, n\frac{(\alpha^2-\beta^2)}{2}\xi^H, \; Q\xi^V = \varepsilon\, n\frac{(\alpha^2-\beta^2)}{2}\xi^V \quad (3.16)$$

Proof:

$$R^H(X^H,Y^H)\xi^H = \nabla^H_{X^H}\nabla^H_{Y^H}\xi^H - \nabla^H_{Y^H}\nabla^H_{X^H}\xi^H - \nabla_{\nabla^H_{X^H}Y^H - \nabla^H_{Y^H}X^H}\xi^H$$

$$= \nabla^H_{X^H}\left\{-\varepsilon\frac{\alpha}{2}\phi Y^H + \frac{\beta}{2}(Y^H - \eta^H(Y^H)\xi^H)\right\} - \nabla^H_{Y^H}\left\{-\varepsilon\frac{\alpha}{2}\phi X^H + \frac{\beta}{2}(X^H - \eta^H(X^H)\xi^H)\right\}$$

$$-\nabla^H_{\nabla^H_{X^H}Y^H}\xi^H + \nabla^H_{\nabla^H_{Y^H}X^H}\xi^H$$

$$= \varepsilon\frac{\alpha}{2}\{(\nabla^H_Y\phi^H)X^H - (\nabla^H_X\phi^H)Y^H\} + \frac{\beta}{2}\{(\nabla^H_Y\eta^H)X^H\xi^H - (\nabla^H_X\eta^H)Y^H\xi^H + \eta^H(X^H)\nabla^H_Y\xi^H - \eta^H(Y^H)\nabla^H_X\xi^H\}$$

then we get following equation

$$= \varepsilon\frac{\alpha}{2}\Big\{-\varepsilon\frac{\alpha}{2}\eta^H(X^H)Y^H + \varepsilon\beta G^H(\phi Y^H, X^H) - \frac{\beta}{2}\eta^H(X^H)\phi Y^H + \varepsilon\frac{\alpha}{2}\eta^H(Y^H)X^H + \frac{\beta}{2}\eta^H(Y^H)\phi X^H\Big\} + \frac{\beta}{2}\Big\{\alpha G^H(Y^H, \phi X^H)\xi^H - \varepsilon\frac{\alpha}{2}\eta^H(X^H)\phi Y^H + \frac{\beta}{2}\eta^H(X^H)Y^H - \frac{\beta}{2}\eta^H(X^H)\eta^H(Y^H)\xi^H + \varepsilon\frac{\alpha}{2}\eta^H(Y^H)\phi X^H - \frac{\beta}{2}\eta^H(Y^H)X^H + \frac{\beta}{2}\eta^H(X^H)\eta^H(Y^H)\xi^H\Big\},$$

If we rearrange last equation, then we have the following one and the proof is completed

$$R(X^H, Y^H)\xi^H = \frac{(\alpha^2 - \beta^2)}{4}\{\eta^H(Y^H)X^H - \eta^H(X^H)Y^H\} + \varepsilon\frac{\alpha\beta}{2}\{\eta^H(Y^H)\phi X^H - \eta^H(X^H)\phi Y^H\}$$

By using similar processing steps, we can obtain the proof for vertical distribution.

Using the equations $G^H(R^H(X^H, Y^H)\xi^H, W^H) = G^H(R^H(\xi^H, W^H)\, X^H, Y^H)$ and $G^V(R^V(X^V, Y^V)\xi^V, W^V) = G^V(R^V(\xi^V, W^V)\, X^V, Y^V)$, we get

$$R^H(\xi^H, W^H)\, X^H = \frac{(\alpha^2-\beta^2)}{4}\{\varepsilon\, G^H(W^H, X^H)\xi^H - \eta^H(X^H)W^H\} + \varepsilon\frac{\alpha\beta}{2}\{\eta^H(X^H)\phi W^H - \varepsilon G^V(\phi W^H, X^H)\xi^H\}.$$

and

$$R^V(\xi^V, W^V)\, X^V = \frac{(\alpha^2-\beta^2)}{4}\{\varepsilon\, G^V(W^V, X^V)\xi^V - \eta^V(X^V)W^V\} + \varepsilon\frac{\alpha\beta}{2}\{\eta^V(X^V)\phi W^V - \varepsilon G^V(\phi W^V, X^V)\xi^V\}.$$

We have from equations (3.7) and (3.8), we get

$$\eta^H(R^H(X^H, Y^H)Z^H) = \varepsilon G(R^H(X^H, Y^H)Z^H, \xi^H) = -\varepsilon G(R^H(X^H, Y^H)\xi^H, Z^H)$$

$$= -\varepsilon G\left(\frac{(\alpha^2-\beta^2)}{4}\{\eta^H(Y^H)X^H - \eta^H(X^H)Y^H\} + \varepsilon\frac{\alpha\beta}{2}\{\eta^H(Y^H)\phi X^H - \eta^H(X^H)\phi Y^H\}, Z^H\right)$$

$$= \varepsilon\,\frac{(\alpha^2-\beta^2)}{4}\{G^H(Y^H, Z^H)\eta^H(X^H) - G^H(X^H, Z^H)\eta^H(Y^H)\}$$

$$+\frac{\alpha\beta}{2}\{\eta^H(X^H)G(\phi Y^H, Z^H) - \eta^H(Y^H)G^H(\phi X^H, Z^H)\}$$

and

$$\eta^V(R^V(X^V, Y^V)Z^V) = \varepsilon G(R^V(X^V, Y^V)Z^V, \xi^V) = -\varepsilon G(R^V(X^V, Y^V)\xi^V, Z^V)$$

$$= -\varepsilon G(\frac{(\alpha^2-\beta^2)}{4}\{\eta^V(Y^V)X^V - \eta^V(X^V)Y^V\} + \varepsilon\frac{\alpha\beta}{2}\{\eta^V(Y^V)\phi X^V - \eta^V(X^V)\phi Y^V\}, Z^V)$$

$$= \varepsilon\frac{(\alpha^2-\beta^2)}{4}\{G^V(Y^V, Z^V)\eta^V(X^V) - G^V(X^V, Z^V)\eta^V(Y^V)\}$$

$$+\frac{\alpha\beta}{2}\{\eta^V(X^V)G(\phi Y^V, Z^V) - \eta^V(Y^V)G^V(\phi X^V, Z^V)\}.$$

Putting $Z^H = \xi^H$ *and* $Z^V = \xi^V$, we get $\eta^H(R^H(X^H, Y^H)\xi^H) = 0, \eta^V(R^V(X^V, Y^V)\xi^V) = 0$.

For the trans-Sasakian indefinite Finsler manifolds $((M^0)^h, \phi^H, \xi^H, \eta^H, G^H)$ and $((M^0)^v, \phi^V, \xi^V, \eta^V, G^V)$, the Ricci tensor S and scalar curvature r is defined by

$$S^H(X^H, Y^H) = \sum_{i=1}^{2n}\varepsilon_i\, G^H(R^H(E_i^H, X^H)Y^H, E_i^H) + \varepsilon\, G^H(R^H(\xi^H, X^H)Y^H, \xi^H),$$

$$r^H = \sum_{i=1}^{2n} S^H(E_i^H, E_i^H),\ r^V = \sum_{i=1}^{2n} S^V(E_i^V, E_i^V),$$

$$S^V(X^V, Y^V) = \sum_{i=1}^{2n}\varepsilon_i\, G^V(R^V(E_i^V, X^V)Y^V, E_i^V) + \varepsilon\, G^V(R^V(\xi^V, X^V)Y^V, \xi^V),$$

where $\{E_1^H, E_2^H, \dots, E_{2n}^H, \ \xi^H\}$ is orthonormal basis field in $(M^0)^h$and $G^H(E_i^H, E_i^H) = \varepsilon_i$

(similarly, $\{E_1^V, E_2^V, \dots, E_{2n}^V, \ \xi^V\}$ is orthonormal basis field in $(M^0)^v$and $G^v(E_i^v, E_i^v) = \varepsilon_i$).

Replacing Y^H by ξ^H, we get

$$S^H(X^H, \ \xi^H) = \sum_{i=1}^{2n} \varepsilon_i \, G^H\left(R^H\left(E_i^H, X^H\right)\xi^H, E_i^H\right) + \varepsilon \, G^H(R^H(\xi^H, X^H)\xi^H, \xi^H)$$

$$= \sum_{i=1}^{2n} \varepsilon_i G^H\left(\frac{(\alpha^2-\beta^2)}{4}\{\eta^H(X^H)E_i^H - \eta^H\left(E_i^H\right)X^H\} + \varepsilon\frac{\alpha\beta}{2}\{\eta^H(X^H)\phi E_i^H - \eta^H\left(E_i^H\right)\phi X^H\}, E_i^H\right) +$$
$$\varepsilon \, G^H\left(\frac{(\alpha^2-\beta^2)}{4}\{\eta^H(X^H)\,\xi^H - \eta^H(\,\xi^H)X^H\} + \varepsilon\frac{\alpha\beta}{2}\{\eta^H(X^H)\phi\xi^H - \eta^H(\,\xi^H)\phi X^H\}, \xi^H\right),$$

where, since $G^H(R^H(\,\xi^H, X^H)\xi^H, \xi^H) = 0$ we get

$$S^H(X^H, \ \xi^H) = n\frac{(\alpha^2-\beta^2)}{2}\eta^H(X^H), \ S^H(\xi^H, \ \xi^H) = n\frac{(\alpha^2-\beta^2)}{2}.$$

The Ricci operatör Q given by

$$S^H(X^H, Y^H) = G^H(QX^H, Y^H) \text{ and } S^V(X^V, Y^V) = G^V(QX^V, Y^V).$$

By using $S^H(X^H, \xi^H) = G^H(QX^H, \xi^H)$ and $S^V(X^V, \xi^V) = G^V(QX^V, \xi^V)$, we obtain

$$QX^H = \varepsilon\frac{n(\alpha^2-\beta^2)}{2}(X^H), \ Q\xi^H = \varepsilon\frac{n(\alpha^2-\beta^2)}{2}\xi^H \text{ and } QX^V = \varepsilon\frac{n(\alpha^2-\beta^2)}{2}(X^V), \ Q\xi^H = \varepsilon\frac{n(\alpha^2-\beta^2)}{2}\xi^V.$$

Example 3.1. Consider the structure of $F^3 = (\mathbb{R}^3, (\mathbb{R}^3)^0, F^*)$ indefinite Finsler manifold. $(\mathbb{R}^3)^0 = \mathbb{R}^6 \setminus \{0\}$ is a real 6-dimensional C^∞ manifold and $T\mathbb{R}^3$ is the tangent bundle of $\mathbb{R}^3$. A coordinate system in $\mathbb{R}^3$can be stated with $\{(U, \varphi): x^1, x^2, x^3\}$, where U is an open subset of $\mathbb{R}^3$; for any $x \in U$, $\varphi: U \to \mathbb{R}^3$ is a diffeomorphism of U onto $\varphi(U)$, and $\varphi(x) = (x^1, x^2, x^3)$. On $\mathbb{R}^3$, denote by π the canonical projection of $T\mathbb{R}^3$ and by T_xM the fibre, at $x \in \mathbb{R}^3$, i.e., $T_x\mathbb{R}^3 = \pi^{-1}(x)$. Through the coordinate system $\{(U, \varphi): x^i\}$ in $\mathbb{R}^3$, we can describe a new coordinate system $\{(U^*, \Phi); x^1, x^2, x^3; y^1, y^2, y^3\}$ or shortly $\{(U^*, \Phi): x^i, y^i\}$ in $T\mathbb{R}^3$, where $U^* = \pi^{-1}(U)$ and $\Phi: U^* \to \mathbb{R}^6$ is a diffeomorphism of U^* on $\varphi(U) \times \mathbb{R}^3$, and $\Phi(y_x) = (x^1, x^2, x^3; y^1, y^2, y^3)$ for any $x \in U$ and $y_x \in T_x\mathbb{R}^3$. Let $(\mathbb{R}^3)^0$ be a non-empty open submanifold of $T\mathbb{R}^3$ such that $\pi((\mathbb{R}^3)^0) = \mathbb{R}^3$ and $\theta(\mathbb{R}^3) \cap (\mathbb{R}^3)^0 = \emptyset$, where θ is the zero section of $T\mathbb{R}^3$. Assume that $(\mathbb{R}^3)^0_x = T_x\mathbb{R}^3 \cap (\mathbb{R}^3)^0$ is a positive conic set, for any $k > 0$ and $y \in (\mathbb{R}^3)^0_x$. we have $ky \in (\mathbb{R}^3)^0_x$. Obviously, the largest $(\mathbb{R}^3)^0$ holding the above circumstances is $T\mathbb{R}^3 \setminus \theta(M)$, ordinarily given with the description of a Finsler manifold. The set of the local vector fields $\left\{\frac{\delta}{\delta x^1}, \frac{\delta}{\delta x^2}, \frac{\delta}{\delta x^3}\right\}$ is a basis in $(T(\mathbb{R}^3)^0)^H$ and $\left\{\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \frac{\partial}{\partial y^3}\right\}$ is a basis in $(T(\mathbb{R}^3)^0)^V$. We get

$X^V = X_1^V(x,y)\frac{\partial}{\partial y^1} + X_2^V(x,y)\frac{\partial}{\partial y^2} + X_3^V(x,y)\frac{\partial}{\partial y^3}$, $X^H = X_1^H(x,y)\frac{\delta}{\delta x^1} + X_2^H(x,y)\frac{\delta}{\delta x^2} + X_3^H(x,y)\frac{\delta}{\delta x^3}$, for any $X^V \in (T(\mathbb{R}^3)^0)^V$and $X^H \in (T(\mathbb{R}^3)^0)^H$. Thus, for any $X \in T(\mathbb{R}^3)^0$, $X = X_i^H(x,y)\frac{\delta}{\delta x^i} + X_i^V(x,y)\frac{\partial}{\partial y^i}$ (i =1, 2, 3). Consider a η, 1-form, $\eta = \eta^H + \eta^V = \eta_i{}^H(x,y)dx^i + \eta_i{}^V(x,y)\delta y^i$ (i =1, 2, 3), $\eta^H \in (T^*(\mathbb{R}^3)^0)^H$ and $\eta^V \in (T^*(\mathbb{R}^3)^0)^V$.

G is a symmetric tensor field of type (0,2), non-degenerate and pseudo-Riemannian metric on $(\mathbb{R}^3)^0$. Then, G is called Sasaki Finsler metric on $(\mathbb{R}^3)^0$. Then, G can be defined as below:

$$G = G^H + G^V = g_{ij}^{F^*} dx^i \otimes dx^j + g_{ij}^{F^*} \delta y^i \otimes \delta y^i \text{ (i=1, 2, 3).}$$

The vector fields

$$E_1^H = x_3 \frac{\delta}{\delta x^1} \ , \ E_2^H = x_3 \frac{\delta}{\delta x^2} \ , \ E_3^H = x_3 \frac{\delta}{\delta x^3} = \xi^H$$

are linear independent at every point of $((\mathbb{R}^3)^0)^h$. Let G be the Sasaki Finsler pseudo-metric given by

$$G^H(E_1^H, \xi^H) = G^H(E_1^H, E_2^H) = G^H(E_2^H, \xi^H) = 0$$

$$G^H(E_1^H, E_1^H) = G^H(E_2^H, E_2^H) = 1, G^H(\xi^H, \xi^H) = \varepsilon = \text{-}1.$$

Let η^H be the 1-form derscribed by

$$\eta^H(Z^H) = -G^H(Z^H, \ \xi^H) = -G^H(z_1 E_1^H + \ z_2 E_2^H + z_3 \,\xi^H, \ \xi^H) = z_3 \ , \ \forall \, Z^H \in (T(\,\mathbb{R}^3)^0)^H.$$

Consider ϕ^H the (1, 1) tensör field stated by

$$\phi^H(E_1^H) = - \ E_2^H \ , \phi^H(\,E_2^H) = E_1^H \ , \ \phi^H(\xi^H) = 0.$$

Then using the linearity of ϕ^H, we have

$$Z^H = z_1 E_1^H + \ z_2 E_2^H + z_3 \,\xi^H \ , W^H = w_1 E_1^H + \ w_2 E_2^H + w_3 \,\xi^H$$

$$\phi^H(Z^H) = \phi^H(z_1 E_1^H + \ z_2 E_2^H + z_3 \,\xi^H \,) = z_1 \phi^H(E_1^H) + z_2 \phi^H(\,E_2^H) + z_3 \,\phi^H(\xi^H)$$

$$\phi^H(Z^H) = -z_1 E_2^H + \ z_2 E_1^H$$

$$\phi^H(W^H) = \phi^H(w E_1^H + \ w E_2^H + w_3 \,\xi^H \,) = w_1 \phi^H(E_1^H) + w_2 \phi^H(\,E_2^H) + w_3 \,\phi^H(\xi^H)$$

$$\phi^H(W^H) = -w_1 E_2^H + \ w_2 E_1^H$$

$$(\phi^H)^2(Z^H) = -z_2 E_2^H - \ z_1 E_1^H = \text{-}\,Z + \eta^H(Z^H)\xi^H$$

Thus we get

$$G^H\big(\phi^H(Z^H), \phi^H(W^H)\big) = \ G^H(Z^H, \ W^H) + \eta^H(Z^H)\,\eta^H(W^H)$$

$\forall \ Z^H \in (T(\,\mathbb{R}^3)^0)^H$ and $\forall \ W^H \in (T(\,\mathbb{R}^3)^0)^H$. Thus the structure $(((\mathbb{R}^3)^0)^h \ , \phi^H, \xi^H, \eta^H, G^H)$ define the almost contact pseudo-metric Finsler structure on $((\mathbb{R}^3)^0)^h$.

Let ∇ be the Levi-Civita connection with respect to pseudo-metric G^H. Then we have

$[E_1^H, E_2^H] = 0 \ , \ [\,E_1^H, \xi^H] = -E_1^H \ , \ [E_2^H, \xi^H] = - \,E_2^H.$

The connection ∇ of the pseudo-metric G^H is given by

$$2G^H\big(\nabla_{X^H} Y^H, Z^H\big) = X^H G^H(Y^H, Z^H) + Y^H G^H(X^H, Z^H) - Z^H G^H(X^H, Y^H)$$
$$-G^H(X^H, [Y^H, Z^H]) - G^H(Y^H, [X^H, Z^H]) \ + G^H(Z^H, [X^H, Y^H]) \qquad (3.17)$$

Which is known as Koszul's formula. Using this formula, we have

$$2G^H\left(\nabla_{E_1^H} \xi^H, \ E_1^H\right) = -G^H(E_1^H, [\xi^H, \ E_1^H \,]\,) - G^H(\xi^H, [\,E_1^H, \ E_1^H]) + G^H(E_1^H \ , [\,E_1^H, \xi^H])$$
$$= 2G^H(-E_1^H, \ E_1^H).$$

Thus,

$$\nabla_{E_1^H} \xi^H = -E_1^H \ , \ \nabla_{\xi^H} E_1^H = 0.$$

Again by using Koszul's formula we obtain

$$2G^H\left(\nabla_{E_2^H} \xi^H, \ E_2^H\right) = -G^H(E_2^H, [\xi^H, \ E_2^H \,]\,) - G^H(\xi^H, [\,E_2^H, \ E_2^H]) + G^H(E_2^H \ , [\,E_2^H, \xi^H])$$
$$= 2G^H(- \,E_2^H, \ E_2^H).$$

Thus,

$$\nabla_{E_2^H}\xi^H = -E_2^H \quad , \nabla_{\xi^H} E_2^H = 0.$$

Also by using Koszul's formula we obtain

$$2G^H\left(\nabla_{E_1^H} E_2^H, \xi^H\right) = G^H(E_1^H, [\xi^H, E_2^H]) + G^H(\xi^H, [E_1^H, E_2^H]) - G^H(E_2^H, [E_1^H, \xi^H]) = 0.$$

Thus,

$$\nabla_{E_1^H} E_2^H = 0\,, \quad \nabla_{E_2^H} E_1^H = 0$$

Similarly we get

$$2G^H\left(\nabla_{E_1^H} E_1^H, \xi^H\right) = -G^H(E_1^H, [E_1^H, \xi^H]) + G^H(\xi^H, [E_1^H, E_1^H]) - G^H(E_1^H, [E_1^H, \xi^H])$$
$$= 2G^H(E_1^H, E_1^H) = -2G^H(\xi^H, \xi^H).$$

Thus,

$$\nabla_{E_1^H} E_1^H = -\xi^H.$$

(3.17) further yields

$$\nabla_{E_2^H} E_2^H = -\xi^H,\ \nabla_{\xi^H} E_1^H = 0,\ \nabla_{\xi^H} E_2^H = 0, \nabla_{E_2^H} E_1^H = 0.$$

If we use the equations we found

$$(\nabla_X^H \xi^H) = x_1 \nabla_{E_1^H}\xi^H + x_2 \nabla_{E_2^H}\xi^H = (-x_1)\, E_1^H - (x_2)E_2^H,$$

$\forall\, X^H \in (T(\mathbb{R}^3)^0)^H$.

The above equations tell us the almost contact pseudo-metric Finsler manifold $\left((\mathbb{R}^3)^0)^h, \phi^H, \xi^H, \eta^H, G^H\right)$ satisfy (3.3) for $\alpha = 0\,, \beta = -2\,,\ \varepsilon = -1$.

With the help of the above results it can be verified that

$$R^H(E_1^H, E_2^H)\, E_2^H = E_1^H, \qquad R^H(\xi^H, E_2^H)\, E_2^H = \xi^H, \qquad R^H(E_1^H, \xi^H)\,\xi^H = -E_1^H$$
$$R^H(E_2^H, \xi^H)\,\xi^H = -E_2^H, \qquad R^H(E_2^H, E_1^H)\, E_1^H = E_2^H, \qquad R^H(\xi^H, E_1^H)\, E_1^H = \xi^H$$

$$S^H(\xi^H, \xi^H) = G^H(R^H(E_1^H, \xi^H)\,\xi^H, E_1^H) + G^H(R^H(E_2^H, \xi^H)\,\xi^H, E_2^H) = G^H(-E_1^H, E_1^H) + G^H(-E_2^H, E_2^H)$$

$$S^H(\xi^H, \xi^H) = n\frac{(\alpha^2-\beta^2)}{2} = -2$$

Example 3.2. Consider the structure of $F^3 = (\mathbb{R}^3, (\mathbb{R}^3)^0, F^*)$ indefinite Finsler manifold. $(\mathbb{R}^3)^0 = \mathbb{R}^6 \setminus \{0\}$ is a real 6-dimensional C^∞ manifold and $T\mathbb{R}^3$ is the tangent bundle of $\mathbb{R}^3$. A coordinate system in $\mathbb{R}^3$ can be stated with $\{(U,\varphi): x^1, x^2, x^3\}$, where U is an open subset of $\mathbb{R}^3$; for any $x \in U$, $\varphi: U \to \mathbb{R}^3$ is a diffeomorphism of U onto $\varphi(U)$, and $\varphi(x) = (x^1, x^2, x^3)$. On $\mathbb{R}^3$, denote by π the canonical projection of $T\mathbb{R}^3$ and by T_xM the fibre, at $x \in \mathbb{R}^3$, i.e., $T_x\mathbb{R}^3 = \pi^{-1}(x)$. Through the coordinate system $\{(U,\varphi): x^i\}$ in $\mathbb{R}^3$, we can describe a new coordinate system $\{(U^*,\Phi); x^1, x^2, x^3; y^1, y^2, y^3\}$ or shortly $\{(U^*,\Phi): x^i, y^i\}$ in $T\mathbb{R}^3$, where $U^* = \pi^{-1}(U)$ and $\Phi: U^* \to \mathbb{R}^6$ is a diffeomorphism of U^* on $\varphi(U) \times \mathbb{R}^3$, and $\Phi(y_x) = (x^1, x^2, x^3; y^1, y^2, y^3)$ for any $x \in U$ and $y_x \in T_x\mathbb{R}^3$. Let $(\mathbb{R}^3)^0$ be a non-empty open submanifold of $T\mathbb{R}^3$ such that $\pi((\mathbb{R}^3)^0) = \mathbb{R}^3$ and $\theta(\mathbb{R}^3) \cap (\mathbb{R}^3)^0 = \emptyset$, where θ is the zero section of $T\mathbb{R}^3$. Assume that $(\mathbb{R}^3)^0_x = T_x\mathbb{R}^3 \cap (\mathbb{R}^3)^0$ is a positive conic set, for any $k > 0$ and $y \in (\mathbb{R}^3)^0_x$. we have $ky \in (\mathbb{R}^3)^0_x$. Obviously, the largest $(\mathbb{R}^3)^0$ holding the above circumstances is $T\mathbb{R}^3 \setminus \theta(M)$, ordinarily given with the description of a Finsler manifold. The set of the local vector fields $\left\{\frac{\delta}{\delta x^1}, \frac{\delta}{\delta x^2}, \frac{\delta}{\delta x^3}\right\}$ is a basis in $(T(\mathbb{R}^3)^0)^H$ and $\left\{\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \frac{\partial}{\partial y^3}\right\}$ is a basis in $(T(\mathbb{R}^3)^0)^V$. We get

$X^V = X_1^V(x,y)\frac{\partial}{\partial y^1} + X_2^V(x,y)\frac{\partial}{\partial y^2} + X_3^V(x,y)\frac{\partial}{\partial y^3}$, $X^H = X_1^H(x,y)\frac{\delta}{\delta x^1} + X_2^H(x,y)\frac{\delta}{\delta x^2} + X_3^H(x,y)\frac{\delta}{\delta x^3}$, for any $X^V \in (T(\mathbb{R}^3)^0)^V$and $X^H \in (T(\mathbb{R}^3)^0)^H$. Thus, for any $X \in T(\mathbb{R}^3)^0$, $X = X_i^H(x,y)\frac{\delta}{\delta x^i} + X_i^V(x,y)\frac{\partial}{\partial y^i}$ (i=1, 2, 3). Consider a η, 1-form, $\eta = \eta^H + \eta^V = \eta_i{}^H(x,y)dx^i + \eta_i{}^V(x,y)\delta y^i$ (i=1, 2, 3), $\eta^H \in (T^*(\mathbb{R}^3)^0)^H$ and $\eta^V \in (T^*(\mathbb{R}^3)^0)^V$.

G is a symmetric tensor field of type (0,2), non-degenerate and pseudo-Riemannian metric on $(\mathbb{R}^3)^0$. Then, G is called Sasaki Finsler metric on $(\mathbb{R}^3)^0$. Then, G can be defined as below:

$$G = G^H + G^V = g_{ij}^{F^*} dx^i \otimes dx^j + g_{ij}^{F^*} \delta y^i \otimes \delta y^i \text{ (i=1, 2, 3).}$$

The vector fields

$$E_1^H = \frac{x_1}{x_3}\frac{\delta}{\delta x^1} \ , \ E_2^H = \frac{x_2}{x_3}\frac{\delta}{\delta x^2} \ , \ E_3^H = \frac{\delta}{\delta x^3} = \xi^H$$

are linear independent at every point of $((\mathbb{R}^3)^0)^h$. Let G^H be the Sasaki Finsler pseudo-metric of index 2 given by

$$G^H(E_1^H,\xi^H) = G^H(E_1^H,E_2^H) = G^H(E_2^H,\xi^H) = 0$$

$$G^H(E_1^H,E_1^H) = G^H(E_2^H,E_2^H) = -1, G^H(\xi^H,\xi^H) = \varepsilon = 1.$$

Let η^H be the 1-form derscribed by

$$\eta^H(Z^H) = G^H(Z^H,\ \xi^H) = G^H(z_1E_1^H + z_2E_2^H + z_3\,\xi^H,\ \xi^H) = z_3\ ,\ \forall\, Z^H \in (T(\mathbb{R}^3)^0)^H.$$

Consider ϕ^H the (1, 1) tensör field stated by

$$\phi^H(E_1^H) = E_2^H \ , \phi^H(E_2^H) = -E_1^H\ ,\ \phi^H(\xi^H) = 0.$$

Then using the linearity of ϕ^H, we have

$$Z^H = z_1E_1^H + z_2E_2^H + z_3\,\xi^H\ , W^H = w_1E_1^H + w_2E_2^H + w_3\,\xi^H$$

$$\phi^H(Z^H) = \phi^H(z_1E_1^H + z_2E_2^H + z_3\,\xi^H) = z_1\phi^H(E_1^H) + z_2\phi^H(E_2^H) + z_3\,\phi^H(\xi^H)$$

$$\phi^H(Z^H) = z_1E_2^H - z_2E_1^H$$

$$\phi^H(W^H) = w_1\phi^H(E_1^H) + w_2\phi^H(E_2^H) + w_3\,\phi^H(\xi^H) = w_1E_2^H - w_2E_1^H$$

$$(\phi^H)^2(Z^H) = -z_2E_2^H - z_1E_1^H = -Z + \eta^H(Z^H)\xi^H$$

Thus we get

$$G^H\big(\phi^H(Z^H),\phi^H(W^H)\big) = G^H(Z^H,\ W^H) - \eta^H(Z^H)\,\eta^H(W^H)$$

$\forall\ Z^H \in (T(\mathbb{R}^3)^0)^H$ and $\forall\ W^H \in (T(\mathbb{R}^3)^0)^H$. Thus the structure $(((\mathbb{R}^3)^0)^h, \phi^H, \xi^H, \eta^H, G^H)$ define the almost contact pseudo-metric Finsler structure on $((\mathbb{R}^3)^0)^h$.

Let ∇ be the Levi-Civita connection with respect to pseudo-metric G^H. Then we have

$$[E_1^H,E_2^H] = 0\ ,\ [E_1^H,\ \xi^H] = \frac{1}{x_3}E_1^H\ ,\ [E_2^H,\ \xi^H] = \frac{1}{x_3}\,E_2^H.$$

The connection ∇ of the pseudo-metric G^H is given by

$$2G^H\big(\nabla_{X^H}Y^H,Z^H\big) = X^HG^H(Y^H,Z^H) + Y^HG^H(X^H,Z^H) - Z^HG^H(X^H,Y^H) - G^H(X^H,[Y^H,Z^H])$$
$$-g(Y^H,[X^H,Z^H]) + G^H(Z^H,[X^H,Y^H])$$

Which is known as Koszul’s formula. Using this formula, we have

$$2G^H\left(\nabla_{E_1^H}\xi^H,\ E_1^H\right) = -G^H(E_1^H,[\xi^H,\ E_1^H\ ]\,) - G^H(\xi^H,[\ E_1^H,\ E_1^H]) + G^H(E_1^H\ ,[\ E_1^H,\ \xi^H])$$

$$= 2G^H(\tfrac{1}{x_3}E_1^H,\ E_1^H).$$

Thus,

$$\nabla_{E_1^H}\xi^H = \tfrac{1}{x_3}E_1^H\ ,\ \nabla_{\xi^H}E_1^H = 0.$$

Again by using Koszul's formula we obtain

$$2G^H\left(\nabla_{E_2^H}\xi^H,\ E_2^H\right) = -G^H(E_2^H,[\xi^H,\ E_2^H\ ]\,) - G^H(\xi^H,[\ E_2^H,\ E_2^H]) + G^H(E_2^H\ ,[\ E_2^H,\xi^H])$$

$$= 2G^H(\tfrac{1}{x_3}\ E_2^H,\ E_2^H).$$

Thus,

$$\nabla_{E_2^H}\xi^H = \tfrac{1}{x_3}\ E_2^H \quad , \quad \nabla_{\xi^H}E_2^H = 0.$$

Also by using Koszul's formula we obtain

$$2G^H\left(\nabla_{E_1^H}\ E_2^H,\ \xi^H\right) = G^H(E_1^H,[\xi^H,\ E_2^H\ ]\,) + (\xi^H,[\ E_1^H,\ E_2^H]) - G^H(E_2^H\ ,[\ E_1^H,\xi^H]) = 0.$$

Thus,

$$\nabla_{E_1^H}\ E_2^H = 0\,, \quad \nabla_{E_2^H}\ E_1^H = 0$$

Similarly we get

$$2G^H\left(\nabla_{E_1^H}\ E_1^H,\ \xi^H\right) = -\ G^H(E_1^H,[E_1^H,\xi^H\ ]\,) + (\xi^H,[\ E_1^H,\ E_1^H]) - G^H(E_1^H\ ,[\ E_1^H,\xi^H])$$

$$= -2G^H\left(\tfrac{1}{x_3}E_1^H\ ,\ E_1^H\right) == \tfrac{2}{x_3} = 2G^H\left(\tfrac{1}{x_3}\xi^H\ ,\ \xi^H\right).$$

Thus,

$$\nabla_{E_1^H}\ E_1^H = \frac{1}{x_3}\ \xi^H\ .$$

If we use the equations we found

$$(\nabla_X^H\xi^H) = x_1\nabla_{E_1^H}\xi^H + x_2\nabla_{E_2^H}\xi^H = x_1\ \tfrac{1}{x_3}\ E_1^H + x_2\ \tfrac{1}{x_3}E_2^H,$$

$\forall\ X^H \in (T(\ \mathbb{R}^3)^0)^H$.

The above equations tell us the almost contact pseudo-metric Finsler manifold $\left((\mathbb{R}^3)^0\right)^h, \phi^H, \xi^H, \eta^H, G^H\right)$ satisfy (3.3) for $\alpha = 0\ ,\ \beta = \frac{2}{x_3},\ \varepsilon =\ 1.$

3.1. α −Sasakian Indefinite Finsler Manifolds

$F^{2n+1} = (M, M^0, F^*)$ be an indefinite Finsler manifold. The almost contact pseudo-metric Finsler structures $(\phi^H, \xi^H, \eta^H, G^H)$ and $(\phi^V, \xi^V, \eta^V, G^V)$ on $(M^0)^h$ and $(M^0)^v$ are the α −Sasakian pseudo-metric Finsler structures if and only if

$$(\nabla_X^H\phi^H)Y^H = \tfrac{\alpha}{2}\{G^H(X^H,Y^H)\xi^H - \varepsilon\eta^H(Y^H)X^H\} \tag{3.18}$$

$$(\nabla_X^V\phi^V)Y^V = \tfrac{\alpha}{2}\{G^V(X^V,Y^V)\xi^V - \varepsilon\eta^V(Y^V)X^V\} \tag{3.19}$$

and

$$(\nabla_X^H \xi^H) = -\varepsilon \frac{\alpha}{2}\, \phi X^H, \qquad (\nabla_X^V \xi^V) = -\varepsilon \frac{\alpha}{2}\, \phi X^V.$$

Moreover, from (3.18) and (3.19) we obtain

$$(\nabla_X^H \eta^H)(Y^H) = \frac{\alpha}{2} \Omega^H (X^H, Y^H) = \frac{\alpha}{2} G^H (X^H, \phi Y^H)$$

$$(\nabla_X^V \eta^V)(Y^V) = \frac{\alpha}{2} \Omega^V (X^V, Y^V) = \frac{\alpha}{2} G^V (X^V, \phi Y^V)$$

Thus, these structures are the α −Sasakian pseudo-metric structures in the α −Sasakian indefinite Finsler manifolds $((M^0)^h, \phi^H, \xi^H, \eta^H, G^H)$ and $((M^0)^v, \phi^V, \xi^V, \eta^V, G^V)$. Also, the following relations hold.

$$R^H(X^H, Y^H)\xi^H = \frac{\alpha^2}{4}\{\eta^H(Y^H)X^H - \eta^H(X^H)Y^H\}$$

$$R^V(X^V, Y^V)\xi^V = \frac{\alpha^2}{4}\{\eta^V(Y^V)X^V - \eta^V(X^V)Y^V\}$$

$$\eta^H(R^H(X^H, Y^H)Z^H) = \varepsilon \frac{\alpha^2}{4}\{G^H(Y^H, Z^H)\eta^H(X^H) - G^H(X^H, Z^H)\eta^H(Y^H)\}$$

$$\eta^V(R^V(X^V, Y^V)Z^V) = \varepsilon \frac{\alpha^2}{4}\{G^V(Y^V, Z^V)\eta^V(X^V) - G^V(X^V, Z^V)\eta^V(Y^V)\}$$

$$(\nabla_Z^H R^H)(X^H, Y^H)\xi^H = \varepsilon \frac{\alpha^2}{8}\{G^H(Y^H, Z^H)X^H - G^H(X^H, Z^H)Y^H\} - \frac{1}{2}\, R^H(X^H, Y^H)Z^H$$

$$(\nabla_Z^V R^V)(X^V, Y^V)\xi^V = \varepsilon \frac{\alpha^2}{8}\{G^V(Y^V, Z^V)X^V - G^V(X^V, Z^V)Y^V\} - \frac{1}{2}\, R^V(X^V, Y^V)Z^V$$

$$R^H(X^H, Y^H)Z^H = \varepsilon \frac{\alpha^2}{4}\{G^H(Y^H, Z^H)X^H - G^H(X^H, Z^H)Y^H\}$$

$$R^V(X^V, Y^V)Z^V = \varepsilon \frac{\alpha^2}{4}\{G^V(Y^V, Z^V)X^V - G^V(X^V, Z^V)Y^V\}$$

$$R^H(X^H, \xi^H)Y^H = \frac{\alpha^2}{4}\{\eta^H(Y^H)X^H - \varepsilon G^H(X^H, Y^H)\xi^H\}$$

$$R^V(X^V, \xi^V)Y^V = \frac{\alpha^2}{4}\{\eta^V(Y^V)X^V - \varepsilon G^V(X^V, Y^V)\xi^V\}$$

$$R^H(\xi^H, X^H)Y^H = \frac{\alpha^2}{4}\{\varepsilon G^H(X^H, Y^H)\xi^H - \eta^H(Y^H)X^H\}$$

$$R^V(\xi^V, X^V)Y^V = \frac{\alpha^2}{4}\{\varepsilon G^V(X^V, Y^V)\xi^V - \eta^V(Y^V)X^V\}$$

$$S^H(\xi^H, \xi^H) = \begin{cases} \alpha^2 \left(\dfrac{2n-q}{4}\right), \xi^H \text{ is a space} - \text{like vector} \\ \alpha^2 \left(\dfrac{2n-q+1}{4}\right), \xi^H \text{ is a time} - \text{like vector} \end{cases}$$

$$S^V(\xi^V,\xi^V)=\begin{cases}\alpha^2\left(\frac{2n-q}{4}\right),\xi^V\ is\ a\ space-like\ vector\\ \alpha^2\left(\frac{2n-q+1}{4}\right),\xi^V\ is\ a\ time-like\ vector\end{cases}$$

$$S^H(X^H,\xi^H)=\begin{cases}\alpha^2\left(\frac{2n-q}{4}\right)\eta^H(X^H),\xi^H\ is\ a\ space-like\ vector\\ \alpha^2\left(\frac{2n-q+1}{4}\right)\eta^H(X^H),\xi^H\ is\ a\ time-like\ vector\end{cases}$$

$$S^V(X^V,\xi^V)=\begin{cases}\alpha^2\left(\frac{2n-q}{4}\right)\eta^V(X^V),\xi^V\ is\ a\ space-like\ vector\\ \alpha^2\left(\frac{2n-q+1}{4}\right)\eta^V(X^V),\xi^V\ is\ a\ time-like\ vector\end{cases}$$

If ξ^H and ξ^V are the space-like vectors, then we get

$$S^H(\phi X^H,\phi Y^H)=S^H(X^H,Y^H)+\alpha^2\left(\frac{q-2n}{4}\right)\eta^H(X^H)\eta^H(Y^H)$$

$$S^V(\phi X^V,\phi Y^V)=S^V(X^V,Y^V)+\alpha^2\left(\frac{q-2n}{4}\right)\eta^V(X^V)\eta^V(Y^V).$$

If ξ^H and ξ^V are the time-like vectors, then we get

$$S^H(\phi X^H,\phi Y^H)=S^H(X^H,Y^H)+\alpha^2\left(\frac{q-2n-1}{4}\right)\eta^H(X^H)\eta^H(Y^H)$$

$$S^V(\phi X^V,\phi Y^V)=S^V(X^V,Y^V)+\alpha^2\left(\frac{q-2n-1}{4}\right)\eta^V(X^V)\eta^V(Y^V).$$

3.2. $\boldsymbol{\beta}$ −Kenmotsu Indefinite Finsler Manifolds

Let $F^{2n+1}=(M,M^0,F^*)$ be an indefinite Finsler manifold with the warped product space $M^{2n+1}=\mathbb{R}\times_f N^{2n}$. We suppose that $(N^0)^{2n}=TN^{2n}\setminus\theta$ is a Kahlerian manifold and $f(t)=ce^{\beta\frac{t}{2}}$. For the almost Kenmotsu pseudo-metric Finsler structures $(\phi^H,\xi^H,\eta^H,G^H)$ and $(\phi^V,\xi^V,\eta^V,G^V)$ on $(M^0)^h$ and $(M^0)^v$ resp., 1-forms η^H and η^V and 2-forms Ω^H and Ω^V satisfy the below conditions.

$$d\eta^H=d\eta^V=0,\qquad d\eta^H=\beta\eta^H\wedge\Omega^H,\qquad d\eta^V=\beta\eta^V\wedge\Omega^V$$

where β being a non-zero real constant.

The almost contact pseudo-metric Finsler structures $(\phi^H,\xi^H,\eta^H,G^H)$ and $(\phi^V,\xi^V,\eta^V,G^V)$ on $(M^0)^h$ and $(M^0)^v$ resp., are the β-Kenmotsu pseudo-metric Finsler structures if and only if

$$(\nabla_X^H\phi)Y^H=\frac{\beta}{2}\{\varepsilon G^H(\phi X^H,Y^H)\xi^H-\eta^H(Y^H)\phi X^H\}\qquad(3.20)$$

$$(\nabla_X^V\phi)Y^V=\frac{\beta}{2}\{\varepsilon G^V(\phi X^V,Y^V)\xi^V-\eta^V(Y^V)\phi X^V\}\qquad(3.21)$$

and

$$(\nabla_X^H\xi^H)=\frac{\beta}{2}(X^H-\eta^H(X^H)\xi^H)=-\frac{\beta}{2}\phi^2X^H$$

$$(\nabla_X^V\xi^V)=\frac{\beta}{2}(X^V-\eta^V(X^V)\xi^V)=-\frac{\beta}{2}\phi^2X^V.$$

Moreover from (3.20) and (3.21) we obtain

$$(\nabla_X^H \eta^H)(Y^H) = \frac{\beta}{2} G^H(\phi X^H, \phi Y^H) = \frac{\beta}{2} \Omega^H(\phi X^H, Y^H)$$

$$(\nabla_X^V \eta^V)(Y^V) = \frac{\beta}{2} G^V(\phi X^V, \phi Y^V) = \frac{\beta}{2} \Omega^V(\phi X^V, Y^V).$$

Thus, these structures are the β −Kenmotsu pseudo-metric Finsler structures.

In the β −Kenmotsu indefinite Finsler manifolds $\left((M^0)^h, \phi^H, \xi^H, \eta^H, G^H\right)$ and$((M^0)^v, \phi^V, \xi^V, \eta^V, G^V)$, the following relations hold.

$$R^H(X^H, Y^H)\xi^H = \frac{\beta^2}{4}\{\eta^H(X^H)Y^H - \eta^H(Y^H)X^H\}$$

$$R^V(X^V, Y^V)\xi^V = \frac{\beta^2}{4}\{\eta^V(X^V)Y^V - \eta^V(Y^V)X^V\}$$

$$\eta^H(R^H(X^H, Y^H)Z^H) = \varepsilon\frac{\beta^2}{4}\{G^H(X^H, Z^H)\eta^H(Y^H) - G^H(Y^H, Z^H)\eta^H(X^H)\}$$

$$\eta^V(R(X^V, Y^V)Z^V) = \varepsilon\frac{\beta^2}{4}\{G^V(Y^V, Z^V)\eta^V(Y^V) - G^V(Y^V, Z^V)\eta^V(X^V)\}$$

$$(\nabla_Z^H R^H)(X^H, Y^H)\xi^H = \varepsilon\frac{\beta^2}{8}\{G^H(X^H, Z^H)Y^H - G^H(Y^H, Z^H)X^H\} - \frac{1}{2} R^H(X^H, Y^H)Z^H$$

$$(\nabla_Z^V R^V)(X^V, Y^V)\xi^V = \varepsilon\frac{\beta^2}{8}\{G^V(X^V, Z^V)Y^V - G^V(Y^V, Z^V)X^V\} - \frac{1}{2} R^V(X^V, Y^V)Z^V$$

$$R^H(X^H, Y^H)Z^H = -\varepsilon\frac{\beta^2}{4}\{G^H(Y^H, Z^H)X^H - G^H(X^H, Z^H)Y^H\}$$

$$R^V(X^V, Y^V)Z^V = -\varepsilon\frac{\beta^2}{4}\{G^V(Y^V, Z^V)X^V - G^V(X^V, Z^V)Y^V\}$$

$$R^H(\xi^H, X^H)Y^H = \varepsilon\frac{\beta^2}{4}\{-G^H(X^H, Y^H)\xi^H + \varepsilon\eta^H(Y^H)X^H\}$$

$$R^V(\xi^V, X^V)Y^V = \varepsilon\frac{\beta^2}{4}\{-G^V(X^V, Y^V)\xi^V + \varepsilon\eta^V(Y^V)X^V\}$$

$$S^H(\xi^H, \xi^H) = \begin{cases} \beta^2\left(\frac{q-2n}{4}\right), \xi^H \text{ is a space} - \text{like vector} \\ \beta^2\left(\frac{q-2n-1}{4}\right), \xi^H \text{ is a time} - \text{like vector} \end{cases}$$

$$S^V(\xi^V, \xi^V) = \begin{cases} \beta^2\left(\frac{q-2n}{4}\right), \xi^V \text{ is a space} - \text{like vector} \\ \beta^2\left(\frac{q-2n-1}{4}\right), \xi^V \text{ is a time} - \text{like vector} \end{cases}$$

$$S^H(X^H, \xi^H) = \begin{cases} \beta^2\left(\frac{q-2n}{4}\right)\eta^H(X^H), \xi^H \text{ is a space} - \text{like vector} \\ \beta^2\left(\frac{q-2n-1}{4}\right)\eta^H(X^H), \xi^H \text{ is a time} - \text{like vector} \end{cases}$$

$$S^V(X^V,\xi^V)=\begin{cases}\beta^2\left(\frac{q-2n}{4}\right)\eta^V(X^V), \xi^V \ is\ a\ space-like\ vector\\ \beta^2\left(\frac{q-2n-1}{4}\right)\eta^V(X^V), \xi^V \ is\ a\ time-like\ vector\end{cases}$$

$$S^H(\phi X^H,\phi Y^H)=S^H(X^H,Y^H)+\beta^2\left(\frac{2n-q}{4}\right)\eta^H(X^H)\eta^H(Y^H)$$

$$S^V(\phi X^V,\phi Y^V)=S^V(X^V,Y^V)+\beta^2\left(\frac{2n-q}{4}\right)\eta^V(X^V)\eta^V(Y^V).$$

4. Conformally Flat Trans-Sasakian Indefinite Finsler Manifolds

We consider conformally flat trans-Sasakian indefinite Finsler manifolds $\left((M^0)^h,\phi^H,\xi^H,\eta^H,G^H\right)$ and $((M^0)^v,\phi^V,\xi^V,\eta^V,G^V)$. The conformal curvature tensor field C is given by

$$C^H(X^H,\ Y^H)Z^H=R^H(X^H,\ Y^H)Z^H-\frac{1}{(2n-1)}[S^H(Y^H,\ Z^H)X^H-\ S^H(X^H,\ Z^H)Y^H+G^H(Y^H,Z^H)QX^H-G^H(X^H,\ Z^H)QY^H]+\frac{r}{2n(2n-1)}[G^H(Y^H,Z^H)X^H-G^H(X^H,Z^H)Y^H] \quad (4.1)$$

and

$$C^V(X^V,\ Y^V)Z^V=R^V(X^V,\ Y^V)Z^V-\frac{1}{(2n-1)}[S^V(Y^V,\ Z^V)X^V-\ S^V(X^V,\ Z^V)Y^V+G^V(Y^V,\ Z^V)QX^V-G^V(X^V,\ Z^V)QY^V]+\frac{r}{2n(2n-1)}[G^V(Y^V,Z^V)X^V-G^V(X^V,\ Z^V)Y^V] \quad (4.2),$$

where R^H, S^H, Q^H and r are the curvature tensor, the Ricci tensor, the Ricci operatör and the scalar curvature tensor of the $(M^0)^h$, respectively. (R^V, S^V, Q^V and r are the curvature tensor, the Ricci tensor, the Ricci operatör and the scalar curvature tensor of the $(M^0)^v$). If the trans-Sasakian indefinite Finsler manifolds $\left((M^0)^h,\phi^H,\xi^H,\eta^H,G^H\right)$ and $((M^0)^v,\phi^V,\xi^V,\eta^V,G^V)$ are conformally flat, i. e.

$C^H=0$ and $C^V=0$, then from (4.1) and (4.2), we have

$$R^H(X^H,\ Y^H)Z^H=\frac{1}{(2n-1)}[S^H(Y^H,\ Z^H)X^H-\ S^H(X^H,\ Z^H)Y^H+G^H(Y^H,Z^H)QX^H-G^H(X^H,\ Z^H)QY^H]-\frac{r}{2n(2n-1)}[G^H(Y^H,Z^H)X^H-G^H(X^H,Z^H)Y^H]$$

$$R^V(X^V,\ Y^V)Z^V=\frac{1}{(2n-1)}[S^V(Y^V,\ Z^V)X^V-\ S^V(X^V,\ Z^V)Y^V+G^V(Y^V,\ Z^V)QX^V-G^V(X^V,\ Z^V)QY^V]-\frac{r}{2n(2n-1)}[G^V(Y^V,Z^V)X^V-G^V(X^V,\ Z^V)Y^V]$$

Now, taking scalar product on both side of above equations with W^H and W^V, we have

$$G^H(R^H(X^H,\ Y^H)Z^H,\ W^H)=G^H(\frac{1}{(2n-1)}[S^H(Y^H,\ Z^H)X^H-\ S^H(X^H,\ Z^H)Y^H+G^H(Y^H,\ Z^H)QX^H-G^H(X^H,\ Z^H)QY^H]+\frac{r}{2n(2n-1)}[G^H(Y^H,\ Z^H)X^H-G^H(X^H,\ Z^H)Y^H]\ ,W^H)$$

and

$$G^V(R^V(X^V,\ Y^V)Z^V,\ W^V)=G^V(\frac{1}{(2n-1)}[S^V(Y^V,\ Z^V)X^V-\ S^V(X^V,\ Z^V)Y^V+G^V(Y^V,\ Z^V)QX^V-G^V(X^V,\ Z^V)QY^V]+\frac{r}{2n(2n-1)}[G^V(Y^V,\ Z^V)X^V-G^V(X^V,\ Z^V)Y^V]\ ,W^V)$$

$$G^H(R^H(X^H,\ Y^H)Z^H,\ W^H)=\frac{1}{(2n-1)}[S^H(Y^H,\ Z^H)G^H(X^H,\ W^H)-\ S^H(X^H,\ Z^H)G^H(Y^H,\ W^H)+G^H(Y^H,\ Z^H)G^H(QX^H,\ W^H)-G^H(X^H,\ Z^H)G^H(QY^H,\ W^H)]$$

$$+\frac{r}{2n(2n-1)}[G^H(Y^H, Z^H)G^H(X^H, W^H) - G^H(X^H, Z^H)G^H(Y^H, W^H)],$$

on putting $W^H = \xi^H$ we get

$$G^H(R^H(X^H, Y^H)Z^H, \xi^H) = \frac{1}{(2n-1)}[S^H(Y^H, Z^H)\varepsilon\,\eta^H(X^H) - S^H(X^H, Z^H)\,\varepsilon\,\eta^H(Y^H) + G^H(Y^H, Z^H)\,S^H(X^H,\xi^H) - G^H(X^H, Z^H)\,S^H(Y^H,\xi^H)] - \varepsilon\,\frac{r}{2n(2n-1)}[G^H(Y^H, Z^H)\,\eta^H(X^H) - G^H(X^H, Z^H)\,\eta^H(Y^H)].$$

Replacing Y^H by ξ^H in equation (3.11) we have

$$G^H(R^H(X^H,\xi^H)Z^H,\xi^H) = \varepsilon\,\eta^H(R^H(X^H,\xi^H)Z^H) = \frac{(\alpha^2-\beta^2)}{4}\{\varepsilon\,\eta^H(X^H)\eta^H(Z^H) - G^H(X^H,Z^H)\}$$

$$-\varepsilon\frac{\alpha\beta}{2}\{G^H(\phi X^H,Z^H)\} = \frac{1}{(2n-1)}[\varepsilon\,S^H(\xi^H, Z^H)\,\eta^H(X^H) - \varepsilon\,S^H(X^H, Z^H) + \varepsilon\,\eta^H(Z^H)S^H(X^H,\xi^H) - G^H(X^H, Z^H)\,S^H(\xi^H, \xi^H)] - \frac{r}{2n(2n-1)}[\eta^H(Z^H)\,\eta^H(X^H) - \varepsilon\,G^H(X^H, Z^H)].$$

by using equations (3.14), (3.15) and (3.16)

$$S^H(X^H, Z^H) = \left[\frac{r}{2n} + \frac{(\alpha^2-\beta^2)}{4}\big((2n-1) - \varepsilon(2n)\big)\right]G^H(X^H,Z^H)$$

$$+[\frac{-\varepsilon\,r}{2n} + \frac{(\alpha^2-\beta^2)}{4}(4n - \varepsilon\,(2n-1)]\,\eta^H(Z^H)\,\eta^H(X^H) + \varepsilon\frac{\alpha\beta}{2}(2n-1)\{G^H(\phi X^H,Z^H)\}$$

and

$$S^V(X^V, Z^V) = \left[\frac{r}{2n} + \frac{(\alpha^2-\beta^2)}{4}\big((2n-1) - \varepsilon(2n)\big)\right]G^V(X^V,Z^V)$$

$$+[\frac{-\varepsilon\,r}{2n} + \frac{(\alpha^2-\beta^2)}{4}(4n - \varepsilon\,(2n-1)]\,\eta^V(Z^V)\,\eta^V(X^V) + \varepsilon\frac{\alpha\beta}{2}(2n-1)\{G^V(\phi X^V,Z^V)\}$$

Hence we have the following theorem

Theorem 4.1. The conformally flat trans-Sasakian indefinite Finsler manifolds $\big((M^0)^h, \phi^H, \xi^H, \eta^H, G^H\big)$ and $((M^0)^v, \phi^V, \xi^V, \eta^V, G^V)$ are the $\eta -$ Einstein manifolds if and only if $\alpha.\beta = 0$, where α, β are constant functions defined on $(M^0)^h$ and $(M^0)^v$.

Corollary 4.1. The conformally flat α -Sasakian indefinite Finsler manifolds $\big((M^0)^h, \phi^H, \xi^H, \eta^H, G^H\big)$ and $((M^0)^v, \phi^V, \xi^V, \eta^V, G^V)$ are the $\eta -$ Einstein manifolds.

Corollary 4.2. The conformally flat β -Kenmotsu indefinite Finsler manifolds $\big((M^0)^h, \phi^H, \xi^H, \eta^H, G^H\big)$ and $((M^0)^v, \phi^V, \xi^V, \eta^V, G^V)$ are the $\eta -$ Einstein manifolds.

5. Conclusion

In this article, we study indefinite trans-Sasakian structures on indefinite Finsler manifolds by using pseudo-Finsler metric. Also, α –Sasakian and β –Kenmotsu indefinite Finsler manifolds are presented. The conformally flat trans-Sasakian indefinite Finsler manifolds $\big((M^0)^h, \phi^H, \xi^H, \eta^H, G^H\big)$ and $((M^0)^v, \phi^V, \xi^V, \eta^V, G^V)$ are the $\eta -$ Einstein manifolds if and only if $\alpha.\beta = 0$, where α, β are constant functions defined on $(M^0)^h$ and $(M^0)^v$.

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