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Determination of the Mathematical Theorems on Ancient Mosaics

Matematik Teoremlerinin Antik Dönem Mozaikleri Üzerinde Tespiti

Erhan AYDOĞDU - Ali Kazım ÖZ*

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Abstract

The aim of this research is to examine and evaluate the reflection of mathematical knowledge on examples of ancient mosaic art. As a result of the comparison between history of mathematics and art of mosaics, a connection has been made between the well-known theorems and patterns through the nature of the forms. For this purpose, patterns like swastika, meander, spiral and cube forms, as well as the forms that can be produced from the graph related to the lunar area calculation of Hippocrates of Chios have been analyzed. In addition, analyzes, discussions and evaluations on the identification of the forms similar to the semi-regular solids of Archimedes and the hexagons of Pappus of Alexandria have been presented. It is thought that the methods used and the information obtained in this study will contribute to the research of mosaic art and history of mathematics, the documentation and evaluation of archaeological artifacts, the museology practices and conservation studies.

Keywords: *Mathematics, geometry, ancient mosaics, theorem, decorative patterns.*

Öz

Bu çalışmanın amacı, Antik Dönem'de matematik bilgisinin mozaik sanatına yansımasının incelenmesi ve değerlendirilmesidir. Matematik tarihi ve mozaik sanatının karşılaştırılması sonucu, en bilinen matematik teoremleri ve mozaik motifleri arasında formların doğası üzerinden bir bağlantı kurulmuştur. Örneğin svastika, meander, spiral, küp ve prizma formları ve Khioslu Hippokrates'in lunar alanlar hesabı ile ilgili grafikten üretilebilen formlar analiz edilmiştir. Ayrıca, İskenderiyeli Pappus'un altıgenleri ve Arkhimedes'in yarı düzgün katıları ile benzerlik gösteren formların tespitine yönelik analizler, tartışmalar ve değerlendirmeler sunulmuştur. Bu çalışmada kullanılan yöntemlerin ve elde edilen bilgilerin; mozaik sanatı ve matematik tarihi araştırmalarına, arkeolojik eserlerin belgelenmesine ve değerlendirilmesine, müzecilik uygulamalarına ve konservasyon çalışmalarına katkı sağlayacağı düşünülmektedir.

Anahtar Kelimeler: *Matematik**,** geometri, mozaik, teorem, desen.*

* Erhan Aydoğdu, Dokuz Eylül University, Department of Archaeology, Tınaztepe Campus, Buca-İzmir, Türkiye. https://orcid.org/0000-0001-6480-0291. E-mail: erhanabdal@gmail.com

Ali Kazım Öz, Dokuz Eylül University, Department of Archaeology, Tınaztepe Campus, Buca-İzmir, Türkiye. https://orcid.org/0000-0002-3005-323X. E-mail: ali.oz@deu.edu.tr

1. Introduction

The initial step towards studying geometric forms in mosaic art involved compiling comprehensive catalogs that allowed for the observation of the chronological development of mosaics. Through this cataloging process, the forms that comprise the geometric motif category were able to be distinguished and defined. Efforts to create motif terminology and study the material necessary to analyze geometric patterns on ancient mosaics have been ongoing since the 1930s, with earlier studies also being present. From the 1930s to 1963, research conducted in this area served as the preparatory process for the recognition of mosaic art as a discipline. These studies highlighted the importance of distinguishing, defining, and analyzing geometric forms in patterns, comparing similar or varying forms in mosaics from different regions, and noting local styles, leading to the definition of workshops and investigation of interactions between them (Aydoğdu 2022: 145). Since 1963, with the first international colloquium on Greek and Roman mosaics, and particularly from around 2000 to 2022, a mature working method has emerged for solving geometric forms, involving geometrical reproduction of basic ornament forms, obtaining complex ornaments with a grid and element pattern approach (Décor I: 10), suggesting algorithms for repetitive use, and comparing variant forms with geometric solutions. However, this working model is insufficient in determining the extent and depth of the relationship between mosaic art and mathematics, as well as establishing the connection of geometric patterns with the history of mathematics and mathematical methods (Aydoğdu 2022: 13-17).

In the present circumstances, the weakest aspect of studies on the relationship between mosaic art and mathematics is the link between the history of mathematics and mathematical methods in analysis. The questioning of the relationship between geometric mosaics and mathematical knowledge is considered to be the weakest link in the chain of these studies.

Geometric mosaics derive their content from geometry and are named in reference to it, leading to questions about the relationship between geometric ornaments and knowledge of geometry. The mosaicist is the actor in this relationship, but considering the various stages of design and construction, the term "mosaicist" is used instead of specific distinctions such as "mosaic artist", "mosaic master", or "mosaic worker". How did the mosaicist establish a relationship with geometry, and did they possess scientific knowledge of it? There are three possible answers to this question: "mosaicist knew geometry", "mosaicist did not know geometry", or "mosaicist had only practical geometry knowledge". However, there is no clear record from ancient times to the present to definitively prove which of these propositions is correct. There are no records showing that a geometric mosaic was designed using geometry and its material equivalent. Therefore, we can only consider these propositions as possibilities.

Mosaicists took their place in history. In one hand, we have the geometric mosaics left over from them, and in the other hand we have scientific geometry. Based on these two things, can we determine which of these propositions might be true? How and to what extent can we detect it? These are the fundamental questions that raise this study. In the context of the fundamental questions, the subject of this study is the identification and evaluation of mosaic decoration showing parallelism with mathematical ideas, theorems, problems and related mathematical graphics in ancient Greek mathematics. Since the subject of the study embodies the problems, discussions and evaluations related to the determination of the relationship between mosaic art and mathematics, with examples showing the parallels between Greek mathematics and geometric mosaics, the title of the study was chosen as "Determination of the mathematical theorems on ancient mosaics".

The analysis of swastika, meander, spiral forms, and quatrefoil have been conducted in this study[1] using mathematical ideas such as measuring the perimeter of a polygon and the graph attributed to Hippocrates of Chios for calculating the areas between circle arcs. The study also includes analyses and discussions on identifying cube forms, Archimedes' semi-regular solids, and Pappus' hexagons.

Through the nature of the forms, this study establishes a connection between mosaic art and the history of mathematics, pointing to mathematics as the inspiration for the decorative repertoire. It offers insight into the relationship between geometry and geometric ornament.

2. Method

In terms of providing the basis for theoretical analysis and in order to observe the relationship between mathematics and mosaic in ancient period, history of mathematics and history of mosaic art were brought together and the parallels between their developments were evaluated.

Squaring the circle, dividing an angle by three (trisection), and doubling the cube are kown as the three famous problems of the ancient period. There are problems which are the extensions of these problems such as drawing polygons within the circle and doubling the square. In addition, there are the problems of not so famous but special in terms of mathematics. The solutions put forward by the ancient mathematicians regarding these problems and the graphics of the solutions were compared with the ornamental repertoire of mosaic art through catalogues. Relationships between the graphics and geometric ornaments were investigated.

3. Analysis and Comparison

3.1. Mathematical Structure of Swastika, Meander and Spiral Forms

The length of a line segment is equal to its own length. The length of the broken line consisting of two line segments is the sum of the lengths of the segments forming the broken line. The perimeter of a triangle is the sum of the lengths of the line segments forming the sides of the triangle. Only itself is sufficient for a line segment. The line segment is equal to itself. To calculate the total length of two line segments forming a broken line, the two lines are placed side by side on the same straight line and added end to end. For this operation, the line segment is transferred with a compass on the linear line carrying the other line segment to which it will be attached. Thus, the total length of the broken line is represented by a line segment equal to the sum of the individual lengths of the lines forming this broken line. In the case of the perimeter of a triangle, the situation is the same as on the broken line. We don't care about the numerical equivalent of length, since we set aside the numerical measure. We are only concerned with what the perimeter of the triangle is, and how long the straight line representing that length is. When the side lengths that make up the triangle are added end to

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end on a linear line, the length equal to the sum of the side lengths of the triangle is obtained. A three-step sequence of operations is performed to geometrically calculate the perimeter of the triangle (Fig. 1).

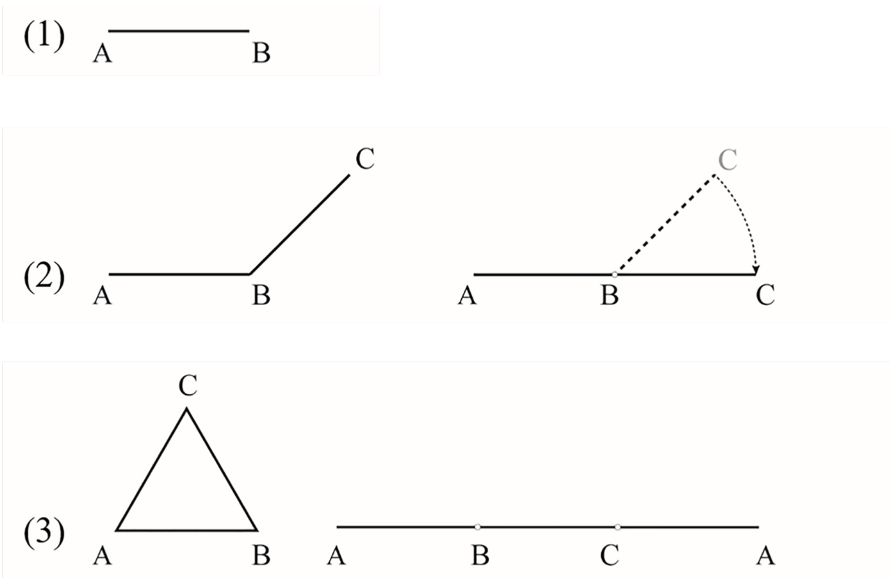


Figure 1
Length transfer (drawing by authors).

The method applied to the broken line and the triangle can be applied to a square. In this case, there are four edges to transfer and four transfer operations. The sides |*AB*|, |*BC*|, |*CD*|, and |*DA*| are extended in the direction *A, B, C,* and *D* respectively. The length |*BC*| is transferred to the axis obtained by extending the |*BC*| in the direction *B*, and the point *T1* is marked. The length |*CT1*| is transferred to the axis obtained by extending the |*CD*| in the direction *C*, and the point *T2* is marked. The length |*DT2*| is transferred to the axis obtained by extending the |*DA*| in the direction *D*, and the point *T3* is marked. The length |*AT3*| is transferred to the axis obtained by extending the |*AB*| in the direction *A*, and the point *T4* is marked. The length |*AT4*| is equal to the perimeter of the square *ABCD*. These transfer operations leave behind four arcs. Their radii are different, and they are added end to end. The broken line connecting the points *A, T1, T2, T3* and *T4* accompanies the arcs (Fig. 2).

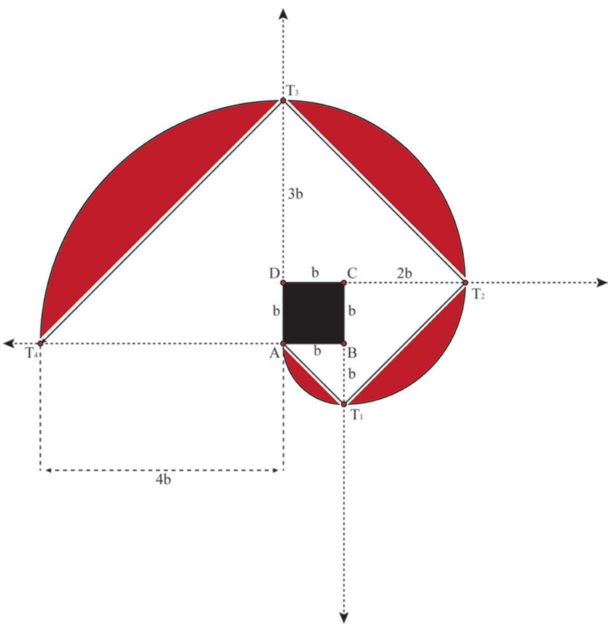


Figure 2
Periphery of a square (drawing by authors).

Once the transferred lengths are distinguished from each other, we can see which length goes where. When the inner tangent circle of the square is included in the graph, the swastika and meander pattern becomes visible. If the loops on the extended edges are continued and expressed in contrasting colors, the geometric structure that explains the meander and swastika patterns is obtained. Line segments that make up the pattern are the segments used in the length transfer. Therefore, the swastika pattern is the main component and the main result of this geometric structure (Fig. 3). The meander pattern consists of areas bounded by the loop lines that form the swastika. The meander is therefore a secondary consequence of the process of length transfer (Fig. 4). Meander and swastika are not the only consequences of the length transfer. The broken line set formed by the line segments performing the length transfer is accompanied by a curve set that imitates the spiral, which is the combination of circle arcs.

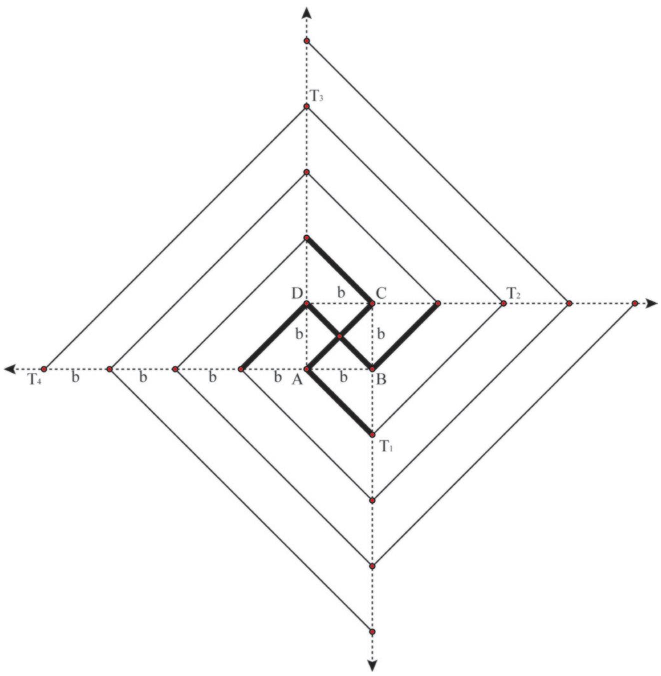


Figure 3
Circumference of square and swastika (drawing by authors).

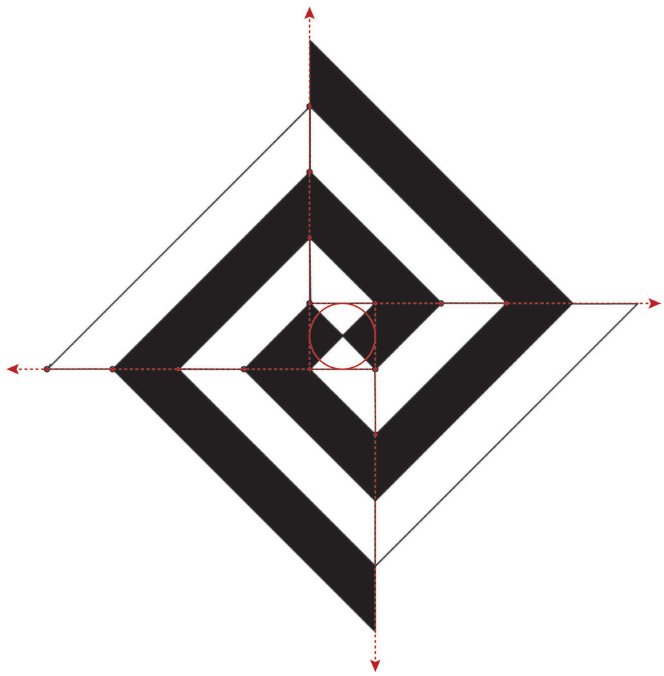


Figure 4
Circumference of square, meander and swastika (drawing by authors).

The broken line set initiated from one corner of the polygon, when applied to all corners, will accompany each broken line set with a pseudo-spiral. There is no doubt that these so-called spirals are not really spirals and can be called as spirals for the sake of convenience. It is reached from the triangle to the triangle spiral. And consequently, the square spiral is reached from the square, and the pentagon spiral is reached from the pentagon (Fig. 5a-c). If the spiral forms are continued towards the center of the polygon, central spirals are obtained (Fig. 5d-f). If clockwise spirals are considered together with counterclockwise spirals, symmetrical spirals are obtained (Fig. 5g-i). Centrally symmetrical spiral structures can be extended to hexagon, octagon and decagon structures.

According to these results, the swastika and meander patterns are linear results of the length transfer process. The curvilinear result of the transfer process is the spiral form. Linear results originate from the ruler, while curvilinear results originate from the caliper. Another example of this is that the regular hexagon drawing also includes the rosette motif. When the length transfer processes applied to polygons such as triangle, square, hexagon, and octagon are continued by increasing the number of polygon sides towards infinity, the result is Archimedes' spiral (Aydoğdu 2022: 168-181) (Fig. 6).

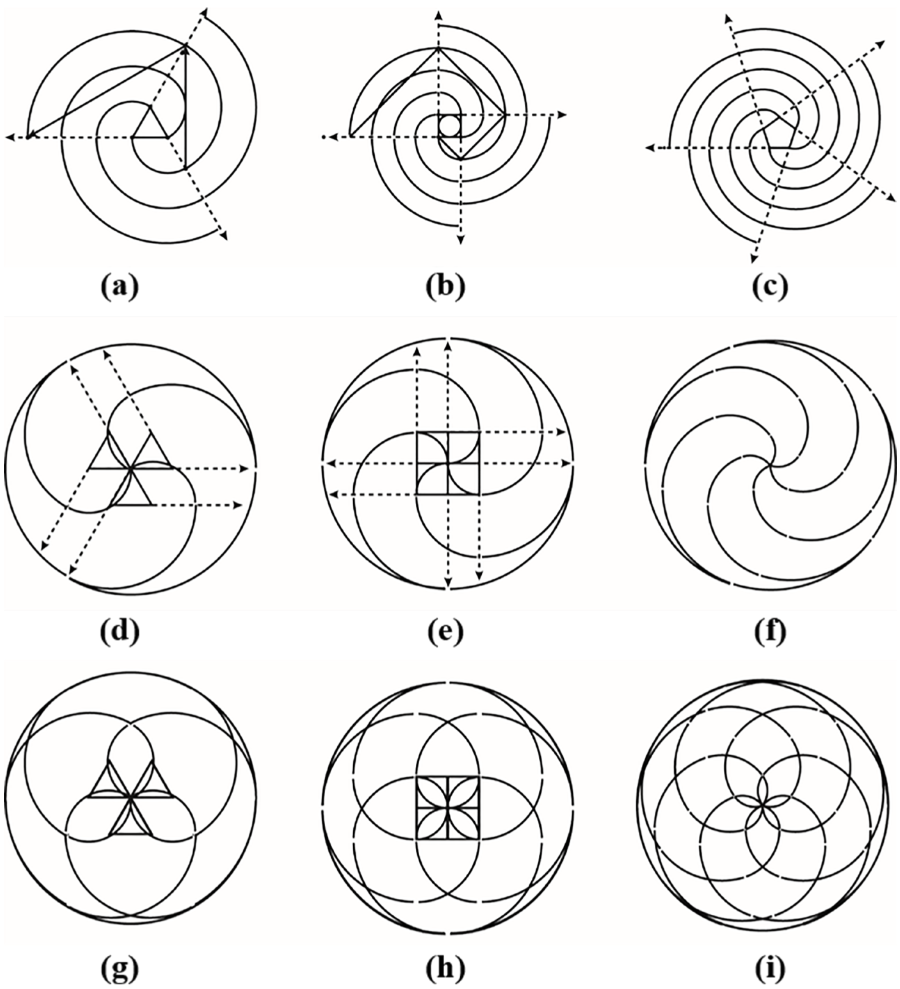


Figure 5
Spiral forms (drawing by authors).

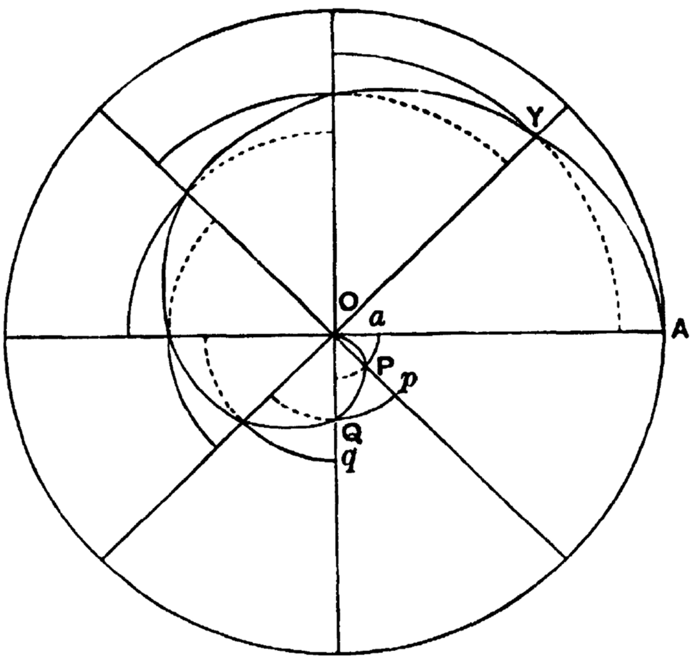


Figure 6
Archimedes' Spiral (Heath 1897: 179).

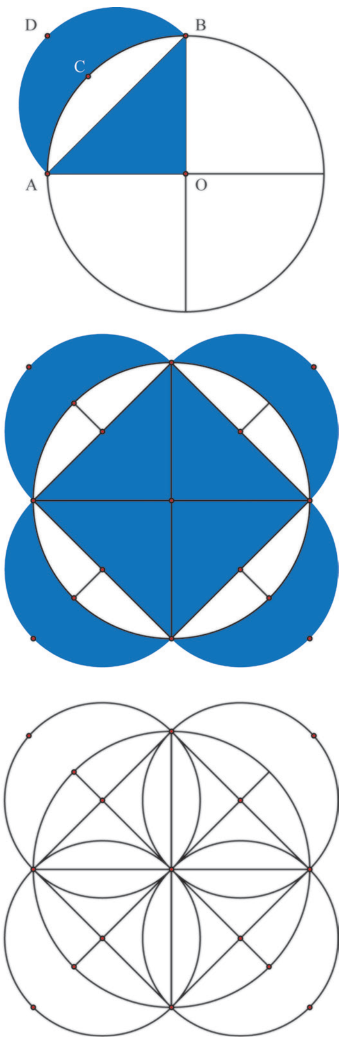
3.2. Hippocrates of Chios and the Lunar Fields

Hippocrates of Chios (circa 430 BC) is best known for three brilliant works such as works on lunar fields (quadrature of lunes), doubling the cube, and his lost work on the *elements of geometry*. The first of these brilliant studies, the calculation of lunar fields, is thought to have emerged in relation to the *squaring of the circle*. His computational work is aimed at calculating areas between arcs of circles and is an early example of investigating areas bounded by more complex curves (Heath 1921a: 183-200; Boyer 1968: 72-74; Aydoğdu 2022: 182-185).

Information about Hippocrates' work on lunar fields comes from two main sources. The first of these sources is the information conveyed by Simplicius, who lived in the 6th century AD, from Eudemos, who was well-known around 320 BC. The second source is the information transferred from the Aristotelian commentator Alexander of Aphrodisias, which is well-known around AD 200 (Boyer 1968: 73). The starting point of the calculation of the areas between the circle arcs is seen as the segments obtained by cutting the circle with a straight line. The starting point of the study is expressed by the following proposition: "Segments of circles with the same ratio of one to the other (cut by the same ratio) are proportional to the squares of their bases" (Heath 1921a: 187). The historical roots of the diagram go back to the 5th century BC, since the diagram given by Eudemos or Alexander is considered to be the earliest examples of curvilinear areas in Greek mathematics and these examples are attributed to Hippocrates of Chios. The diagram must have pioneered other diagrams that were directly its own consequence. Therefore, it is valuable to reveal the diagram's relevance to the elemental pattern concept of mosaic art.

Figure 7
Extending the lunar area account to the circle (drawing by authors).

(a)
(b)
(c)



The graph attributed to Hippocrates of Chios, in its current form, corresponds to a quarter of a circle. In order to obtain a complete view, the relationship of the lunar field with the quarter circle has been reflected to the other quarters of the circle, so that the lunar field calculation has extended to the whole circle. The result obtained reveals *the lunar four leaves form* or *four spindle leaves*, namely *quatrefoil* (Décor II: 42) (Fig. 7). The resulting l*unar four leaves form* can be used repetitively on the horizontal or vertical axis as an elemental pattern. Repeated use reveals another elemental pattern, the form of four lunar circles within a square or *circle of four spindles* (Décor II: 38; Vargas Vázquez 2017: 347-364 figs. 2-3).

3.3. Platonic Solids and Cube Examples

In the *Timaeus* dialogue, Plato tells his thoughts on the creation of the ideally abstract universe and beings as material reality, and the realization of material beings by ideal forms. Firstly, the requirements for the construction of a visible and tangible universe are determined (Plat. Tim. 31 b 4-32 c 4). *The maker of the universe* begins by shaping each of the four elements (fire, air, earth and water) that will make up the universe to be as complete and perfect as possible (Plat. Tim. 53 b 5-6). *The maker* determines three regular polygons as equilateral triangle, square, and pentagon, as the parts that will form the solid bodies to instantiate the elements. The rules that will work in the formation of the solid that will represent the four basic elements and the universe as a whole, are that the number of surfaces joining at each corner of the object is the same and each surface is a regular polygon. As a result of these evaluations, the model chosen for the *fire* is a regular four-faced (tetrahedron), each facet of which is equilateral triangle. For the *air*, the eight-faced (octahedron), each facet of which is equilateral triangle, is chosen. The *water* is represented by the regular twenty-faced (icosihedron), each facet of which is equilateral triangle. The *earth* will be represented by the regular six-faced (hexahedron, namely cube), each facet of which is square. Finally, the *dodecahedron* is chosen to represent the *universe* as a whole, as it is the shape closest to the sphere (Plat. Tim. 55 c 4-6). Thus, Plato determines the ideal forms of the elements, namely the five regular solids, that will bring into being complex, shapeless and formless beings (Aydoğdu 2022: 60-73, 195-198) (Fig. 8).

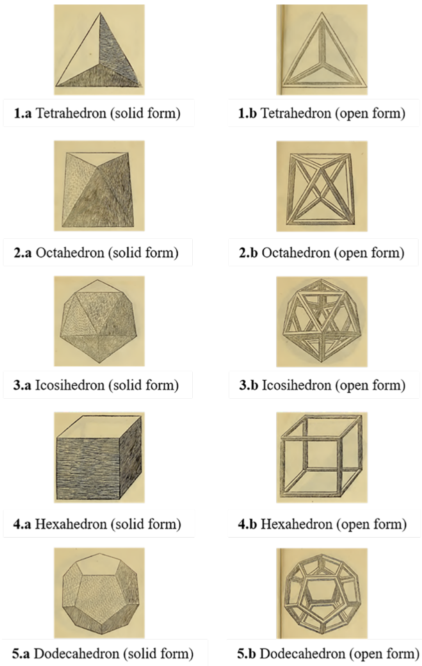


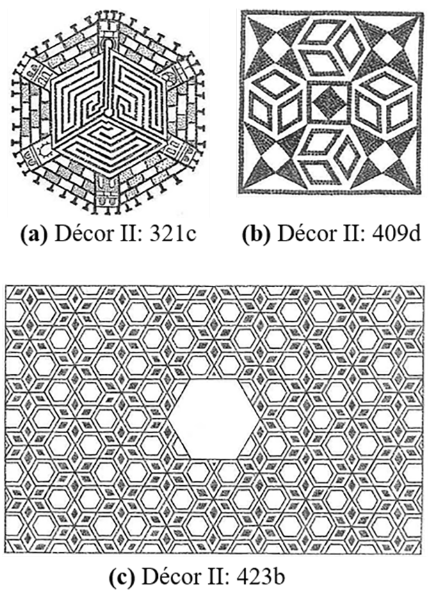
Figure 8
Platonic solids (Pacioli 1509: pls. I, II, VII, VIII, XV, XVI, XXI, XXII, XXVII, XXVIII).

Whether these solids were introduced by Plato or earlier by Pythagoras or the early Pythagoreans is debatable. Based on Plato's *Timaeus*, the earliest authority on the five solids is thought to be Plato, and these solids are named as the Platonic solids (Heath 1921a: 158).

In the 13th book of Euclid, the method of constructing the octahedron, hexahedron (cube), icosihedron, and dodecahedron in the sphere is given, respectively (Eukl. elem. XIII. 6. 14-17). An indefinite dated marginal note (scholium) in Euclid's book states that Pythagoreans knew only three of the five solids as the tetrahedron, hexahedron, and dodecahedron, and that the octahedron and icosihedron were introduced by Plato's friend *Theaetetus* of Athens (414-369 BC) (Boyer 1968: 715). Heron of Alexandria (AD 10-85) has also gave his calculations for Plato's solids in the second book of his *Metrika*, in which he calculated the volumes of solids (Procl. *On Euclid* I; Eukl. elem. XIII. 438-439; Heath 1921a: 220; Boyer 1968: 54, 159).

In Décor II (Décor II: pls. 287b, 294b), there are examples of cube-in-hexagonal representation. An ornament from Gamzigrad has been described as a hexagonal labyrinth of 3 lozenge-shaped sections (Décor II: 128 pl. 321c). An ornament from Ouzouer-sur-Trézée shows the cube form in four hexagons around a square and the use of a four-pointed star in the spaces between the cubes (Décor II: 230-231 pl. 409d). Another example from Ouzouer-sur-Trézée demonstrates the repeated use of the form introduced by Kepler and the hexagonal spaces between them. The star of six lozenges is a structure highlighted by rhombuses. Since only the parts of the star form are colored in Kepler form, the parts that allow the perception of the cubes combine to form hollow hexagons. Thus, instead of three-dimensional cubes, two-dimensional rectangular leaves forming the star are brought to the fore (Décor II: 251 pl. 423b) (Fig. 9).

Figure 9
Examples of the cube forms and their repetitive use (Décor II: pls. 321c, 409d, 423b).



3.4. Semi-regular Solids of Archimedes

The first of the ideas that play a role in the construction of the Platonic solids is the filling of space with the repetitive use, without spacing in between them, of the same solid; and the second thought is that each surface of the solid is made from the same polygon. Platonic solids are expressed as *regular solids* by referring to these properties. Archimedes removes the constraint of making all the surfaces of the solid from the same polygon and allows the surfaces of the solid to be made from different regular polygons. In addition, he changes the condition that there should be no spacing between the repeated use of solids and accepts that there may be spacing. These tolerances are the reasons for the transition from being *regular* to being *semi-regular* (Aydoğdu 2022: 98-101, 199).

Pappus of Alexandria (AD 290-350) states that Archimedes examined 13 semi-regular solids apart from Plato's regular solids (Thompson 1925: 181-188). Solids became the focus of attention again in the Renaissance Period. Luca Pacioli (AD 1445-1514), in the *Divina Proportione* (Divine Ratio) of AD 1509, gives the ratio later known as the golden ratio, polygons, and drawings of solids attributed to Leonardo Da Vinci (Pacioli 1509: pls. I-II, VII, VIII, XV-XVI, XXI-XXII, XXVII, XXVIII; Taylor 1942). Albrecht Dürer (AD 1471-1528), in his book *Underweysung (Unterweisung) der Messung mit dem Zirkel und Richtscheyt*, published in 1525, gives a drawing of seven of the Archimedes solids. Five solids given by Pacioli are also among these drawings (Dürer 1525: 32, 34, 93). Danielle Barbaro (AD 1513-1570), who was also the commentator of Vitruvius, gives drawings of eleven of the Archimedes' solids in his book *La Pratica della perspettiva*, published in 1568 (Barbaro 1568). Johannes Kepler (AD 1571-1630), in his book *Harmonicus Mundi*, published in 1619, shows how solids could be drawn, and gives their names used today. While examining the method of constructing solids, Kepler also considers the ability of regular polygons to span the two-dimensional plane without spacing. Kepler's graphs show that the first designs of three-dimensional objects were developed in the two-dimensional plane, and then were dealt with the third dimension (Keppleri 1619: 61-65 [Libri II. Prop. XXVIII]; Kepleri 1864: 123-126) (Figs. 10-12).

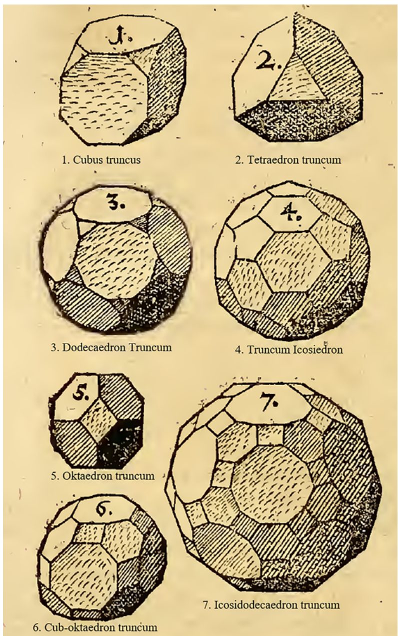


Figure 10
Archimedes' semi-regular solids (solids 1-7) (Keppleri 1619: Libri II. 64).

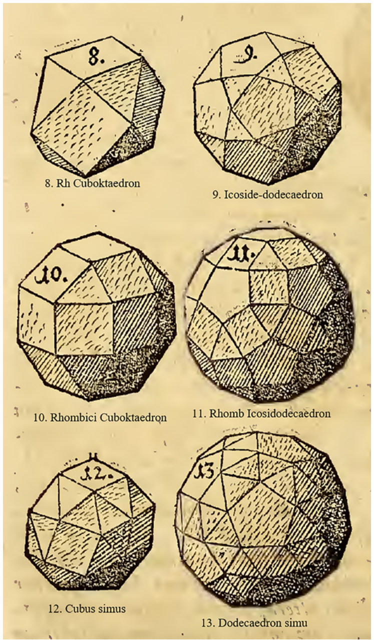


Figure 11
Archimedes' semi-regular solids (solids 8-13) (Keppleri 1619: Libri II. 62).

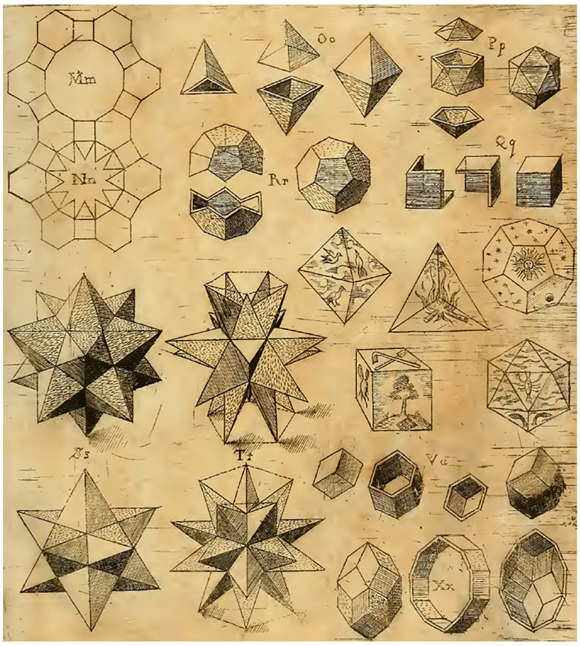


Figure 12
Polyhedrons (Keppleri 1619: Libri II. 58-59).

The mosaics provide examples showing similarities with Archimedes' semi-regular solids. *Décor II* allows to observe three of these examples. The first example in catalog order comes from Brescello (Italy) (Fig. 13a). This example is defined in *Décor II* as: "Centralized pattern, in a circle and around a hexagon, in 2 registers, of 6 squares adjacent to the hexagon, and of 12 squares contiguous to the circle, all of the squares contiguous by a point and forming lozenges and triangles (here outlined)." (Décor II: 189 pl. 375b). The second example of this combined use is the example of Sainte Colombe (France), which is defined as the development of the Brescello example (Décor II: 190 pl. 376a). Enriching the insides of squares and rhombus patterns by painting them in contrasting colors is considered as a state of development (Fig. 13b). Geometrically, its difference from the Brescello example is that the circle surrounding the pattern is expanded a little more, and the empty spaces as a result of the expansion are filled with triangles. It should be noted that the upper base of the *Rhombici cuboctaedron* is also square. The third example comes from the Mataró (Espagne) (Décor II: 190 pl. 376b). The squares have been decorated with guilloche patterns.

Geometrically, its differences from the Brescello example are that the circle surrounding the pattern is further expanded compared to the Sainte Colombe example, and the empty spaces as a result of the expansion are filled with a square organization. The most obvious difference is that the upper base of the *Rhombici cuboctaedron* has been depicted as a hexagon instead of a square (Fig. 13c).

The pattern is geometrically composed of a hexagon in the center, equilateral triangles that combine with the sides of the hexagon, each side of which carries a square, and a circle that carries this geometric structure. In the structure consisting of hexagons, squares and triangles, the spaces between the squares produce motifs in the form of rhombuses. The construction of the pattern begins with a regular hexagon. Setting up a square on each side of the hexagon is the second step. Equilateral triangles, the sides of which are equal to the sides of the square, are added to the squares. Squares are set on the two exposed sides of these equilateral triangles. The ends of the last squares touch the circle surrounding the pattern. The result obtained is the geometric texture of the Brescello sample (Fig. 14).

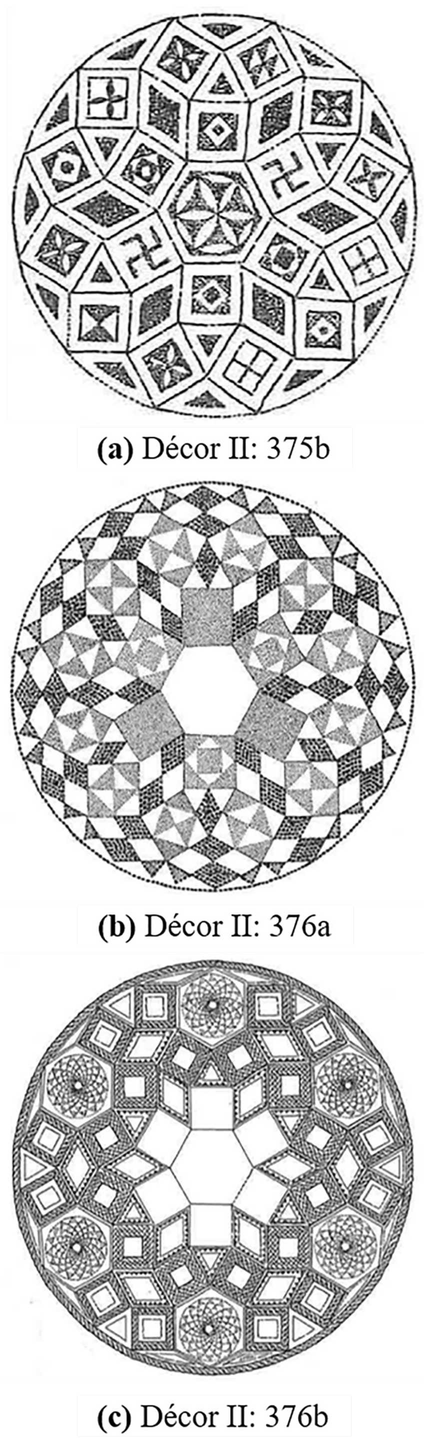


Figure 13
Examples from Brescello, Sainte Colombe, and Mataró (Décor II: 189 pl. 375b, 376a, b).

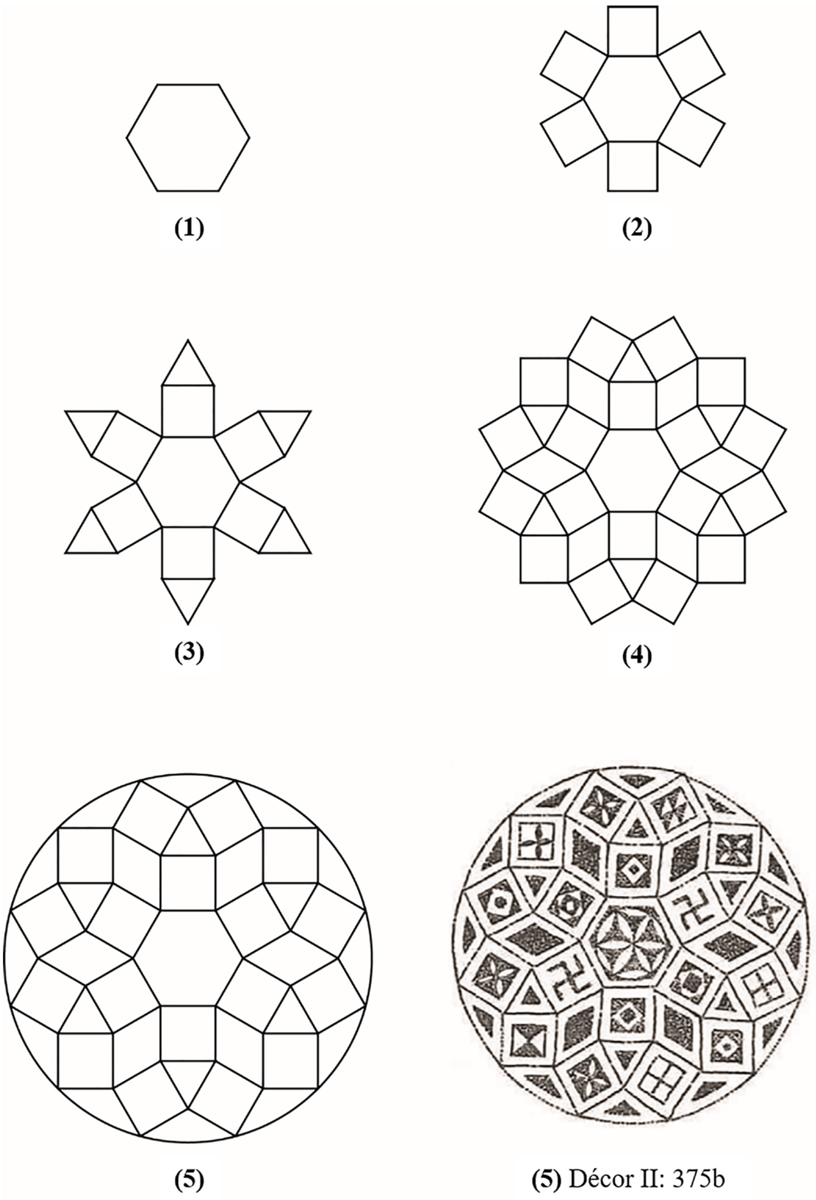
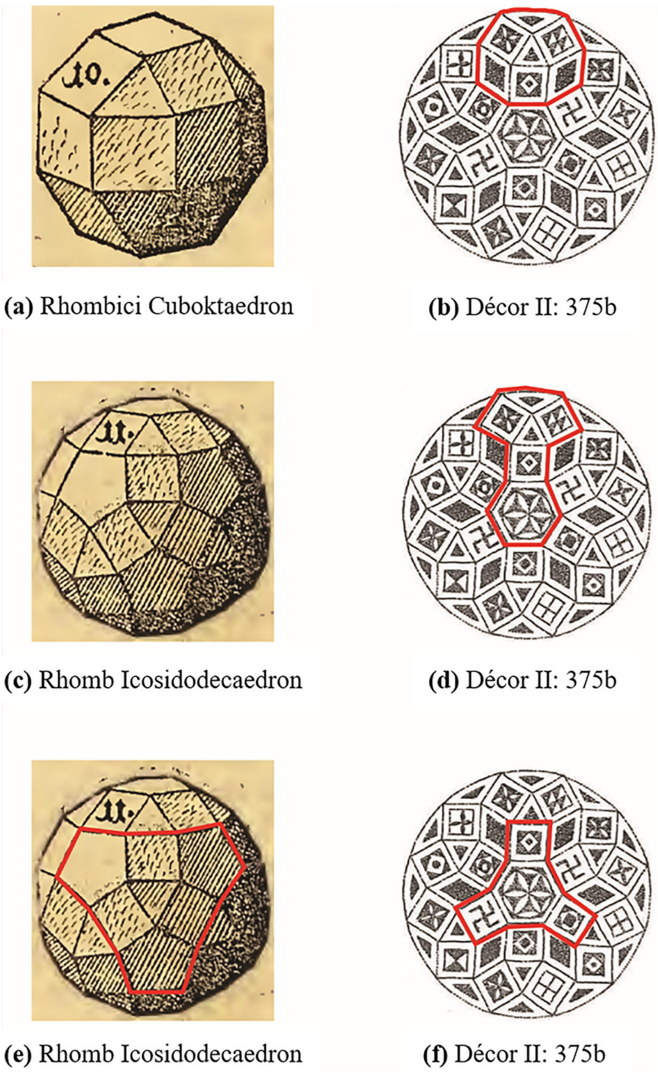


Figure 14
Geometric construction of Brescello example (drawing by authors).

The geometric texture of the Brescello sample seems to be related to the semi-regular solids of Archimedes for two reasons. The first reason is that it shares the same plan with the *Rhombici cuboctaedron*, one of Archimedes' 13 semi-regular solids. The *Rhombici cuboctaedron* has 26 faces consisting of 8 triangles and 18 squares (Keppleri 1619: Liber II fig. 10). Each triangle connects three squares. Its view from six directions is octagonal. Namely, the lateral borders of the solid have an octagonal view from six directions: top, bottom, left, right, opposite and rear. It has a square at its lower base, eight squares at its equator, and the upper base is a square plane. Equatorial squares are connected to the upper base by squares in the midline and by triangles in the lateral lines. Thus, the equatorial squares are connected to the upper base by four triangles at the four corners and four squares at the midline.

In perspective, the part that is distinguished by sharp lines is 3 of the 8 squares of the equator, a square connecting the left and right of these three squares to the upper base, and a triangle between these connected squares. The perspective view, in this context, in Kepler's drawing covers the middle line and the top of the drawing. This perspective view is repeated six times around the hexagon in the center of the motif in the Brescello example. In each iteration, the squares and triangular structure in the perspective image are preserved, while the patterns that adorn the surface of these shapes vary, suggesting that the solid is given a perspective view from a different angle each time (Fig. 15a, b).

Figure 15
Semi-regular solids and
Brescello Example (drawing by authors).



In Kepler's drawing, except the middle one of the three squares on the equator of the solid; the other two squares on the equator, the two connecting squares, and the upper base, a total of five squares are in the form of a rhombus. In the Brescello example, the three squares, the middle square and the two connecting squares are perfect squares, while the squares to the left and right at the equator are rhombic, and the upper base is a rhombus truncated by the circle. The Brescello example more closely models the actual structure of the solid (Fig. 15a, b).

Each triangle connecting the equatorial squares to the upper base is coincident with a square on each of its three sides. This connection form is also seen in the *Rhomb Icosidodecaedron*. In this solid, in front view, a pentagon is connected to the surrounding decagon by a similar connecting line. In the midline, the edge of the decagon extends upwards with a square. A triangle that accepts the upper edge of the square as the base is connected and the remaining two sides of this triangle unite with a square to reach the pentagon at the top. From the decagon upwards, the square-triangle-two square-upper base connection is the same as the Brescello example (Fig. 15c, d).

The exit from a triangle located at the equator of the *Rhomb Icosidodecaedron* takes place with squares added to its sides. In the Brescello example, there is a hexagon at the equator, and the exit from the hexagon takes place by adding squares to the sides of the hexagon. According to the type of solid, the number of upper bases targeted by the exit from the equator varies according to the number of sides of the geometric shape that is the exit point.

In the *Rhomb Icosidodecaedron* structure, since the exit from the center starts from the triangle, the triangular region with concave sides and truncated ends, which shaped by the connection lines of the exit directions from the equator and accepting the target geometric shape as the end region is in three direction. In the Brescello example (Blake 1930: 113 pl. 41.4), since the exit from the center starts from the hexagon, this region is in six directions but symmetrically in three directions. The *Rhomb Icosidodecaedron* and the Brescello example are therefore structurally similar (Fig. 15e, f). According to these evaluations, the Brescello sample combines two of the Archimedes solids in the same plane.

The Brescello example can be read as solid views repeated around the central hexagon. Couldn't another polygon be chosen instead of the central hexagon? Of course it could be chosen. Each of these choices can be formulated as "central polygon-square-triangle-squares". We can see some of the examples that may arise depending on the selection of polygons in Kepler's drawings. For example, in the arrangement around the central triangle, solid appearances lead to pentagonal spaces (Fig. 15c). In the case of arranging around the central square, it can be predicted that the lozenge-shaped forms that give the perspective view of the solid shape will be distorted. The case of using a central pentagon contains difficulties specific to the regular pentagon (Fig. 15c). The regular heptagon falls outside the polygons that can be constructed with an measureless ruler and compass. In the case of using a central octagon, it can be predicted that the lozenge-shaped forms, which give three-dimensional effect, will be compressed and the organization will be difficult. The case of using a central nonagon is also the same as using a heptagon. Given these possible options, the Brescello example represents the option of arranging around a regular hexagon as a reasonable option.

Considering the geometric plan of the Brescello example, the Sainte-Colombe and Mataro examples are not faithful to this plan. In order to obtain variants with faithfully to the plan, it is expected that the circle surrounding the model is kept constant and the details are modified. However, this has not been done. The plan itself has been modified. These differences show that what created the Brescello example was not a standard geometric plan describing the example, but a simple, easily modified plan.

The geometric plan is about dividing the circumference of the circle first by 6, then by 12, and then by 24. However, the organization from the central hexagon to reach the upper base of the *Rhombici cuboctaedron* is not a natural element of the plan. To describe this organization on surfaces of varying sizes, it is necessary to determine the ratio between the outer circle and the central hexagon. Reaching the outer circle from the center of the central hexagon and reaching the triangle and square arrangements in the intermediate layers requires complex calculations. Of course, these operations can be performed by transferring lengths instead of complex calculation. This practical solution works by adding a square to a hexagon, then adding a triangle. The geometric plan points to the Archimedes solids as a solution.

3.5. Hexagons of Pappus of Alexandria

Pappus of Alexandria (AD 284-305) is known circa AD 320. Pappus' historical role is to reconsider and give a summary of the mathematical knowledge that reached up to his time, to put an end to the long stagnation in the theory of Greek mathematics, and to bring about a revival (Heath 1921b: 355; Boyer 1968: 196-215, 686). In book eight of his work named as *Synagoge*, Pappus tackles an interesting problem on drawing seven identical hexagons in a circle. One of the hexagons will be at the center of the circle and the others will be around the center of the circle. While one side of the hexagons drawn around is the same as one of the sides of the central hexagon, the side opposite the same side will cut the circumference of the circle as a chord. Along with solving the problem, Pappus also gives a drawing that graphically describes the problem. Therefore, this pattern can be named as *Pappus form* (Pappus. Synagoge VIII. c. 23. Prop. 19; Heath 1921b: 438-439; Aydoğdu 2022: 185-187) (Fig. 16).

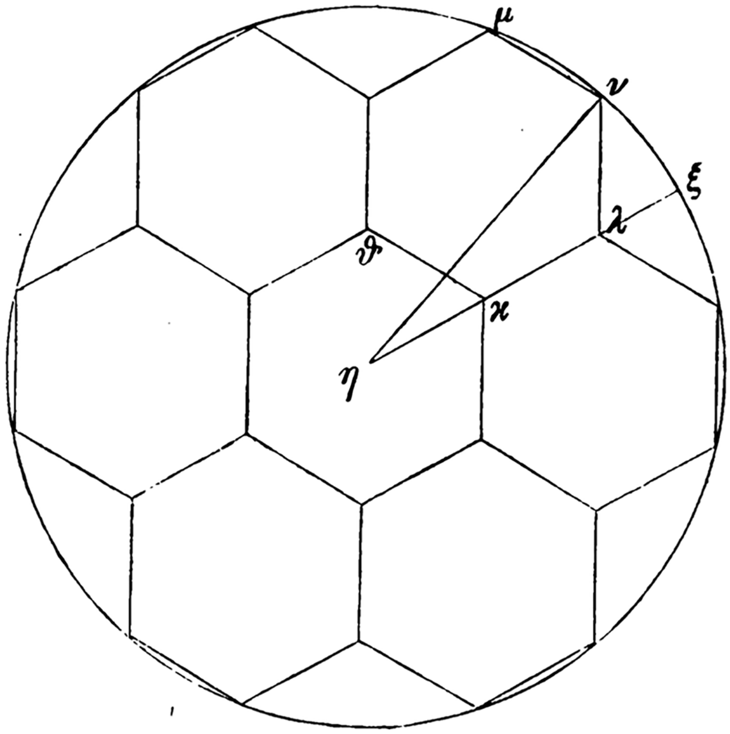


Figure 16
Pappus' hexagons
(Pappus. Synagog VIII. c. 23. Prop. 19).

In *Décor II*, patterns similar to the graphic presented by Pappus are found in the examples of Saint-Bertrand-de-Comminges (France), Nîmes (France), and Vaison la Romaine (France) from the Gaul region (Décor II: 239 pl. 415a-d) (Fig. 17). These patterns, as an extension of the regular hexagon (honey comb) form (Décor I: 321 pl. 204a), are defined in the category of triaxial patterns as "in a circle and around a hexagon, 6 adjacent hexagons and 6 truncated hexagons (as triangles with concave base) on the periphery" (Décor II: 239 pl. 415a).

When Pappus' design of hexagons around a hexagon is considered as a pattern, it is naturally expected that the pattern will lead to different variations. In the examples of Fliessem (Germany), Autun (France) and Vienne (France), quadrilaterals have been arranged around the hexagon (Décor II: 243 pl. 418a-c). In the examples of Nîmes (France), Sens (France), and Carthage (Tunisia), concave hexagons are arranged around the concave hexagon and can also be perceived as lunar six leaves around lunar six leaves (Décor II: 240 pl. 416a-c) (Fig. 18).

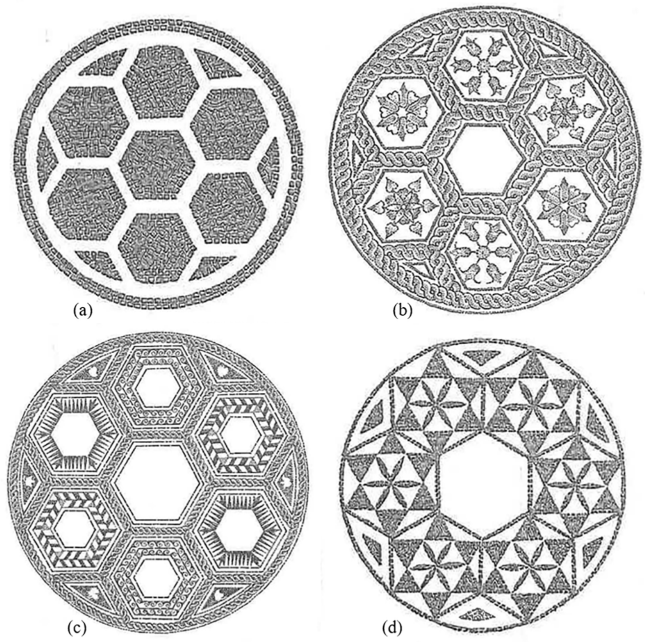


Figure 17
Use of Pappus' hexagons as decorations (Décor II: pl. 415a-d).

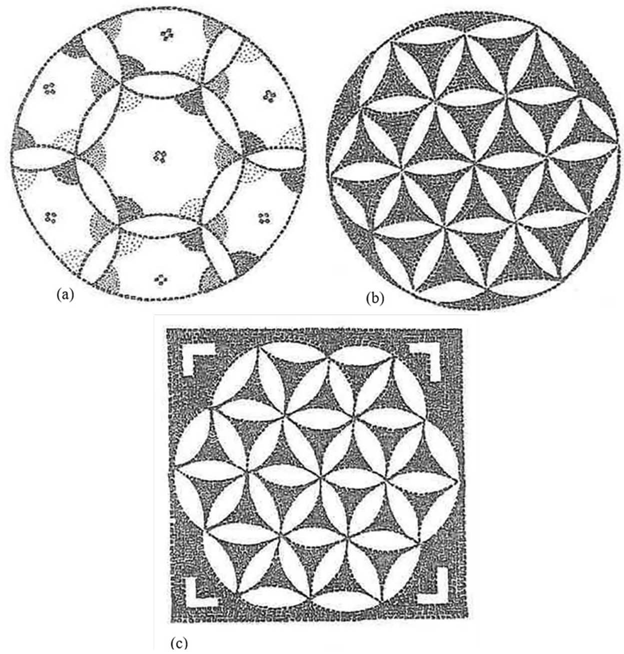


Figure 18
Concave hexagonal applications (Décor II: 240 pl. 416a-c).

Planeterium Hause mosaic (Spain, Italica) bears hexagonal ornaments. The mosaic is dated to the middle of the second century AD (Penedo 1993: 73-78 kat. no: 13-14 pls. VIII-IX; López Monteagudo - Neira 2010: 17-189 fig. 205). This mosaic is contemporary with Ptolemy's Almagest. It is thought that Pappus was alive circa AD 284-305 and was known around AD 320, and his work called *Synagoge* belongs to this period. In this case, there is a time gap of about a century and a half between the Planetarium House mosaic and Pappus' graphic. The same is true for Peacock mosaic from the Gaul region (Lassus 1971: 45-72 fig. 48; Décor II: pl. 415c) (Fig. 17c). Another example, attributed to the mid-1st century AD from Pompeii (De Vos 1991: 36-60 fig. 27), shows that the *Pappus form* existed in mosaic art two and a half centuries before Pappus.

4. Discussion

The determination of mathematical theorems in mosaic art is an area of study that can establish a direct relationship between mathematics and mosaic art. If a mathematician's drawing is depicted in a mosaic, it raises the question of whether there was contact between mosaicists and mathematicians, their works, or people who were knowledgeable about these works. However, the identification of a mathematician's drawing in a mosaic is not enough to prove the existence of such a contact. Even if the existence of direct or indirect contact was guaranteed, the geometric ornament in the mosaic would remain a crucial factor in proving the exchange of information or graphics. Therefore, while identification is important, it is not sufficient for the determination. It is also important to acknowledge the difficulties that may arise during identification and to discuss the arguments leading to the identification. To provide a healthy framework for discussion, this study explores topics such as Greek mathematics or scientific geometry, practical geometry, the source of the mosaicist's knowledge, difficulties in constructing forms and their consequences, representation of solids on the plane, identification of two-dimensional form, and their results, all presented under separate headings.

4.1. Greek Mathematics or Scientific Geometry and Practical Geometry

It is thought that geometry emerged and developed long before the emergence of writing (Boyer 1968: 7). The Nile Valley and Mesopotamia provide examples of primitive writing from before the end of the 4th millennium BC. But few of them have been associated with the theme of mathematics (Boyer 1968: 1-10). In Egypt and Mesopotamia, mathematics is at the level of the craft. Measurements and calculations made to meet the needs of daily life constitute the content of mathematics. Formulas had undoubtedly established in problem solutions, but the operations and calculations could not turn into theory (Boyer 1968: 12). The fact that mathematics, which reached a certain level of development in Egypt and Mesopotamia, became a science discipline, is the feature that characterizes the Greek mathematics (Heath 1921a: 65-66; Boyer 1968: 48-67). It is thought that geometry, which started as a craft practice in this line of development, turned into scientific geometry. The period of Greek mathematics as a whole covers the 12 centuries, which historically fall between the 6th century BC and the 6th century AD. The 26 cities, which are mentioned together with the philosophers and scientists who revealed the Greek mathematics, determine the Mediterranean environment and the southwestern shores of the Black Sea as the geography where mathematics spread (Boyer 1968: 196). We can see the point that Greek mathematics reached in three centuries by looking at the the *Elements* of Eucleides (ca. 300 BC). Elements consists of 13 books. The *Elements* contain almost all basic mathematical knowledge such as higher arithmetic, number theory, synthetic geometry (point, line, plane, circle, sphere) and algebra (Heath 1921a: 354-360). The forms used in geometric mosaics correspond to a very small set of this content.

Greek geometry or scientific geometry treats forms based on logical inference, and defines the mathematical properties of forms, their own properties and their relations with other forms. The expression "practical geometry" is used to express functional and easily applicable techniques and knowledge in craft practice. It is known that the information presented in prescription form is widely used, especially in the Roman world. Vitruvius does this when he recommends using a 3-4-5 foot long triangle to make right angles at the corners (Vitr. IX, 6-8). The source of this practical knowledge is mathematical knowledge of the nature of the right triangle.

Practical knowledge of geometric forms, no matter where or how it comes from, stems from the nature of those forms. For example, using nails and string to draw circles, it is an example of practical geometry that reflects the relationship of center, radius and perimeter, and this is an ancient knowledge. To obtain an equilateral triangle, it is sufficient to join three laths of equal length at their end points: This operation is the result of equilateral and triangular properties. Similarly, a square can be obtained. Equilateral triangles can be combined to form a hexagon. These examples can be multiplied for physical construction. When forms and their configurations within a certain scale need to be planned for physical construction (for example on a piece of paper), "practical knowledge" becomes knowledge about the maneuvers of the ruler and compass.

Greek mathematics is concerned not only with the abstract study of geometry, but with its practice. The search for polygons that can only be drawn with a measureless ruler and a compass (or string and nail) is concerned with determining the possibilities of realizing abstract forms with very simple tools and simple methods. For example, polygon drawing is obtained by dividing the circumference of the circle at equal intervals. That is, the circle is divided into equal segments by rays extending outward from the center. In abstract analyzes of scientific geometry, the ruler and compass maneuvers come from theory. These maneuvers are extremely practical for many polygons. From this point of view, it can be thought that "practical geometry" should not be treated as a completely separate category from the "scientific geometry" context. The content, scope and limits of the set of "practical geometry" or "practical knowledge" can become more understandable by examining the works of mosaicists on the axis of scientific geometry.

4.2. Source of Geometry Knowledge of Mosaicists

In the context of "practical geometry" and "scientific geometry", the issue of how geometric mosaics were made is controversial. At the center of the discussions, we can see that the relationship between the mosaicist and scientific geometry is questioned. The question to be answered is: Where did the mosaicists' knowledge of geometry come from?

While seeking the answer to this question, the social status of mosaicists has also been taken into account. Because the source of knowledge of geometry may be the academic environment or mosaic workshops (master-apprentice relationship). On the other hand, it has been thought that there were two separate processes in the creation of geometric mosaics, such as the design process and the construction process (Daszewski - Michaelides 1989: 14; Duran-Kremer 2012: 59-70). In this case, it can be thought that the actors of the design and workmanship processes were different people and that these people did not have the same opportunities to access geometry information. The design demands of the customers and their guiding effects on the process are also open to discussion. Relationships and interaction between different categories of people that interfere with the labor process interaction emerge as key considerations. It is thought that mosaic worker's access to geometry knowledge through academia was not so possible. Access to practical geometry knowledge through the master-apprentice relationship is seen as a more reasonable option (Balmelle - Darmon 1986: 247; Bruneau 1987: 154).

Geometric forms used in the decoration of geometric mosaics belong to the geometry repertoire. There are successful representation of geometric forms and complex configurations in mosaics. It is known that rulers and compasses (or nail and string) were used in the making of mosaics (Vitr. VII.I.3). Based on these reasons, focusing on the possibility that mosaicists knew geometry and investigating the content, limits and possible sources of their knowledge can contribute to the understanding and evaluation of geometric mosaics. This approach does not lead to an underestimation of the originality of mosaicist's designs, their creativity, their artistry and their ability to solve practical problems they encounter. On the contrary, it can serve to develop a healthier understanding of these issues. At this point, it would be useful to clarify the relationship between scientific geometry and practical geometry as follows.

4.3. Difficult or Impossible Forms, Practical Difficulties and Degenerate Forms

Scientific geometry says that all regular polygons are theoritically possible in space (or in plane). Difficulties are faced when it comes to drawing or physically constructing polygons. Straight line and circle are embodied as ruler and compass without measure. Maneuvers with these instruments are under the control of theory. Therefore, "measureless ruler and compass" are extensions of mathematical thought to the physical world. It starts with an equilateral triangle and continues with a square. As for the five-sided regular polygon (pentagon), it is seen that it is not so easy, but it is succeeded. The construction of the regular hexagon is quite practical. As for the seven-sided polygon (heptagon), there is a silence. Octagon is easy. There is silence again in the nine-sided polygon (nonagon). These silences continue until the 19th century AD: It is proved that the regular heptagon and nonagon, which are possible in theory, cannot be constructed with measureless ruler and compass (Wantzel 1837: 366-372; Cajori 1918: 339-347; Tavares - Freitas 2018: 187-194).

The regular pentagon is included in the repertoire of polygons that can be drawn with measureless ruler and compass, but it is difficult to consume, so it is not that practical. Therefore, we should never miss the opportunity to be surprised if we come across the regular pentagon form in mosaic art. As for heptagon and nonagon, there is no place for them in the repertoire, within the possibilities of drawing with measureless ruler and compass. If we come across a regular heptagon or nonagon in mosaic art, we should be doubly surprised: First, because this work cannot be accomplished with measureless ruler and compass; secondly, because they somehow managed to do this job.

Couldn't the mosaicists have built the heptagon or nonagon in some other practical way? Of course, they could. They could join seven or nine slats of equal length at the ends. They could also adjust the equality of the angles with a circle. It might not be perfect, but it would look aesthetic. So, apart from the tools and methods used in the construction of other polygons, and therefore with a practice that is detached from the theoretical integrity, we can expect these polygons to be built and include them in the repertoire.

We can summarize the conclusion to be reached from this short discussion as follows: Pentagon, heptagon and nonagon are theoretically possible; but, in the context of construction with measureless ruler and compass, regular pentagon is difficult to construct, regular heptagon or nonagon is impossible.

Difficulty or impossibility in the construction of geometric forms may arise directly from the nature of the polygon (pentagon, heptagon, nonagon), as well as from the approach chosen for repetitive use, as in the *square grid-element pattern* approach. Consider the regular hexagon form. It is quite easy to draw with a ruler and compass. However, when trying to create a honeycomb texture, it is faced with the fact that the regular hexagon cannot be repeated in the square grid structure. Because the height and width of a regular hexagon are not equal. This problem can be solved by using equilateral triangle grid instead of square grid (Décor I: 321 pl. 204e). The resulting graphic constitutes an open infrastructure for creative designs (Fig. 9c). If the *square grid* cannot be dispensed with, the "regularity" requirement should be abandoned. That is, the regular hexagon must be squeezed into the square. When this is done, the result obtained is again a hexagon, but this time an irregular hexagon (Décor I: 321 pl. 204f). It can be called a "degenerate hexagon" in reference to its rule-related deformation. With the deformation of the regular hexagon, the equilateral triangles dividing the hexagon are also deformed and become "degenerate triangles". Within the square grid, degenerate triangles repeated as parts of a hexagon naturally give rise to designs such as a zig-zag pattern repeated throughout the strip, a lozenge pattern repeated in one direction, or repeated cubes (Décor I: 321 pl. 204d; Aydoğdu 2022: 195-198 figs. 105-112).

4.4. Representation of Solids in Plane and Identification Problem

In two-dimensional representation, two-dimensional geometric forms are depicted as they are. A perspective view of the form is used to represent a three-dimensional form on the plane (in two dimensions). The form is depicted as it appears, not as it is. When the perspective changes, the image also changes. In freehand drawing, a three-dimensional form can be depicted from infinitely different angles. When the drawing possibilities are restricted to regular geometric forms, the depiction options are reduced.

Let's take the cube form. Different methods can be followed for the cube depiction. Adding lozenges to the two edges of a square that connects to the same corner is a method. In this method, the boundaries of the design form an non-regular hexagon (Fig. 8.4.a). Another method is to divide the regular hexagon into three lozenges (Fig. 9a-c, Fig. 12.Qq-Vv). This method makes places where the regular hexagon can be represented also suitable for cube design. The use of "degenerate hexagon" instead of regular hexagon in the cube design is also a practical option and the resulting cube forms can be seen as "degenerate cubes". Similar to the depiction of the squares forming the cube surface as diamonds in perspective view, the polygons forming the surface of the solid in other solids are deformed in accordance with the perspective (Figs. 10-12).

If the closed and open forms of the cube (Fig. 8.4.a, b) are compared, from the observer's point of view (top-right), the differences between the depiction of the visible surfaces and the depiction of the skeleton of the form can be seen. In closed form, surfaces have been distinguished by the difference in color and texture. In the open form, the skeleton of the cube has been drawn. The surfaces between the square, which is understood to be the front face, and the square, which is understood to be the back face, are in the shape of a lozenge. In open form, the surfaces have not been distinguished by color and texture difference. If the thick lines forming the skeleton were not textured and the light-shadow effect was not emphasized, if the lines were linear, since there is no significant size difference between the squares (closeness-distance), we wouldn't be able to distinguish the front and the back. In other words, our perception would be confused as if we were seeing the form from two different angles (left-bottom and right-upper). It could also be argued that the drawing was not three-dimensional but two-dimensional.

In the mosaics, we can see that the closed form of the cube is depicted instead of the open form, and the boundaries of the form consist of six sides. The visible surfaces of the cube give us six edges, naturally also the method for its construction. In the example of the cube form, we can distinguish whether the design is two-dimensional or three-dimensional, by distinguishing the surfaces by color, texture or decoration in addition to the geometric structure that embodies the perspective view.

In this study, the similarities and differences between Archimedes's solids (*Rhombici cuboctaedron* and *Rhomb Icosidodecaedron*) and the mosaic ornaments from Brescello (Italy), Sainte Colombe (France) and Mataró (Espagne) (Fig. 13a-c) have been evaluated. In the evaluations about ornaments, the solids have been taken as reference because they are ideal forms. It is worth noting that the differences between the ornament and the ideal form do not attribute a defect or deficiency to the work, and enable a healthier evaluation of the differences.

The common feature of these mosaics geometrically is that they have a design consisting of hexagon-square-triangle-square layers from the center outwards. The configuration repeated around the central hexagon in this design is the same as seen in the *Rhombici cuboctaedron* (Fig. 15a). The organization seen in the *Rhomb Icosidodecaedron* and the organization seen in the geometric design of these mosaics are similar in terms of the arrangement of the forms from the center to the outward (central form-square-triangle-squares) (Fig. 15b, c). Based on these determinations, in terms of similarity, a relationship has been established between the mosaic ornaments in question and the semi-regular solids of Archimedes.

Whether the geometric organization represented in these mosaics represents a two-dimensional design or a three-dimensional object from different angles is open to debate. The design has been shown step by step in the geometric plan of the Brescello mosaic (Fig. 14). When there is only a central hexagon, when squares are added to the hexagon, or when it comes to the hexagon-square-triangle structure, the design is unquestionably two-dimensional (Fig. 14.1-3). When the *hexagon-square-triangle-squares* structure is reached, the discussion can begin. Because the lozenges, which give the perspective effect, are changing the situation by emphasizing three dimensions (Fig. 14.4). The two-dimensional effect is getting a little stronger when a circle is drawn around the design (Fig. 14.5). When the surfaces are distinguished by decorations, the repetitions of the *Rhombici cuboctaedron* form from the perspective view become evident (Fig. 14.6). From another point of view, since the three-dimensional appearance repetitions occur around the hexagon, the design can also be viewed as a mixed structure that processes two and three dimensions together.

The reason why the discussion about being two or three-dimensional, which is not a problem in the identification of cube design, is a problem in Archimedes solids, is that the design in these mosaics meets the *Rhombici coboctaedron* form in accordance with the perspective, but on a partial appearance scale instead of the whole view. Therefore, what is done in this study is an identification based on the partial appearance of the solid.

For other solids, as in the cube depiction, it can be expected that the closed form to be depicted and the boundaries of the form consist of the visible edges of the solid. Difficulties due to the polygon forming the surfaces of the solid (for example, the regular pentagon), difficulties due to the number of visible edges of the closed form (10 edges for the *Rhombici cuboctaedron*), difficulties due to the organization and deformation of the surfaces of the solid in accordance with the perspective view can be estimated. In addition, it can be thought that the solid cannot be represented as a whole due to the restrictive effect of the design with regular geometric forms instead of free drawing. An approach such as preserving the geometric form organization of the solid (square-triangle-squares) but choosing an easy polygon instead of a difficult to construct, thus degenerating the solid is also included in the possibilities.

4.5. The *Pappus Form*, Adequacy and Inadequacy of Identification, Contributions

Pappus lived around the rule of Diocletianus (AD 284-305). He was a well-known person circa AD 320. Pappus has done more than contribute to the history of mathematics by giving a list of Greek mathematicians and mentioning their work. His work is more like a handbook for Greek geometry than an encyclopedia (Heath 1921b: 355-358; Boyer 1968: 196-215, 686). Pappus gives a summary of the information that has reached his time. This is an important part of his historical role. Was the problem and drawing of hexagons in a circle first posed by Pappus or did he inherit it? The exact answer to this question is unknown. The limited resources on mathematical manuscripts do not allow to determine when this graph first appeared in mathematics. The mosaics (from Italica, Gaul, Pompeii) present examples of the form of hexagons in a circle long before Pappus' time. Mosaicists who lived before Pappus's time cannot have learned this form from Pappus. This situation is consistent with Pappus' historical role. The mosaics document that the form was in use long before Pappus. In this respect, the identification of the Pappus form on mosaic artifacts is important for the history of mathematics.

An ornament similar to the graphic given by Pappus of Alexandria is seen on a polychrome mosaic from the Planetarium House in Italica (Spain). Considering that Spanish mosaics are associated with Alexandria (Dunbabin 1999: 149-150), identification of a mathematical graphic on a mosaic work may be of value as a finding in inferences about the relationships between artifacts from different cities.

Patterns similar to the *Pappus form* have been defined in the category of triaxial patterns (Décor I: 12, Décor II: 239) as an extension of the regular hexagon (honey comb) form (Décor I: 321 pl. 204a). Similar to the "two-dimensional versus three-dimensional" discussion, a discussion can also be made for the *Pappus form*: Did the mosaicist want to draw hexagons in a circle, or did he want to draw a circle around the hexagons? From this point of view, it can be thought that the mosaicist's approach to the subject, unlike Pappus' approach, may be as simple as drawing a circle around the hexagons instead of drawing hexagons in the circle or placing the honeycomb in the circle. So, it can be claimed that there is no relationship between the *Pappus form* and the form in the mosaic works. Or, it can be claimed that the mosaicists and mathematicians came up with the same form independently and unaware of each other, without any contact between them. It can also be thought that a mathematician or Pappus saw the form in the mosaic and dealt with it as a mathematical problem.

The essence of the problem is to draw a central hexagon and hexagons around the central hexagon in a square area. Geometrically, the circle surrounding the hexagons states that the drawing will be performed in a square shaped area. Therefore, it is necessary to establish a proportion between the square frame and the size of the hexagon to be repeated. Since the dimensions of the area allocated to the drawing may change, it is important that the drawing is repeatable at the desired scale. If there was such a skill for applying the honeycomb form consisting of regular hexagons at different scales, this situation shows that mosaicists had already faced and solved the problem expressed by Pappus, whether or not the circle is drawn around the hexagons. Analysis of the geometric plans of the mosaics bearing these forms, together with the actual measurements, can illuminate how the mosaicists solved the problem and deepen the understanding of the content of practical knowledge. A connection can be established between the *Pappus form* and its counterparts in mosaic art, in terms of the fact that they are essentially related to the same problem and result in the same graphic, apart from their similarities only in shape.

5. Conclusion

The analysis of the swastika, meander, and spiral forms has been conducted through the measurement of the perimeter of a polygon using only a ruler and compass, without the use of arithmetic operations. The resulting graphs provide a theoretical explanation for the design of the swastika and meander forms, as well as clarification on Archimedes' spiral design, highlighting the mathematically natural, necessary, and inevitable relationship between these forms. This analysis establishes a mathematical infrastructure and a theoretical framework for proposed algorithms aimed at determining the grid structure necessary to produce the ornament and the repetition order in grid cells for ornamental surfaces created through the repetitive use of meander and swastika forms.

The *Décor* catalogue (Décor I-II) provides a collection of geometric ornaments dating back to the 1st century AD through the 6th century AD. It is observed that polygons like triangles, squares, rectangles, hexagons, and octagons were frequently used in the repertoire of geometric ornaments, but regular pentagon and regular heptagon forms did not receive the same level of attention. The forms such as regular pentagon and regular heptagon are theoretically possible, but in the context of construction with measureless ruler and compass, regular pentagon is difficult to construct, regular heptagon is impossible. The scarcity of these forms in ornaments is attributed to the limitations posed by the nature of these geometric forms. In lieu of regular and semi-regular solids with regular pentagons in their composition, combined forms that imitate solids, as well as easy-to-construct degenerate forms, were more commonly used, which can be interpreted as a consequence of these difficulties.

The study of original and degenerate forms has shed light on the use of mathematical objects, such as equilateral triangles, regular hexagons, and cubes, as they appear in ornaments, as well as the use of degenerate forms for practical convenience. Additionally, insights have been gained regarding the use of square grids and degenerate grids. Geometric mosaics hold a significant value in the history of mathematics, as they provide valuable documentation that contributes to the evaluation of mathematical graphics related to the forms and patterns they carry as ornaments. The *Pappus form* is a prime example of this. The identification of geometric forms from Greek mathematics in mosaic works, such as Hippocrates of Chios' diagram of lunar areas, Plato's regular solids, Archimedes' semi-regular solids, and Pappus' hexagons, adds new dimensions to studies and evaluations of mosaic art. However, the identification alone is not sufficient to prove the existence of a relationship between the mathematician and the mosaicist. The identification and related discussions provide insights into the relationship between geometry and geometric ornament, which is a necessary element for the existence of such a relationship, particularly in instances where the relationship between mathematician and mosaicist is subject to controversy.

Geometric models used in creating ornamental forms allow for a clear distinction and objective comparison between original and variant forms, facilitate the evaluation of the connection between planning and reality, and enable the reproduction of damaged mosaic ornamentation. The general framework of mathematical analysis introduces the concepts of geometric reintegration and analytical restoration, which are relevant in the context of conservation studies. The sample analyses presented in this study could serve as an analysis model for comprehensive catalog analyses that extend to the limits of the Greek and Roman mosaic art decor repertoire. Analyzing the geometric forms and configurations that decorate mosaics with scientific geometry can lead to a better understanding of these forms and contribute to the evaluation of findings related to the mosaic-making process, such as traces, drawings, and techniques.

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