

Miao-Tam Equation and Ricci Solitons on Three-Dimensional Trans-Sasakian Generalized Sasakian Space-Forms

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Abstract

The aim of the present article is to characterize some properties of the Miao-Tam equation on three-dimensional generalized Sasakian space-forms with trans-Sasakian structures. It has been proved that in such space-forms if the Miao-Tam equation admits non-trivial solution, then the metric of the space form must be a gradient Ricci soliton. We have derived that a non-trivial solution of the Fischer-Marsden equation does not exist on the said space-forms. We have also investigated certain features of Ricci solitons and gradient Ricci solitons. At the end of the article, we construct an example to verify the obtained results

1. Introduction

Miao-Tam equation on f -cosymplectic manifolds was investigated by X. Chen [1]. He proved that under certain restrictions such a manifold is either locally the product of a Kähler manifold and an interval or a unit circle, or, the manifold is of constant scalar curvature. He also established that if the manifold is connected and satisfies the Miao-Tam equation, then the manifold is Einstein under certain conditions. Since an Einstein manifold or a manifold of constant curvature is model of some interesting physical systems, geometers are naturally motivated to find the conditions under which a manifold will be Einstein or, a manifold of constant scalar curvature. To this end we study Miao-Tam equation on generalized Sasakian space-forms with trans-Sasakian structure and established that if a generalized Sasakian space-form with trans-Sasakian structure admits a non-trivial solution of the Miao-Tam equation, then the scalar curvature is constant and the manifold is Einstein or the structure is β -Kenmotsu. Several researchers [2–10] have investigated the Miao-Tam equation for some classes of contact manifolds.

Let $(M^n, g), n > 2$ be a compact orientable Riemannian manifold with a smooth boundary ∂M and $\lambda : M^n \rightarrow \mathbb{R}$ be a smooth function on the manifold. Then the Miao-Tam equation on M^n is given by

$$\text{Hess}\lambda = (\Delta\lambda)g + \lambda S + g, \quad (1.1)$$

on M and $\lambda = 0$ on ∂M , Hess , Δ being respectively the Hessian operator and Laplacian with respect to the metric g . S indicates the Ricci curvature and λ indicates the potential function. The metrics satisfying the equation (1.1) are known as Miao-Tam critical metrics [11]. A sub-class of the Miao-Tam equation is the Fischer-Marsden equation which is given by

$$\text{Hess}\lambda = (\Delta\lambda)g + \lambda S.$$

The Fischer-Marsden equation (FME, in short) was constructed by A.E. Fischer and J. Marsden in [12]. The authors [12] in their paper conjectured that a compact Riemannian manifold that admits a non-trivial solution of the FME is necessarily Einstein. This statement is known as Fischer-Marsden conjecture. Later Kobayashi [13] pointed out that the said conjecture is not true in general. They are valid only in some special cases. After that a huge number of works has been done to analyze Fischer-Marsden conjecture on Riemannian manifolds admitting several structures.

R. S. Hamilton [14] introduced the notion of the Ricci flow in 1988. On a Riemannian or semi-Riemannian manifold,

$$\frac{\partial g}{\partial t} + 2S = 0$$

denotes the Ricci flow equation. A self-similar solution of the above equation is called the Ricci soliton and the soliton equation is given by

$$\mathcal{L}_V g + 2S + 2\psi g = 0, \quad (1.2)$$

\mathcal{L} denotes the Lie-derivative operator. Here V is called the potential vector field and ψ is the soliton constant. If the sign of ψ is positive then the soliton is known as expanding and for the cases where ψ is zero or negative, the soliton is steady or shrinking, respectively. For details about Ricci solitons see the articles [15–18]. If the potential vector field V is the gradient of a smooth function ζ , then it is called the gradient Ricci soliton. Thus the gradient Ricci soliton is given by

$$\text{Hess}(\zeta) + S + \psi g = 0, \quad (1.3)$$

here Hess is the Hessian operator.

The theory of generalized Sasakian space-forms came into existence after the work of Alegre et al. [19]. A generalized Sasakian space-form (GSSF, in short) is such a manifold whose Riemann curvature R is given by

$$R(V_1, V_2)V_3 = f_1 R_1(V_1, V_2)V_3 + f_2 R_2(V_1, V_2)V_3 + f_3 R_3(V_1, V_2)V_3, \quad (1.4)$$

f_1, f_2 and f_3 are smooth functions on M and

$$\begin{aligned} R_1(V_1, V_2)V_3 &= g(V_2, V_3)V_1 - g(V_1, V_3)V_2, \\ R_2(V_1, V_2)V_3 &= g(V_1, \phi V_3)\phi V_2 - g(V_2, \phi V_3)\phi V_1 + 2g(V_1, \phi V_2)\phi V_3, \\ R_3(V_1, V_2)V_3 &= \eta(V_1)\eta(V_3)V_2 - \eta(V_2)\eta(V_3)V_1 + g(V_1, V_3)\eta(V_2)\xi - g(V_2, V_3)\eta(V_1)\xi. \end{aligned}$$

Such a manifold admitting different almost contact structures like Sasakian, K -contact, trans-Sasakian, etc. was analyzed by Alegre and Carriazo. GSSF is now drawing attention of several geometers. In [20], it is proved that any GSSF with dimension greater than or equal to five must be Sasakian-space-form. It is also proved in the same article that a K -contact GSSF is a Sasakian manifold. For more details we cite the papers [21–25].

The present paper is organized as follows: After the introduction, we give some preliminaries in the Section 2. In Section 3, we have studied Miao-Tam equation on three dimensional GSSFs with trans-Sasakian structure. In the same section we have proved that if a non-trivial solution of the Miao-Tam equation exists then the metric must be a gradient Ricci soliton and non-existence of the non-trivial solution of the Fischer-Marsden equation is also deduced. In the next section, we have derived some new results of Ricci solitons and gradient Ricci solitons on the same space-forms. In the last section, we give an example to verify the deduced results.

2. Preliminaries

A smooth manifold M^{2n+1} is known as an almost contact manifold (ACM) if there exists a structure (ϕ, θ, η) , where ϕ , θ and η are, respectively, a $(1, 1)$ -tensor field, a $(1, 0)$ type vector field and a 1-form, such that

$$\phi^2 V_1 = -V_1 + \eta(V_1)\theta, \quad \eta(\theta) = 1, \quad \phi\theta = 0, \quad \eta \cdot \phi = 0 \quad \text{rank}(\phi) = 2n,$$

for every vector field V_1 on M^{2n+1} [26, 27].

An ACM M^{2n+1} is called an almost contact metric manifold (ACMM) if it admits a Riemannian metric g such that

$$g(\phi V_1, \phi V_2) = g(V_1, V_2) - \eta(V_1)\eta(V_2), \quad (2.1)$$

for every vector fields V_1, V_2 on M^{2n+1} . Equation (2.1) gives

$$g(\phi V_1, V_2) = -g(V_1, \phi V_2).$$

An ACMM is called a contact metric manifold if there exists a 2-form Φ such that $d\eta = \Phi$, where $\Phi(V_1, V_2) = g(V_1, \phi V_2)$. An ACMM is called normal if Nijenhuis torsion tensor $[\phi, \phi](V_1, V_2) + 2d\eta(V_1, V_2)\theta$ vanishes, where $[\phi, \phi](V_1, V_2) = \phi^2[V_1, V_2] + [\phi V_1, \phi V_2] - \phi[\phi V_1, V_2] - \phi[V_1, \phi V_2]$. A normal contact metric manifold is called a Sasakian manifold. An ACMM is called a trans-Sasakian manifold [28] if there exist two smooth functions α and β such that

$$(\nabla_{V_1}\phi)V_2 = \alpha(g(V_1, V_2)\theta - \eta(V_2)V_1) + \beta(g(\phi V_1, V_2)\theta - \eta(V_2)\phi V_1), \quad (2.2)$$

for every vector fields V_1, V_2 on M^{2n+1} . Actually, trans-Sasakian manifolds are the generalizations of Sasakian manifolds and Kenmotsu manifolds, that means, if $\beta = 0$ (res. $\alpha = 0$) then the manifold reduces to α -Sasakian (res. β -Kenmotsu) manifold. For more details please follow the articles [29–33]. From equation (2.2), one can obtain

$$\nabla_{V_1}\theta = -\alpha\phi V_1 + \beta(V_1 - \eta(V_1)\theta). \quad (2.3)$$

In view of (1.4), we have

$$S(V_2, V_3) = (2f_1 + 3f_2 - f_3)g(V_2, V_3) - (3f_2 + f_3)\eta(V_2)\eta(V_3), \quad (2.4)$$

which gives

$$QV_2 = (2f_1 + 3f_2 - f_3)V_2 - (3f_2 + f_3)\eta(V_2)\theta, \quad (2.5)$$

Q is the Ricci operator. Again, contracting V_2 in the foregoing equation, we get the scalar curvature as

$$r = 2(3f_1 + 3f_2 - 2f_3). \quad (2.6)$$

Lemma 2.1. For a trans-Sasakian GSSF M , the following relation holds:

$$f_1 - f_3 + \theta(\alpha) + \theta(\beta) - \alpha^2 + \beta^2 = 0. \tag{2.7}$$

Proof. According to the equations (2.2) and (2.3), we obtain

$$R(V_1, \theta)\theta = (\theta(\alpha) + \alpha\beta)\phi V_1 + (-\theta(\beta) - \beta^2 + \alpha^2 + \alpha\beta)(V_1 - \eta(V_1)\theta). \tag{2.8}$$

On the other hand, from equation (1.4), it can be easily seen that

$$R(V_1, \theta)\theta = (f_1 - f_3)(V_1 - \eta(V_1)\theta). \tag{2.9}$$

Comparing (2.8) and (2.9), we have

$$\theta(\alpha) + \alpha\beta = 0$$

and

$$-\theta(\beta) - \beta^2 + \alpha^2 + \alpha\beta = f_1 - f_3.$$

Combining the last two equations, we obtain the equation (2.7). □

Definition 2.2 ([34, 35]). A vector field V on a Riemannian manifold is called an infinitesimal contact transformation if

$$\mathcal{L}_V \eta = \kappa \eta, \tag{2.10}$$

for some smooth function κ on the manifold. If $\kappa = 0$, then the vector field is called a strict infinitesimal contact transformation.

3. Miao-Tam Equation (MTE) on Trans-Sasakian Generalized Sasakian Space-forms

The prime aim of the present section is to study the Miao-Tam equation (MTE, in short) on three-dimensional trans-Sasakian GSSFs and make a bridge between MTE and Ricci solitons. Before going to main topic, we proof the following lemma.

Lemma 3.1. Let M^3 be a trans-Sasakian GSSF of dimension three, then

$$\begin{aligned} (\nabla_{V_1} Q)V_2 = & V_1(2f_1 + 3f_2 - f_3)V_2 - V_1(3f_2 + f_3)\eta(V_2)\theta - (3f_2 + f_3)(-\alpha g(\phi V_1, V_2)\theta + \beta(g(V_1, V_2)\theta - \eta(V_1)\eta(V_2)\theta)) \\ & - (3f_2 + f_3)(-\alpha\phi V_1 + \beta(V_1 - \eta(V_1)\theta))\eta(V_2), \end{aligned} \tag{3.1}$$

$$\frac{1}{2}V_2(r) = V_2(2f_1 + 3f_2 - f_3) - \theta(3f_2 + f_3)\eta(V_2) - 2\beta(3f_2 + f_3)\eta(V_2), \tag{3.2}$$

and

$$\theta(r) = 4(\theta(f_1 - f_3) - \beta(3f_2 + f_3)), \tag{3.3}$$

for every vector fields V_1, V_2 on M^3 .

Proof. Differentiating the equation (2.5) covariantly and using (2.3), one can obtain the equation (3.1). Contracting the equation (3.1) with respect to V_1 , we obtain (3.2). Putting $V_2 = \xi$ in (3.2), we get the equation (3.3). □

Theorem 3.2. If a three-dimensional trans-Sasakian GSSF admits non-trivial solution of the Miao-Tam equation then the scalar curvature is a constant.

Proof. Let us suppose that the said space form admits non-trivial solution of the Miao-Tam equation. Then, from (1.1), we obtain

$$(\Delta\lambda)g(V_1, V_2) = (Hess\lambda)(V_1, V_2) - \lambda S(V_1, V_2) - g(V_1, V_2). \tag{3.4}$$

Let $\{u_1, u_2, \xi\}$ be an orthonormal set of tangent vector fields on M^3 . Substituting $V_1 = V_2 = u_i$ in the previous equation and summing over i , we have

$$(\Delta\lambda) = -(3f_1 + 3f_2 - 2f_3)\lambda - \frac{3}{2}. \tag{3.5}$$

Using (3.5) in (3.4), we obtain

$$\nabla_{V_1} D\lambda = \lambda QV_1 - (3f_1 + 3f_2 - 2f_3)\lambda V_1 - \frac{1}{2}V_1. \tag{3.6}$$

The covariant derivative of the equation (3.6) in the direction of V_2 gives

$$\nabla_{V_2} \nabla_{V_1} D\lambda = V_2(\lambda)QV_1 + \lambda \nabla_{V_2} QV_1 - V_2(3f_1 + 3f_2 - 2f_3)\lambda V_1 - (3f_1 + 3f_2 - 2f_3)(V_2(\lambda)V_1 + \lambda \nabla_{V_2} V_1) - \frac{1}{2} \nabla_{V_2} V_1. \tag{3.7}$$

Interchanging V_1 and V_2 in (3.7), one can obtain

$$\nabla_{V_1} \nabla_{V_2} D\lambda = V_1(\lambda)QV_2 + \lambda \nabla_{V_1} QV_2 - V_1(3f_1 + 3f_2 - 2f_3)\lambda V_2 - (3f_1 + 3f_2 - 2f_3)(V_1(\lambda)V_2 + \lambda \nabla_{V_1} V_2) - \frac{1}{2} \nabla_{V_1} V_2. \tag{3.8}$$

Again, equation (3.6) gives

$$\nabla_{[V_1, V_2]} D\lambda = \lambda Q[V_1, V_2] - (3f_1 + 3f_2 - 2f_3)\lambda[V_1, V_2] - \frac{1}{2}[V_1, V_2]. \quad (3.9)$$

Using (3.7)-(3.9), we get the curvature tensor as

$$R(V_1, V_2)D\lambda = V_1(\lambda)QV_2 - V_2(\lambda)QV_1 + \lambda((\nabla_{V_1}Q)V_2 - (\nabla_{V_2}Q)V_1) - V_1(3f_1 + 3f_2 - 2f_3)\lambda V_2 + V_2(3f_1 + 3f_2 - 2f_3)\lambda V_1 - (3f_1 + 3f_2 - 2f_3)(V_1(\lambda)V_2 - V_2(\lambda)V_1). \quad (3.10)$$

Contracting (3.10) along the vector field V_1 , we obtain

$$S(V_2, D\lambda) = (2f_1 + 3f_2 - f_3)V_2(\lambda) - (3f_2 + f_3)\theta(\lambda)\eta(V_2) + \lambda\{V_2(2f_1 + 3f_2 - f_3) - \theta(3f_2 + f_3)\eta(V_2) - 2\beta(3f_2 + f_3)\eta(V_2)\}. \quad (3.11)$$

According to (2.4), we find

$$S(V_2, D\lambda) = (2f_1 + 3f_2 - f_3)V_2(\lambda) - (3f_2 + f_3)\theta(\lambda)\eta(V_2). \quad (3.12)$$

Comparing (3.11) and (3.12), we get

$$V_2(2f_1 + 3f_2 - f_3) - \theta(3f_2 + f_3)\eta(V_2) - 2\beta(3f_2 + f_3)\eta(V_2) = 0, \quad (3.13)$$

where we have used $\lambda \neq 0$. Substituting (3.13) in (3.2), we see that $V_2(r) = 0$, that is, r is a constant.

This completes the proof. \square

Theorem 3.3. *If a three-dimensional trans-Sasakian GSSF admits non-trivial solution of the Miao-Tam equation then either the structure is β -Kenmotsu or, the manifold is Einstein.*

Proof. Replacing V_1 by ξ and taking inner product with V_1 of the equation (3.10), we have

$$g(R(\theta, V_2)D\lambda, V_1) = \theta(\lambda)\{-(f_1 - f_3)g(V_1, V_2) - (3f_2 + f_3)\eta(V_1)\eta(V_2)\} + (f_1 + 3f_3)V_2(\lambda)\eta(V_1) + \lambda\{-\theta(f_1 - f_3)g(V_1, V_2) - \theta(3f_2 + f_3)\eta(V_1)\eta(V_2) + V_2(f_1 + 3f_2)\eta(V_1) + (3f_2 + f_3)(-\alpha g(V_1, \phi V_2) + \beta(g(V_1, V_2) - \eta(V_1)\eta(V_2)))\}. \quad (3.14)$$

Putting $V_1 = \xi$ in (1.4) and then taking inner product with $D\lambda$, one can obtain

$$g(R(\theta, V_2)V_1, D\lambda) = (f_1 - f_3)(\theta(\lambda)g(V_1, V_2) - V_2(\lambda)\eta(V_1)). \quad (3.15)$$

Comparing (3.14) and (3.15), we find

$$\theta(\lambda)\{-(f_1 - f_3)g(V_1, V_2) - (3f_2 + f_3)\eta(V_1)\eta(V_2)\} + (f_1 + 3f_3)V_2(\lambda)\eta(V_1) + \lambda\left\{-\theta(f_1 - f_3)g(V_1, V_2) - \theta(3f_2 + f_3)\eta(V_1)\eta(V_2) + V_2(f_1 + 3f_2)\eta(V_1) + (3f_2 + f_3)(-\alpha g(V_1, \phi V_2) + \beta(g(V_1, V_2) - \eta(V_1)\eta(V_2)))\right\} = (f_3 - f_1)(\theta(\lambda)g(V_1, V_2) - V_2(\lambda)\eta(V_1)). \quad (3.16)$$

Interchanging V_1 and V_2 in the foregoing equation, we find

$$\theta(\lambda)\{-(f_1 - f_3)g(V_1, V_2) - (3f_2 + f_3)\eta(V_1)\eta(V_2)\} + (f_1 + 3f_3)V_1(\lambda)\eta(V_2) + \lambda\left\{-\theta(f_1 - f_3)g(V_1, V_2) - \theta(3f_2 + f_3)\eta(V_1)\eta(V_2) + V_1(f_1 + 3f_2)\eta(V_2) + (3f_2 + f_3)(\alpha g(V_1, \phi V_2) + \beta(g(V_1, V_2) - \eta(V_1)\eta(V_2)))\right\} = (f_3 - f_1)(\theta(\lambda)g(V_1, V_2) - V_1(\lambda)\eta(V_2)). \quad (3.17)$$

Subtracting (3.17) from (3.16), one can obtain

$$(3f_2 + f_3)(V_2(\lambda)\eta(V_1) - V_1(\lambda)\eta(V_2)) + \lambda\{V_2(f_1 + 3f_2)\eta(V_1) - V_1(f_1 + 3f_2)\eta(V_2) - 2(3f_2 + f_3)\alpha g(V_1, \phi V_2)\} = 0.$$

Replacing V_1 and V_2 by ϕV_1 and ϕV_2 , respectively, in the last equation, we obtain

$$(3f_2 + f_3)\alpha g(V_1, \phi V_2) = 0,$$

which implies that either $3f_2 + f_3 = 0$ or, $\alpha = 0$, i.e., the structure is β -Kenmotsu.

Let us now discuss the case when $3f_2 + f_3 = 0$. Then from (2.6), we get $r = 6(f_1 - f_3)$. With the help of (2.4), (3.1), equation (3.16) can be written as

$$\theta(\lambda)(S(V_1, V_2) - (f_3 - f_1)g(V_1, V_2)) - 3(f_1 - f_3)V_2(\lambda)\eta(V_1) + \theta(f)g(V_1, V_2) - V_2(f)\eta(V_1) = 0,$$

where $f = -\frac{r\lambda+1}{2}$ and

$$\nabla_{V_1} D\lambda = \lambda QV_1 + fV_1. \quad (3.18)$$

As r is a constant, $2V_2(f) = -rV_2(\lambda)$ and so, $2\theta(f) = -r\theta(\lambda)$. Applying these relations in the above equation, we obtain

$$\theta(\lambda)\{S(V_1, V_2) - 2(f_1 - f_3)g(V_1, V_2)\} = 0,$$

where we have used $r = 6(f_1 - f_3)$. From the foregoing equation we obtain either $\theta(\lambda) = 0$ or, $S(V_1, V_2) = 2(f_1 - f_3)g(V_1, V_2)$. If we consider $\theta(\lambda) = 0$, i.e., $g(\theta, D\lambda) = 0$, then by covariant derivative

$$g(\nabla_{V_1}\theta, D\lambda) + g(\theta, \nabla_{V_1}D\lambda) = 0.$$

Using (2.3) and (3.18) in the foregoing equation, we have

$$-\alpha\phi V_1(\lambda) + \beta V_1(\lambda) + \lambda S(V_1, \theta) + f\eta(V_1) = 0, \tag{3.19}$$

where we have used $\theta(\lambda) = 0$. Applying (1.5), $r = 6(f_1 - f_3)$ and $f = -\frac{r\lambda+1}{2}$ in (3.19), we obtain

$$-\alpha\phi V_1(\lambda) + \beta V_1(\lambda) - \left\{ \lambda(f_1 - f_3) + \frac{1}{2} \right\} \eta(V_1) = 0. \tag{3.20}$$

Replacing V_1 by θ , equation (3.20) gives $\lambda(f_1 - f_3) + \frac{1}{2} = 0$, as $\theta(\lambda) = 0$. Thus we find that $f = 1$, a constant and hence λ is also a non-zero constant. Applying these data in (3.4), we see that $S(V_1, V_2) = -\frac{1}{\lambda}g(V_1, V_2)$, i.e., $S(V_1, V_2) = 2(f_1 - f_3)g(V_1, V_2)$, as $\lambda(f_1 - f_3) + \frac{1}{2} = 0$. Thus for every cases, the space-form obeys $S(V_1, V_2) = 2(f_1 - f_3)g(V_1, V_2)$. Hence the manifold is Einstein. Thus the proof is completed. \square

A consequence of the above theorem is

Corollary 3.4. *There does not exist a non-cosymplectic three-dimensional GSSF with β -Kenmotsu structure obeying non-trivial solution of the MTE, where β is a constant.*

Proof. Putting $V_1 = V_2 = u_i$ in (3.16), where $\{u_i\}$, ($i = 1, 2, 3$) being an orthonormal frame of the tangent space, and summing over i , we find

$$\theta(f_1 - f_3) - \beta(3f_2 + f_3) = 0. \tag{3.21}$$

Comparing (3.3) and (3.21), we obtain $\theta(r) = 0$. Using (2.7) in (3.21) and considering β as a constant, we find

$$\beta(3f_2 + f_3) = 0,$$

which gives $\beta = 0$, as $3f_2 + f_3 \neq 0$. Hence the structure is cosymplectic. \square

Corollary 3.5. *Let a trans-Sasakian GSSF be an Einstein manifold and the space form admit non-trivial solution of MTE. Then the metric is a gradient Ricci soliton.*

Proof. Using $S(V_1, V_2) = 2(f_1 - f_3)g(V_1, V_2)$ in (3.18), we see that

$$\nabla_{V_1}D\lambda = \{2(f_1 - f_3) + 1\}V_1.$$

The foregoing equation can be written as

$$Hess(\lambda)(V_1, V_2) + S(V_1, V_2) - \{2(f_1 - f_3)(\lambda + 1) + 1\}g(V_1, V_2) = 0,$$

which is the gradient Ricci soliton, where the soliton constant is $2(f_1 - f_3)(\lambda + 1) + 1$. \square

Theorem 3.6 ([36]). *If $\tilde{\lambda}$ is a solution of the Fischer-Marsden equation (FME, in short) on a three-dimensional trans-Sasakian GSSF, then the curvature tensor R is given by*

$$R(V_1, V_2)D\tilde{\lambda} = V_1(\tilde{\lambda})QV_2 - V_2(\tilde{\lambda})QV_1 + \tilde{\lambda}\{(\nabla_{V_1}Q)V_2 - (\nabla_{V_2}Q)V_1\} + V_1(\tilde{f})V_2 - V_2(\tilde{f})V_1, \tag{3.22}$$

for every vector fields V_1, V_2 on M and $\tilde{f} = -\frac{r\tilde{\lambda}}{2}$.

Moreover,

$$\nabla_{V_1}D\tilde{\lambda} = \tilde{\lambda}QV_1 + \tilde{f}V_1. \tag{3.23}$$

Theorem 3.7. *In a three-dimensional trans-Sasakian GSSF, if the FME admits a solution then either the solution is trivial or, the scalar curvature is a constant.*

Proof. Using (2.4) in (3.22), one can obtain

$$R(V_1, V_2)D\tilde{\lambda} = (2f_1 + 3f_2 - f_3)V_1(\tilde{\lambda})V_2 - (3f_2 + f_3)V_1(\tilde{\lambda})\eta(V_2)\theta - (2f_1 + 3f_2 - f_3)V_2(\tilde{\lambda})V_1 + (3f_2 + f_3)V_2(\tilde{\lambda})\eta(V_1)\theta + \tilde{\lambda}\{(\nabla_{V_1}Q)V_2 - (\nabla_{V_2}Q)V_1\} + V_1(\tilde{f})V_2 - V_2(\tilde{f})V_1. \tag{3.24}$$

Contracting (3.24) along V_1 , we infer

$$S(V_2, D\tilde{\lambda}) = (2f_1 + 3f_2 - f_3)V_2(\tilde{\lambda}) - (3f_2 + f_3)\theta(\tilde{\lambda})\eta(V_2) + \frac{\tilde{\lambda}}{2}V_2(r), \tag{3.25}$$

where we have used $\tilde{f} = -\frac{r\tilde{\lambda}}{2}$. Comparing (3.25) with (3.12), we find that $\tilde{\lambda}V_2(r) = 0$, which gives either $\tilde{\lambda} = 0$, i.e., the solution is trivial or, $V_2(r) = 0$, i.e., the scalar curvature is a constant.

This establishes the theorem. \square

Theorem 3.8. *In a three-dimensional trans-Sasakian GSSF, if the FME admits a solution then either the structure is β -Kenmotsu or, the manifold is Einstein or, the solution is trivial.*

Proof. Taking inner product of (3.22) with θ , we find that

$$\begin{aligned} g(R(V_1, V_2)D\tilde{\lambda}, \theta) &= 2(f_1 - f_3)\{V_1(\tilde{\lambda})\eta(V_2) - V_2(\tilde{\lambda})\eta(V_1)\} \\ &\quad + \tilde{\lambda}\{2V_1(f_1 - f_3)\eta(V_2) - 2V_2(f_1 - f_3)\eta(V_1) + 2(3f_2 + f_3)\alpha g(\phi V_1, V_2)\} \\ &\quad + V_1(\tilde{f})\eta(V_2) - V_2(\tilde{f})\eta(V_1). \end{aligned} \quad (3.26)$$

Replacing V_1 by ϕV_1 and V_2 by ϕV_2 in (3.26), one can obtain

$$g(R(\phi V_1, \phi V_2)D\tilde{\lambda}, \theta) = -2\tilde{\lambda}(3f_2 + f_3)\alpha g(V_1, \phi V_2). \quad (3.27)$$

Also, from (1.4), we have

$$g(R(\phi V_1, \phi V_2)D\tilde{\lambda}, \theta) = 0. \quad (3.28)$$

Comparing (3.27) and (3.28), we obtain

$$\tilde{\lambda}(3f_2 + f_3)\alpha g(V_1, \phi V_2) = 0.$$

Thus three possibility arise: (1) $\tilde{\lambda} = 0$, (2) $(3f_2 + f_3) = 0$ and (3) $\alpha = 0$.

Let us discuss the case when $(3f_2 + f_3) = 0$. Then, from (2.6), we find that $r = 6(f_1 - f_3)$. From (3.22), we get

$$g(R(\theta, V_2)D\tilde{\lambda}, V_1) = \theta(\tilde{\lambda})S(V_1, V_2) - V_2(\tilde{\lambda})S(V_1, \theta) + \theta(\tilde{f})g(V_1, V_2) - V_2(\tilde{f})\eta(V_1). \quad (3.29)$$

Also, from (1.4), we infer

$$g(R(\theta, V_2)D\tilde{\lambda}, V_1) = -(f_1 - f_3)\{\theta(\tilde{f})g(V_1, V_2) - V_2(\tilde{f})\eta(V_1)\}. \quad (3.30)$$

Comparing (3.29) and (3.30) and using $r = 6(f_1 - f_3)$, $f = -\frac{r\tilde{\lambda}}{2}$ and the equation (2.4), one can obtain

$$\theta(\tilde{\lambda})(S(V_1, V_2) - 2(f_1 - f_3)g(V_1, V_2)) = 0,$$

which implies either $S(V_1, V_2) = 2(f_1 - f_3)g(V_1, V_2)$, i.e., the manifold is Einstein or, $\theta(\tilde{\lambda}) = 0$. Let us discuss the case when $(\theta\tilde{\lambda}) = 0$. Then we have $g(\theta, D\tilde{\lambda}) = 0$, which gives

$$g(\nabla_{V_2}\theta, D\tilde{\lambda}) + g(\theta, \nabla_{V_2}D\tilde{\lambda}) = 0.$$

Applying (2.3), (2.4), (3.23) and $\tilde{f} = -\frac{r\tilde{\lambda}}{2}$ in the foregoing equation, we see that

$$-\alpha\phi V_2(\tilde{\lambda}) + \beta V_2(\tilde{\lambda}) - (f_1 - f_3)\tilde{\lambda}\eta(V_2) = 0, \quad (3.31)$$

where we have used $\theta(\tilde{\lambda}) = 0$. Replacing V_2 by θ and taking $f_1 \neq f_3$ in (3.31), we find that $\tilde{\lambda} = 0$, i.e., the solution is trivial. This ensures the validity of the theorem. \square

4. Ricci Solitons on Three-Dimensional Generalized Sasakian Space-forms with Trans-Sasakian Structures

In the present section, we study Ricci solitons on three-dimensional generalized Sasakian space-forms with trans-Sasakian structure.

Theorem 4.1. *In a three-dimensional trans-Sasakian GSSF obeying Ricci solitons, the potential vector field is an infinitesimal contact transformation.*

Proof. From (1.2), we have

$$(\mathcal{L}_V g)(V_1, V_2) + 2S(V_1, V_2) + 2\psi g(V_1, V_2) = 0.$$

Applying $V_2 = \theta$ in the foregoing equation and using (2.4), we have

$$(\mathcal{L}_V g)(V_1, \theta) = -2(2(f_1 - f_3) + \psi)\eta(V_1). \quad (4.1)$$

Again, changing V_1 by θ in (4.1), we get

$$\mathcal{L}_V \theta = (2(f_1 - f_3) + \psi)\theta. \quad (4.2)$$

Applying Lie derivative of $\eta(V_1) = g(V_1, \theta)$ with respect to V and then using (4.1) and (4.2), we find that

$$(\mathcal{L}_V \eta)(V_1) = -(2(f_1 - f_3) + \psi)\eta(V_1),$$

an infinitesimal contact transformation. \square

From the above theorem, we prove the following:

Theorem 4.2. *In a three-dimensional trans-Sasakian GSSF obeying Ricci solitons, the soliton is shrinking, expanding or steady if $f_1 - f_3$ is positive, negative or zero, respectively.*

Proof. We have

$$(\mathcal{L}_V d\eta)(V_1, V_2) = (\mathcal{L}_V g)(V_1, \phi V_2) + g(V_1, (\mathcal{L}_V \phi)V_2).$$

Using (2.4) and (1.2) in the foregoing equation, we infer

$$(\mathcal{L}_V d\eta)(V_1, V_2) = -2(2f_1 + 3f_2 - f_3 + \psi)g(V_1, \phi V_2) + g(V_1, (\mathcal{L}_V \phi)V_2). \tag{4.3}$$

According to Theorem 4.1, V is an infinitesimal contact transformation. Also, since \mathcal{L} and d commutes, equation (2.10) gives

$$(\mathcal{L}_V d\eta)(V_1, V_2) = ((d\kappa) \wedge \eta)(V_1, V_2) + \kappa g(V_1, \phi V_2) = \frac{1}{2}(V_1(\kappa)\eta(V_2) - V_2(\kappa)\eta(V_1)) + \kappa g(V_1, \phi V_2). \tag{4.4}$$

Comparing (4.3) and (4.4), we have

$$g(V_1, (\mathcal{L}_V \phi)V_2) = \frac{1}{2}(V_1(\kappa)\eta(V_2) - V_2(\kappa)\eta(V_1)) + (2(2f_1 + 3f_2 - f_3 + \psi) + \kappa)g(V_1, \phi V_2),$$

which gives

$$(\mathcal{L}_V \phi)V_2 = \frac{1}{2}(\eta(V_2)D\kappa - V_2(\kappa)\theta) + (2(2f_1 + 3f_2 - f_3 + \psi) + \kappa)\phi V_2.$$

Changing V_2 by θ in the previous equation, we find

$$(\mathcal{L}_V \phi)\theta = \frac{1}{2}(D\kappa - \theta(\kappa)\theta). \tag{4.5}$$

But

$$(\mathcal{L}_V \phi)\theta = \mathcal{L}_V \phi \theta - \phi(\mathcal{L}_V \theta) = 0, \tag{4.6}$$

where we used (4.2) and $\phi \theta = 0$. Using (4.6) in (4.5), we obtain

$$D\kappa = \theta(\kappa)\theta,$$

which gives

$$d\kappa = \theta(\kappa)\eta. \tag{4.7}$$

By exterior derivative we find from (4.7) that

$$0 = d^2\kappa = d(\theta(\kappa)) \wedge \eta + \theta(\kappa)d\eta.$$

Taking wedge product with η in the foregoing equation, we get

$$\theta(\kappa)\eta \wedge d\eta = 0.$$

As $\eta \wedge d\eta \neq 0$, the previous equation gives $\theta(\kappa) = 0$. Thus, from (4.7), we have $d\kappa = 0$, i.e., κ is a constant.

Due to Cartan's formula, for the closed volume form $\Omega (= \eta \wedge d\eta)$, we have

$$\mathcal{L}_V \Omega = (divV)\Omega, \tag{4.8}$$

where div is the divergence operator. Again, taking Lie derivative of the volume form $\Omega (= \eta \wedge d\eta)$ and using (4.4) and (4.8), we get

$$(divV)\Omega = 2\kappa\Omega,$$

which implies

$$divV = 2\kappa.$$

Integrating the above equation and using divergence theorem, we see that $\kappa = 0$. Thus V is the strict infinitesimal contact transformation and hence, we get $\psi = -2(f_1 - f_3)$.

This establishes the theorem. □

Theorem 4.3. *In a three dimensional trans-Sasakian GSSF obeying gradient Ricci solitons, either the structure is β -Kenmotsu or, the potential function is constant, i.e., the soliton is trivial.*

Proof. Let us suppose that a three dimensional trans-Sasakian generalized Sasakian space-form admit gradient Ricci solitons. Then, from (1.3), we can write

$$\nabla_{V_1} D\zeta = -QV_1 - \psi V_1. \quad (4.9)$$

Applying covariant derivative on (4.9), we get

$$\nabla_{V_2} \nabla_{V_1} D\zeta = -\nabla_{V_2} QV_1 - \psi \nabla_{V_2} V_1. \quad (4.10)$$

Interchanging V_1 and V_2 in the previous equation, we obtain

$$\nabla_{V_1} \nabla_{V_2} D\zeta = -\nabla_{V_1} QV_2 - \psi \nabla_{V_1} V_2. \quad (4.11)$$

Also, equation (4.9) gives

$$\nabla_{[V_1, V_2]} D\zeta = -Q[V_1, V_2] - \psi[V_1, V_2]. \quad (4.12)$$

Using (4.10)-(4.12), we get the curvature tensor as

$$\begin{aligned} R(V_1, V_2)D\zeta = & -\{V_1(2f_1 + 3f_2 - f_3)V_2 - V_1(3f_2 + f_3)\eta(V_2)\theta - V_2(2f_1 + 3f_2 - f_3) + V_2(3f_2 + f_3)\eta(V_1)\theta \\ & + 2(3f_2 + f_3)g(\phi V_1, V_2)\theta - (3f_2 + f_3)(-\alpha\phi V_1 + \beta(V_1 - \eta(V_1)\theta))\eta(V_2) \\ & + (3f_2 + f_3)(-\alpha\phi V_2 + \beta(V_2 - \eta(V_2)\theta))\eta(V_1)\}. \end{aligned} \quad (4.13)$$

Replacing V_1 by θ in (4.13) and then taking inner product with V_1 , we see that

$$\begin{aligned} g(R(\theta, V_2)D\zeta, V_1) = & -\{\theta(2f_1 + 3f_2 - f_3)g(V_1, V_2) - \theta(3f_2 + f_3)\eta(V_1)\eta(V_2) - 2V_2(f_1 - f_3)\eta(V_1) \\ & + (3f_2 + f_3)(-\alpha g(V_1, \phi V_2) + \beta(g(V_1, V_2) - \eta(V_1)\eta(V_2)))\}. \end{aligned} \quad (4.14)$$

Also, the equation (1.4) can be written as

$$g(R(\theta, V_2)D\zeta, V_1) = (f_1 - f_3)\{V_2(\zeta)\eta(V_1) - \theta(\zeta)g(V_1, V_2)\}. \quad (4.15)$$

Comparing (4.14) and (4.15), we obtain

$$\begin{aligned} \theta(2f_1 + 3f_2 - f_3)g(V_1, V_2) - \theta(3f_2 + f_3)\eta(V_1)\eta(V_2) - 2V_2(f_1 - f_3)\eta(V_1) \\ + (3f_2 + f_3)(-\alpha g(V_1, \phi V_2) + \beta(g(V_1, V_2) - \eta(V_1)\eta(V_2))) + (f_1 - f_3)\{V_2(\zeta)\eta(V_1) - \theta(\zeta)g(V_1, V_2)\} = 0. \end{aligned} \quad (4.16)$$

Interchanging V_1 and V_2 in (4.16), we have

$$\begin{aligned} \theta(2f_1 + 3f_2 - f_3)g(V_1, V_2) - \theta(3f_2 + f_3)\eta(V_1)\eta(V_2) - 2V_1(f_1 - f_3)\eta(V_2) \\ + (3f_2 + f_3)(\alpha g(V_1, \phi V_2) + \beta(g(V_1, V_2) - \eta(V_1)\eta(V_2))) + (f_1 - f_3)\{V_1(\zeta)\eta(V_2) - \theta(\zeta)g(V_1, V_2)\} = 0. \end{aligned} \quad (4.17)$$

Subtracting (4.17) from (4.16), we see that

$$2V_1(f_1 - f_3)\eta(V_2) - 2V_2(f_1 - f_3)\eta(V_1) - 2(3f_2 + f_3)\alpha g(V_1, \phi V_2) + (f_1 - f_3)\{V_2(\zeta)\eta(V_1) - V_1(\zeta)\eta(V_2)\} = 0. \quad (4.18)$$

Replacing V_1 by ϕV_1 and V_2 by ϕV_2 in (4.18), we obtain

$$(3f_2 + f_3)\alpha g(\phi V_1, V_2) = 0,$$

which indicates that either $\alpha = 0$, i.e., the structure is β -Kenmotsu or, $3f_2 + f_3 = 0$. For the later case, with the help of (2.6) and (3.2), we get

$$V_1(f_1 - f_3) = 0, \quad (4.19)$$

for every vector field V_1 , i.e., $f_1 - f_3$ is a constant. Thus, from (4.18), we obtain

$$(f_1 - f_3)\{V_2(\zeta)\eta(V_1) - V_1(\zeta)\eta(V_2)\} = 0,$$

which gives either $f_1 = f_3$ or

$$V_2(\zeta)\eta(V_1) = V_1(\zeta)\eta(V_2). \quad (4.20)$$

Let us discuss the second possibility. Putting $V_2 = \theta$ in (4.20), we obtain

$$D\zeta = \theta(\zeta)\theta. \quad (4.21)$$

Taking covariant derivative of (4.21) with respect to V_1 and using (2.3), we obtain

$$\nabla_{V_1} D\zeta = V_1(\theta(\zeta))\theta + \theta(\zeta)(-\alpha\phi V_1 + \beta(V_1 - \eta(V_1)\theta)). \quad (4.22)$$

Comparing (4.22) with (4.9), we find that

$$V_1(\theta(\zeta))\eta(V_2) = -S(V_1, V_2) - \psi g(V_1, V_2) - \theta(\zeta)(-\alpha g(\phi V_1, V_2) + \beta(g(V_1, V_2) - \eta(V_1)\eta(V_2))).$$

Since $3f_2 + f_3 = 0$, using (2.4) in the above equation, we get

$$V_1(\theta(\zeta))\eta(V_2) = -\{2(f_1 - f_3) + \psi\}g(V_1, V_2) - \theta(\zeta)(-\alpha g(\phi V_1, V_2) + \beta(g(V_1, V_2) - \eta(V_1)\eta(V_2))). \tag{4.23}$$

Replacing V_2 by ϕV_2 in (4.23), we see that

$$\{2(f_1 - f_3) + \psi\}g(V_1, \phi V_2) + \theta(\zeta)(-\alpha(g(V_1, V_2) - \eta(V_1)\eta(V_2)) + \beta g(V_1, \phi V_2)) = 0.$$

Contracting the above equation and using $\text{tr}\phi = 0$, we get

$$\alpha\theta(\zeta) = 0,$$

which gives $\theta(\zeta) = 0$, as we consider $\alpha \neq 0$. Thus, from (4.21), we see that $D\zeta = 0$, i.e., ζ is a constant. Hence the proof is completed. □

From the equation(4.19), we can state the following corollary

Corollary 4.4. *If a three-dimensional trans-Sasakian GSSF admits gradient Ricci solitons, then either the structure is β -Kenmotsu or, $f_1 - f_3$ is a constant.*

5. Example

Let $M = \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}$ be a three-dimensional manifold, where (x, y, z) are the standard co-ordinates in \mathbb{R}^3 . We choose the basis vectors on M as

$$u_1 = e^{-2z} \frac{\partial}{\partial x}, \quad u_2 = e^{-2z} \frac{\partial}{\partial y}, \quad u_3 = \frac{\partial}{\partial z}.$$

Then we find by direct computation that

$$[u_1, u_2] = 0, \quad [u_1, u_3] = 2u_1, \quad [u_2, u_3] = 2u_2.$$

Let g be the metric tensor defined by

$$g(u_1, u_1) = 1, \quad g(u_2, u_2) = 1, \quad g(u_3, u_3) = 1, \quad g(u_1, u_2) = 0, \quad g(u_1, u_3) = 0, \quad g(u_2, u_3) = 0.$$

The 1-form η is given by $\eta(V_1) = g(V_1, u_3)$ for all V_1 on M . Let us define the $(1, 1)$ -tensor field ϕ as

$$\phi u_1 = -u_2, \quad \phi u_2 = u_1, \quad \phi u_3 = 0.$$

Then we see that

$$\eta(u_3) = 1, \quad \phi^2 V_1 = -V_1 + \eta(V_1)u_3, \quad g(\phi V_1, \phi V_2) = g(V_1, V_2) - \eta(V_1)\eta(V_2), \quad d\eta(V_1, V_2) = g(V_1, \phi V_2).$$

Thus the given manifold admits a contact metric structure (ϕ, u_3, η, g) . Now, using Koszul's formula, we obtain

$$\begin{aligned} \nabla_{u_1} u_1 &= -2u_3, & \nabla_{u_1} u_2 &= 0, & \nabla_{u_1} u_3 &= 2u_1, & \nabla_{u_2} u_1 &= 0, & \nabla_{u_2} u_2 &= -2u_3, & \nabla_{u_2} u_3 &= 2u_2, & \nabla_{u_3} u_1 &= 0, \\ \nabla_{u_3} u_2 &= 0, & \nabla_{u_3} u_3 &= 0. \end{aligned}$$

Thus the given structure is a trans-Sasakian structure with $\alpha = 0, \beta = 2$. The components of the curvature tensor are given by

$$\begin{aligned} R(u_1, u_2)u_2 &= -4u_1, & R(u_2, u_1)u_1 &= -4u_2, & R(u_1, u_3)u_3 &= -4u_1, & R(u_2, u_3)u_3 &= -4u_2, & R(u_3, u_1)u_1 &= -4u_3, \\ R(u_3, u_2)u_2 &= -4u_3, & R(u_1, u_2)u_3 &= 0, & R(u_1, u_3)u_2 &= 0, & R(u_2, u_3)u_1 &= 0. \end{aligned}$$

From the above expressions, the given manifold is a generalized Sasakian space-form with $f_1 = \omega - 1, f_2 = -\frac{\omega+3}{3}$ and $f_3 = \omega + 3$, where ω is a smooth function on M .

The non-zero components of the Ricci tensor are given by

$$S(u_1, u_1) = -8, \quad S(u_2, u_2) = -8, \quad S(u_3, u_3) = -8.$$

Thus we see that $S(V_1, V_2) = -8g(V_1, V_2)$, for every vector fields V_1, V_2 on M . Hence the space-form is an Einstein manifold. The scalar curvature of the manifold is -24 .

Let $\lambda = e^{-\frac{az}{2}} + b$, where a and b are scalars, so that, $e^{-\frac{az}{2}} = \lambda - b$. Now $D\lambda = -\frac{a}{2}e^{-\frac{az}{2}}u_3 = -\frac{a}{2}(\lambda - b)u_3$. Then

$$\nabla_{u_1} D\lambda = -a(\lambda - b)u_1, \quad \nabla_{u_2} D\lambda = -a(\lambda - b)u_2, \quad \nabla_{u_3} D\lambda = \frac{a^2}{4}(\lambda - b)u_3.$$

Thus $(\Delta_g \lambda) = (\frac{a^2}{4} - 2a)(\lambda - b)$. Now $-(\Delta_g \lambda)g(u_i, u_j) + g(\nabla_{u_i} D\lambda, u_j) - \lambda S(u_i, u_j) = g(u_i, u_j), \quad i, j = 1, 2, 3$, gives the following two equations

$$(a - \frac{a^2}{4})(\lambda - b) + 8\lambda = 1$$

and

$$2a(\lambda - b) + 8\lambda = 1.$$

Comparing the above two equations, we see that $a = 0$, $b = -\frac{7}{8}$ and $\lambda = \frac{1}{8}$ or $a = -4$, $b = \frac{1}{8}$ and $\lambda = e^{2z} + \frac{1}{8}$. Thus the non-trivial solution of the Miao-Tam equation exists on the given manifold. Since the manifold is Einstein and the structure is β -Kenmotsu (as $\alpha = 0$), the Theorem 3.3 holds good.

Again, let $\tilde{\lambda} = e^{-\frac{az}{2}} + b$, where a and b are scalars, so that, $e^{-\frac{az}{2}} = \tilde{\lambda} - b$. Now $D\tilde{\lambda} = -\frac{a}{2}e^{-\frac{az}{2}}u_3 = -\frac{a}{2}(\tilde{\lambda} - b)u_3$. Then

$$\nabla_{u_1}D\tilde{\lambda} = -a(\tilde{\lambda} - b)u_1, \quad \nabla_{u_2}D\tilde{\lambda} = -a(\tilde{\lambda} - b)u_2, \quad \nabla_{u_3}D\tilde{\lambda} = \frac{a^2}{4}(\tilde{\lambda} - b)u_3.$$

Thus $(\Delta_g \tilde{\lambda}) = (\frac{a^2}{4} - 2a)(\tilde{\lambda} - b)$. Now $-(\Delta_g \tilde{\lambda})g(u_i, u_j) + g(\nabla_{u_i}D\tilde{\lambda}, u_j) - \tilde{\lambda}S(u_i, u_j) = 0$, $i, j = 1, 2, 3$, gives the following two equations

$$(a - \frac{a^2}{4})(\lambda - b) + 8\lambda = 0$$

and

$$2a(\lambda - b) + 8\lambda = 0.$$

Solving the last two equations, we see that $\tilde{\lambda} = 0$, i.e., the solution is trivial, which ensures the validity of the Theorem 3.8.

Let us consider the potential vector field $V = xe^{2z}u_1 + ye^{2z}u_2 + \frac{1}{2}(e^{2z} - 1)u_3$. Then equation (1.2) is satisfied for that V with $\psi = 8 - e^{2z}$, i.e., the soliton is steady at $z = \frac{3}{2} \log 2$ and it is expanding or shrinking if z is less than or greater than $\frac{3}{2} \log 2$, respectively. Also $(\mathcal{L}_V \eta)(V_1) = e^{2z}\eta(V_1)$, for any vector field V_1 on M . Hence V is an infinitesimal contact transformation. In this way Theorem 4.1 is satisfied. Next, we suppose that the potential vector field V is the gradient of a smooth function ζ , i.e., $V = D\zeta$. Then

$$D\zeta = e^{-2z} \frac{\partial \zeta}{\partial x} u_1 + e^{-2z} \frac{\partial \zeta}{\partial y} u_2 + \frac{\partial \zeta}{\partial z} u_3.$$

Therefore,

$$\begin{aligned} \nabla_{u_1}D\zeta &= e^{-4z} \frac{\partial^2 \zeta}{\partial x^2} u_1 - 2e^{-2z} \frac{\partial \zeta}{\partial x} u_3 + e^{-4z} \frac{\partial^2 \zeta}{\partial x \partial y} u_2 + e^{-2z} \frac{\partial^2 \zeta}{\partial x \partial z} u_3 + 2 \frac{\partial \zeta}{\partial z} u_1, \\ \nabla_{u_2}D\zeta &= e^{-4z} \frac{\partial^2 \zeta}{\partial y^2} u_2 - 2e^{-2z} \frac{\partial \zeta}{\partial y} u_3 + e^{-4z} \frac{\partial^2 \zeta}{\partial y \partial x} u_1 + e^{-2z} \frac{\partial^2 \zeta}{\partial y \partial z} u_3 + 2 \frac{\partial \zeta}{\partial z} u_2, \\ \nabla_{u_3}D\zeta &= -2e^{-2z} \frac{\partial \zeta}{\partial x} u_1 + e^{-2z} \frac{\partial^2 \zeta}{\partial z \partial x} u_1 - 2e^{-2z} \frac{\partial \zeta}{\partial y} u_2 + e^{-2z} \frac{\partial^2 \zeta}{\partial z \partial y} u_2 + \frac{\partial^2 \zeta}{\partial z^2} u_3. \end{aligned}$$

Thus the equation $\nabla_{V_1}D\zeta + QV_1 + \psi V_1 = 0$ gives

$$\begin{aligned} e^{-4z} \frac{\partial^2 \zeta}{\partial x^2} + 2 \frac{\partial \zeta}{\partial z} - 8 + \psi &= 0, \\ e^{-4z} \frac{\partial^2 \zeta}{\partial y^2} + 2 \frac{\partial \zeta}{\partial z} - 8 + \psi &= 0, \end{aligned}$$

and

$$\frac{\partial^2 \zeta}{\partial z^2} - 8 + \psi = 0.$$

The last three equations satisfy simultaneously only when ζ is a constant. Thus we see that the soliton is trivial, which verifies the Theorem 4.3.

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