Konuralp Journal of Mathematics
Volume 5 No. 2 pp. 78–86 (2017) ©KJM

DERIVATIVES WITH RESPECT TO HORIZONTAL AND VERTICAL LIFTS OF THE CHEEGER-GROMOLL METRIC ${}^{CG}g$ ON THE $(1,1)-$TENSOR BUNDLE $T_1^1(M)$.

HAŞIM ÇAYIR AND MOHAMMAD NAZRUL ISLAM KHAN

Abstract. In this paper, we define the Cheeger-Gromoll metric in the $(1,1)$ $-$tensor bundle $T_1^1(M)$, which is completely determined by its action on vector fields of type X^H and ω^V. Later, we obtain the covarient and Lie derivatives applied to the Cheeger-Gromoll metric with respect to the horizontal and vertical lifts of vector and kovector fields, respectively.

1. Introduction

Let M be a differentiable manifold of class C^∞ and finite dimension n. Then the set $T_1^1(M) = \cup_{P\in M} T_1^1(P)$ is, by definition, the tensor bundle of type $(1,1)$ over M, where $\cup$ denotes the disjoint union of the tensor spaces $T_1^1(P)$ for all $P \in M$. For any point $\tilde{P}$ of $T_1^1(M)$ such that $\tilde{P} \in T_1^1(M)$, the surjective correspondence $\tilde{P} \to P$ determines the natural projection $\pi : T_1^1(M) \to M$. The projection π defines the natural differentiable manifold structure of $T_1^1(M)$, that is, $T_1^1(M)$ is a $C^\infty-$ manifold of dimension $n+n^2$. If x^j are local coordinates in a neighborhood U of $P \in M$, then a tensor t at P which is an element of $T_1^1(M)$ is expressible in the form (x^j, t_j^i), where t_j^i are components of t with respect to the natural base. We may consider $(x^j, t_j^i) = (x^j, x^{\bar{j}}) = (x^J)$, $j = 1,...,n$, $\bar{j} = n+1,...,n+n^2$, $J = 1,...,n+n^2$ as local coordinates in a neighborhood $\pi^{-1}(U)$.

Let $X = X^i \frac{\partial}{\partial x^i}$ and $A = A_j^i \frac{\partial}{\partial x^i} \otimes dx^j$ be the local expressions in U of a vector field X and a $(1,1)$ tensor field A on M, respectively. Then the vertical lift A^V of A and the horizontal lift X^H of X are given, with respect to the induced coordinates,by

$$ {}^V A = \begin{pmatrix} {}^V A^j \\ {}^V A^{\bar{j}} \end{pmatrix} = \begin{pmatrix} 0 \\ A_j^i \end{pmatrix} \tag{1.1} $$

Date: January 6, 2017 and, in revised form, May 31, 2017.
2000 *Mathematics Subject Classification.* 15A72; 47B47; 53A45; 53C15.
Key words and phrases. $(1,1)-$tensor bundle, Covarient Derivative, Lie Derivative, Cheeger-Gromoll metric, Horizontal Lift, Vertical Lift.
This study is supported by Giresun University Scientific Projects Office (GBAP)(Project No:FEN-BAP-A-160317-49, 2017).

and

$$^{H}X = \begin{pmatrix} ^{H}X^{j} \\ ^{H}X^{\bar{j}} \end{pmatrix} = \begin{pmatrix} X^{j} \\ X^{s}(\Gamma_{sj}^{m} t_{m}^{i} - \Gamma_{sm}^{i} t_{j}^{m}) \end{pmatrix} \tag{1.2}$$

where Γ_{ij}^{h} are the coefficient of the connection ∇ on M [9].

Let $\varphi \in \Im_1^1(M)$. The global vector fields $\gamma\varphi$ and $\tilde{\gamma}\varphi \in \Im_0^1(\Im_1^1(M))$ are respectively defined by

$$\gamma\varphi = \begin{pmatrix} 0 \\ t_j^m \varphi_m^i \end{pmatrix}, \tilde{\gamma}\varphi = \begin{pmatrix} 0 \\ t_m^i \varphi_j^m \end{pmatrix}$$

with respect to the coordinates $\left(x^i, x^{\bar{j}}\right)$ in $T_1^1(M)$, where φ_j^i are the components of φ [9].

The Lie bracket operation of vertical and horizontal vector fields on $T_1^1(M)$ is given by

$$\begin{aligned} \left[^{H}X, ^{H}Y\right] &= {}^{H}[X,Y] + (\tilde{\gamma} - \gamma)R(X,Y) \\ \left[^{H}X, ^{V}A\right] &= {}^{V}(\nabla_X A) \\ \left[^{V}A, ^{V}B\right] &= 0 \end{aligned} \tag{1.3}$$

for any $X, Y \in \Im_0^1(M)$ and $A, B \in \Im_1^1(M)$, where R is the curvature tensor field of the connection ∇ on M defined by $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ and $(\tilde{\gamma} - \gamma)\, R(X,Y) = \binom{0}{t_m^i R_{klj}^m X^k Y^l - t_j^m R_{klm}^i X^k Y^l}$ (for details, see [7, 17] and for sufraces [3, 4]).

1.1. **Cheeger-Gromoll type metric on the (1, 1)-tensor bundle.** An n-dimensional manifold M in which a (1,1) tensor field φ satisfying $\varphi^2 = id$, $\varphi \neq \pm id$ is given is called an almost product manifold. A Riemannian almost product manifold (M, φ, g) is a manifold M with an almost product structure φ and a Riemannian metric g such that [1, 2, 10, 11]

$$g(\varphi X, Y) = g(X, \varphi Y) \tag{1.4}$$

for all $X, Y \in \Im_0^1(M)$. Also, the condition (3.1) is referred to as purity condition for g with respect to φ [9]. The almost product structure φ is integrable, i.e. the Nijenhuis tensor N_φ determined by

$$N_\varphi(X,Y) = [\varphi X, \varphi Y] - \varphi\,[\varphi X, Y] - \varphi\,[X, \varphi Y] + [X,Y]$$

for all $X, Y \in \Im_0^1(M)$ is zero then the Riemannian almost product manifold. (M, φ, g) is called a Riemannian product manifold. A locally decomposable Riemannian manifold can be defined as a triple (M, φ, g) which consists of a smooth manifold M endowed with an almost product structure φ and a pure metric g such that $\nabla\varphi = 0$,where ∇ is the Levi-Civita connectian of g [9].

Definition 1.1. Let $T_1^1(M)$ be the $(1,1)-$tensor bundle over a Riemannian manifold (M, g). For each $P \in M$, the extension of scalar product g (marked by G) is defined on the tensor space $\pi^{-1}(P) = T_1^1(P)$ by $G(A, B) = g_{ij} g^{jl} A_j^i B_l^t$ for all $A, B \in \Im_1^1(P)$. The Cheeger-Gromoll type metric ^{CG}g is defined on $T_1^1(M)$ by the following three equations:

$$^{CG}g(X^H, Y^H) = (g(X,Y))^V \tag{1.5}$$

$$ {}^{CG}g(A^V, Y^H) = 0 \tag{1.6} $$

$$ {}^{CG}g(A^V, B^V) = \frac{1}{\alpha}(G(A,B) + G(A,t)G(B,t))^V \tag{1.7} $$

for any $X, Y \in \Im_0^1(M)$ and $A, B \in \Im_1^1(M)$, where $r^2 = G(t,t) = g_{it}g^{jl}t_j^i t_l^t$ and $\alpha = 1 + r^2$ [9].

2. Main Results

Definition 2.1. Let M be an $n-$dimensional diferentiable manifold. Differantial transformation of algebra $T(M)$, defined by

$$ D = \nabla_X : T(M) \to T(M),\ X \in \Im_0^1(M) $$

is called as covariant derivation with respect to vector field X if

$$ \begin{aligned} \nabla_{fX+gY}t &= f\nabla_X t + g\nabla_Y t, \\ \nabla_X f &= Xf, \end{aligned} \tag{2.1} $$

where $\forall f, g \in \Im_0^0(M)$, $\forall X, Y \in \Im_0^1(M)$, $\forall t \in \Im(M)$ (see [13], p.123).
On the other hand, a transformation defined by

$$ \nabla : \Im_0^1(M) \times \Im_0^1(M) \to \Im_0^1(M) $$

is called as an affin connection (see for details [13, 16]).

Definition 2.2. The horizontal lift ${}^H\nabla$ of any connection ∇ on the tensor bundle $T_1^1(M)$ is defined by

$$ \begin{aligned} {}^H\nabla_{{}^VA}\,{}^VB &= 0,\ {}^H\nabla_{{}^VA}\,{}^HY = 0, \\ {}^H\nabla_{{}^HX}\,{}^VB &= {}^V(\nabla_X B),\ {}^H\nabla_{{}^HX}\,{}^HY = {}^H(\nabla_X Y) \end{aligned} \tag{2.2} $$

for all vector fields $X, Y \in \Im_0^1(M)$ and $A, B \in \Im_1^1(M)$ (see [8, 14, 15, 17]).

Theorem 2.1. *Let ${}^{CG}g$ be the Cheeger-Gromoll type metric ${}^{CG}g$ defined by (1.5),(1.6),(1.7) and the horizontal lift ${}^H\nabla$ of any connection ∇ on the tensor bundle $T_1^1(M)$ is defined by (2.2). From Definintion 1.1 and Definintion 2.1, we get the following* *results*

$$\begin{array}{rcl}
i)\ ({}^{H}\nabla_{{}^{V}C}{}^{CG}g)({}^{V}A,{}^{V}B) &=& 0,\\
ii)\ ({}^{H}\nabla_{{}^{V}C}{}^{CG}g)({}^{V}A,{}^{H}Y) &=& 0,\\
iii)\ ({}^{H}\nabla_{{}^{V}C}\ {}^{CG}g)({}^{H}X,B^{V}) &=& 0,\\
iv)\ ({}^{H}\nabla_{{}^{V}C}{}^{CG}g)({}^{H}X,{}^{H}Y) &=& 0,\\
v)\ ({}^{H}\nabla_{{}^{H}Z}{}^{CG}g)({}^{V}A,{}^{H}Y) &=& 0,\\
vi)\ ({}^{H}\nabla_{{}^{H}Z}{}^{CG}g)({}^{H}X,{}^{V}B) &=& 0,\\
vii)\ ({}^{H}\nabla_{{}^{H}Z}{}^{CG}g)({}^{H}X,{}^{H}Y) &=& {}^{V}((\nabla_{Z}g)(X,Y)),\\
viii)\ ({}^{H}\nabla_{{}^{H}Z}{}^{CG}g)({}^{V}A,{}^{V}B) &=& {}^{V}(\nabla_{Z}\frac{1}{\alpha})^{V}(G(A,B)+G(A,t)G(B,t))\\
&& +\frac{1}{\alpha}^{V}((\nabla_{Z}G)(A,B))+\frac{1}{\alpha}^{V}(\nabla_{Z}(G(A,t)G(B,t)))\\
&& -\frac{1}{\alpha}^{V}(G((\nabla_{Z}A),t)G(B,t))\\
&& -\frac{1}{\alpha}^{V}(G(A,t)G((\nabla_{Z}B),t)),
\end{array}$$

where the vertical lift ${}^{V}A \in \Im_{0}^{1}(T_{1}^{1}M)$ of $A \in \Im_{1}^{1}(M)$ and the horizontal lifts ${}^{H}X \in \Im_{0}^{1}(T_{1}^{1}M)$ of $X \in \Im_{0}^{1}(M)$ defined by (1.1) and (1.2), respectively.

Proof. *i)*

$$\begin{array}{rcl}
({}^{H}\nabla_{{}^{V}C}{}^{CG}g)({}^{V}A,{}^{V}B) &=& {}^{H}\nabla_{{}^{V}C}{}^{CG}g({}^{V}A,{}^{V}B)-{}^{CG}g({}^{H}\nabla_{{}^{V}C}{}^{V}A,{}^{V}B)\\
&& -{}^{CG}g({}^{V}A,{}^{H}\nabla_{{}^{V}C}{}^{V}B)\\
&=& {}^{H}\nabla_{{}^{V}C}\frac{1}{\alpha}^{V}(G(A,B)+G(A,t)G(B,t))\\
&=& 0
\end{array}$$

ii)

$$\begin{array}{rcl}
({}^{H}\nabla_{{}^{V}C}{}^{CG}g)({}^{V}A,{}^{H}Y) &=& {}^{H}\nabla_{{}^{V}C}{}^{CG}g({}^{V}A,{}^{H}Y)-{}^{CG}g({}^{H}\nabla_{{}^{V}C}{}^{V}A,{}^{H}Y)\\
&& -{}^{CG}g({}^{V}A,{}^{H}\nabla_{{}^{V}C}{}^{H}Y)\\
&=& -{}^{CG}g({}^{V}A,{}^{H}\nabla_{{}^{V}C}{}^{H}Y)\\
&=& 0
\end{array}$$

iii)

$$\begin{array}{rcl}
({}^{H}\nabla_{{}^{V}C}\ {}^{CG}g)({}^{H}X,B^{V}) &=& {}^{H}\nabla_{{}^{V}C}\ {}^{CG}g({}^{H}X,{}^{V}B)-{}^{CG}g({}^{H}\nabla_{{}^{V}C}{}^{H}X,{}^{V}B)\\
&& -{}^{CG}g({}^{H}X,{}^{H}\nabla_{{}^{V}C}{}^{V}B)\\
&=& -{}^{CG}g({}^{H}\nabla_{{}^{V}C}{}^{H}X,{}^{V}B)\\
&=& 0
\end{array}$$

iv)

$$\begin{aligned}
({}^{H}\nabla_{{}^{V}C}{}^{CG}g)({}^{H}X,{}^{H}Y) &= {}^{H}\nabla_{{}^{V}C}{}^{CG}g({}^{H}X,{}^{H}Y) -{}^{CG}g({}^{H}\nabla_{{}^{V}C}{}^{H}X,{}^{H}Y) \\
&\quad -{}^{CG}g({}^{H}X,{}^{H}\nabla_{{}^{V}C}{}^{H}Y) \\
&= {}^{H}\nabla_{{}^{V}C}{}^{V}(g(X,Y)) \\
&= {}^{V}C^{V}(g(X,Y)) \\
&= 0
\end{aligned}$$

v)

$$\begin{aligned}
({}^{H}\nabla_{{}^{H}Z}{}^{CG}g)({}^{V}A,{}^{H}Y) &= {}^{H}\nabla_{{}^{H}Z}{}^{CG}g({}^{V}A,{}^{H}Y) -{}^{CG}g({}^{H}\nabla_{{}^{H}Z}{}^{V}A,{}^{H}Y) \\
&\quad -{}^{CG}g({}^{V}A,{}^{H}\nabla_{{}^{H}Z}{}^{H}Y) \\
&= {}^{CG}g({}^{V}(\nabla_{Z}A),{}^{H}Y) -{}^{CG}g({}^{V}A,{}^{H}(\nabla_{Z}Y)) \\
&= 0
\end{aligned}$$

vi)

$$\begin{aligned}
({}^{H}\nabla_{{}^{H}Z}{}^{CG}g)({}^{H}X,{}^{V}B) &= {}^{H}\nabla_{{}^{H}Z}{}^{CG}g({}^{H}X,{}^{V}B) -{}^{CG}g({}^{H}\nabla_{{}^{H}Z}{}^{H}X,{}^{V}B) \\
&\quad -{}^{CG}g({}^{H}X,{}^{H}\nabla_{{}^{H}Z}{}^{V}B) \\
&= -{}^{CG}g({}^{H}(\nabla_{Z}X),{}^{V}B) -{}^{CG}g({}^{H}X,{}^{V}(\nabla_{Z}B)) \\
&= 0
\end{aligned}$$

vii)

$$\begin{aligned}
({}^{H}\nabla_{{}^{H}Z}{}^{CG}g)({}^{H}X,{}^{H}Y) &= {}^{H}\nabla_{{}^{H}Z}{}^{CG}g({}^{H}X,{}^{H}Y) -{}^{CG}g({}^{H}\nabla_{{}^{H}Z}{}^{H}X,{}^{H}Y) \\
&\quad -{}^{CG}g({}^{H}X,{}^{H}\nabla_{{}^{H}Z}{}^{H}Y) \\
&= {}^{H}\nabla_{{}^{H}Z}{}^{V}(g(X,Y)) -{}^{CG}g({}^{H}(\nabla_{Z}X),{}^{H}Y) \\
&\quad -{}^{CG}g({}^{H}X,{}^{H}(\nabla_{Z}Y)) \\
&= {}^{V}(\nabla_{Z}g(X,Y)) -{}^{V}(g((\nabla_{Z}X),Y)) -{}^{V}(g(X,(\nabla_{Z}Y))) \\
&= {}^{V}((\nabla_{Z}g)(X,Y))
\end{aligned}$$

viii)

$$\begin{aligned}
({}^H\nabla_{{}^H Z}{}^{CG}g)({}^V A,{}^V B) &= {}^H\nabla_{{}^H Z}{}^{CG}g({}^V A,{}^V B) -{}^{CG} g({}^H\nabla_{{}^H Z}{}^V A,{}^V B) \\
&\quad -{}^{CG}g({}^V A,{}^H \nabla_{{}^H Z}{}^V B) \\
&= {}^H\nabla_{{}^H Z}\frac{1}{\alpha}^V (G(A,B)+G(A,t)G(B,t)) \\
&\quad -{}^{CG}g({}^V(\nabla_Z A),{}^V B) -{}^{CG} g({}^V A,{}^V(\nabla_Z B)) \\
&= {}^V(\nabla_Z\frac{1}{\alpha})^V (G(A,B)+G(A,t)G(B,t)) \\
&\quad +\frac{1}{\alpha}^V (\nabla_Z(G(A,B)+G(A,t)G(B,t))) \\
&\quad -\frac{1}{\alpha}^V (G((\nabla_Z A),B)+G((\nabla_Z A),t)G(B,t)) \\
&\quad -\frac{1}{\alpha}^V (G(A,(\nabla_Z B))+G(A,t)G((\nabla_Z B),t)) \\
&= {}^V(\nabla_Z\frac{1}{\alpha})^V (G(A,B)+G(A,t)G(B,t)) \\
&\quad +\frac{1}{\alpha}^V ((\nabla_Z G)(A,B))+\frac{1}{\alpha}^V (\nabla_Z(G(A,t)G(B,t))) \\
&\quad -\frac{1}{\alpha}^V (G((\nabla_Z A),t)G(B,t))-\frac{1}{\alpha}^V (G(A,t)G((\nabla_Z B),t))
\end{aligned}$$

□

Definition 2.3. Let M be an $n-$dimensional differentiable manifold. Differential transformation $D = L_X$ is called as Lie derivation with respect to vector field $X \in \Im_0^1(M)$ if

$$\begin{aligned}
L_X f &= Xf, \forall f \in \Im_0^0(M), \\
L_X Y &= [X,Y], \forall X,Y \in \Im_0^1(M).
\end{aligned} \tag{2.3}$$

$[X,Y]$ is called by Lie bracked. The Lie derivative $L_X F$ of a tensor field F of type $(1,1)$ with respect to a vector field X is defined by [5, 6, 12, 18]

$$(L_X F)Y = [X,FY] - F[X,Y]. \tag{2.4}$$

Definition 2.4. The bracket operation of vertical and horizontal vector fields is given by the formulas

$$\begin{cases} [{}^V A,{}^V B] = 0\ , \\ [{}^H X,{}^V A] = {}^V(\nabla_X A), \\ [{}^H X,{}^H Y] = {}^H[X,Y]+(\tilde{\gamma}-\gamma)R(X,Y), \end{cases} \tag{2.5}$$

where R denotes the curvature tensor field of the connection ∇, and $\tilde{\gamma}-\gamma : \varphi \to \Im_0^1(T_1^1(M))$ is the operator defined by

$$(\tilde{\gamma}-\gamma)\varphi = \begin{pmatrix} 0 \\ t_m^i\varphi_j^m - t_j^m\varphi_m^i \end{pmatrix}$$

for any $\varphi \in \Im_1^1(M)$ [17].

Theorem 2.2. *Let ${}^{CG}g$ be the Cheeger-Gromoll type metric ${}^{CG}g$ defined by (1.5),(1.6),(1.7) and L_X the operator Lie derivation with respect to X. From Definintion 2.3 and Definintion 2.4, we get the following results*

$$\begin{aligned}
i)\ (L_{{}^VC}{}^{CG}g)({}^VA, {}^VB) &= 0\\
ii)\ (L_{{}^VC}{}^{CG}g)({}^HX, {}^HY) &= 0\\
iii)\ (L_{{}^HZ}{}^{CG}g)({}^VA, {}^HY) &= -{}^{CG}g({}^VA, (\tilde{\gamma}-\gamma)R(Z,Y))\\
iv)\ (L_{{}^HZ}{}^{CG}g)({}^HX, {}^VB) &= -{}^{CG}g((\tilde{\gamma}-\gamma)R(Z,X), {}^VB)\\
v)\ (L_{{}^VC}{}^{CG}g)({}^VA, {}^HY) &= \frac{1}{\alpha}^V(G(A,(\nabla_Y C)) + G(A,t)G((\nabla_Y C),t))\\
vi)\ (L_{{}^VC}{}^{CG}g)({}^HX, {}^VB) &= \frac{1}{\alpha}^V(G((\nabla_X C),B) + G((\nabla_X C),t)G(B,t))\\
vii)\ (L_{{}^HZ}{}^{CG}g)({}^HX, {}^HY) &= {}^V((L_Z g)(X,Y)) - {}^{CG}g((\tilde{\gamma}-\gamma)R(Z,X), {}^HY)\\
&\quad -{}^{CG}g({}^HX, (\tilde{\gamma}-\gamma)R(Z,Y))\\
viii)\ (L_{{}^HZ}{}^{CG}g)({}^VA, {}^VB) &= {}^V(\nabla_Z \frac{1}{\alpha})^V(G(A,B) + G(A,t)G(B,t))\\
&\quad +\frac{1}{\alpha}^V((\nabla_Z G)(A,B)) + \frac{1}{\alpha}^V(\nabla_Z(G(A,t)G(B,t)))\\
&\quad -\frac{1}{\alpha}^V(G(A,t)G((\nabla_Z B),t))\\
&\quad -\frac{1}{\alpha}^V(G((\nabla_Z A),t)G(B,t))
\end{aligned}$$

where the vertical lift ${}^VA \in \Im^1_0(T^1_1M)$ of $A \in \Im^1_1(M)$ and the horizontal lifts ${}^HX \in \Im^1_0(T^1_1M)$ of $X \in \Im^1_0(M)$ defined by (1.1) and (1.2), respectively.

Proof. $i)$

$$\begin{aligned}
(L_{{}^VC}{}^{CG}g)({}^VA, {}^VB) &= L_{{}^VC}{}^{CG}g({}^VA, {}^VB) - {}^{CG}g(L_{{}^VC}{}^VA, {}^VB) - {}^{CG}g({}^VA, L_{{}^VC}{}^VB)\\
&= 0
\end{aligned}$$

$ii)$

$$\begin{aligned}
(L_{{}^VC}{}^{CG}g)({}^HX, {}^HY) &= L_{{}^VC}{}^{CG}g({}^HX, {}^HY) - {}^{CG}g(L_{{}^VC}{}^HX, {}^HY) - {}^{CG}g({}^HX, L_{{}^VC}{}^HY)\\
&= L_{{}^VC}{}^V(g(X,Y)) + {}^{CG}g({}^V(\nabla_X C), {}^HY) + {}^{CG}g({}^HX, {}^V(\nabla_Y C))\\
&= {}^VC^V(g(X,Y))\\
&= 0
\end{aligned}$$

$iii)$

$$\begin{aligned}
(L_{{}^HZ}{}^{CG}g)({}^VA, {}^HY) &= L_{{}^HZ}{}^{CG}g({}^VA, {}^HY) - {}^{CG}g(L_{{}^HZ}{}^VA, {}^HY) - {}^{CG}g({}^VA, L_{{}^HZ}{}^HY)\\
&= -{}^{CG}g({}^V(\nabla_Z A), {}^HY) - {}^{CG}g({}^VA, {}^H[Z,Y] + (\tilde{\gamma}-\gamma)R(Z,Y))\\
&= -{}^{CG}g({}^VA, {}^H(L_Z Y)) - {}^{CG}g({}^VA, (\tilde{\gamma}-\gamma)R(Z,Y))\\
&= -{}^{CG}g({}^VA, (\tilde{\gamma}-\gamma)R(Z,Y))
\end{aligned}$$

$iv)$

$$\begin{aligned}
(L_{{}^HZ}{}^{CG}g)({}^HX, {}^VB) &= L_{{}^HZ}{}^{CG}g({}^HX, {}^VB) - {}^{CG}g(L_{{}^HZ}{}^HX, {}^VB) - {}^{CG}g({}^HX, L_{{}^HZ}{}^VB)\\
&= -{}^{CG}g({}^H[Z,X] + (\tilde{\gamma}-\gamma)R(Z,X), {}^VB) - {}^{CG}g({}^HX, {}^V(\nabla_Z B))\\
&= -{}^{CG}g((\tilde{\gamma}-\gamma)R(Z,X), {}^VB)
\end{aligned}$$

v)

$$\begin{aligned}
(L_{^VC}{}^{CG}g)(^VA,^HY) &= L_{^VC}{}^{CG}g(^VA,^HY) -^{CG} g(L_{^VC}{}^VA,^HY) -^{CG} g(^VA, L_{^VC}{}^HY) \\
&= {}^{CG}g(^VA,^V(\nabla_Y C)) \\
&= \frac{1}{\alpha}^V (G(A,(\nabla_Y C)) + G(A,t)G((\nabla_Y C),t)))
\end{aligned}$$

vi)

$$\begin{aligned}
(L_{^VC}{}^{CG}g)(^HX,^VB) &= L_{^VC}\,{}^{CG}g(^HX,^VB) -^{CG} g(L_{^VC}{}^HX,^VB) -^{CG} g(^HX, L_{^VC}{}^VB) \\
&= +^{CG}g(^V(\nabla_X C),^VB) \\
&= \frac{1}{\alpha}^V (G((\nabla_X C),B) + G((\nabla_X C),t)G(B,t))
\end{aligned}$$

vii)

$$\begin{aligned}
(L_{^HZ}\,{}^{CG}g)(^HX,^HY) &= L_{^HZ}\,{}^{CG}g(^HX,^HY) -^{CG} g(L_{^HZ}{}^HX,^HY) -^{CG} g(^HX, L_{^HZ}{}^HY) \\
&= {}^HZ^V(g(X,Y)) -^{CG} g(^H[Z,X] + (\tilde{\gamma}-\gamma)R(Z,X),^HY) \\
&\quad -^{CG}g(^HX,^H[Z,Y] + (\tilde{\gamma}-\gamma)R(Z,Y)) \\
&= {}^V(L_Z g(X,Y)) -^V (g((L_Z X),Y)) -^V (g(X,(L_Z Y))) \\
&\quad -^{CG}g((\tilde{\gamma}-\gamma)R(Z,X),^HY) -^{CG} g(^HX,(\tilde{\gamma}-\gamma)R(Z,Y)) \\
&= {}^V((L_Z g)(X,Y)) -^{CG} g((\tilde{\gamma}-\gamma)R(Z,X),^HY) \\
&\quad -^{CG}g(^HX,(\tilde{\gamma}-\gamma)R(Z,Y))
\end{aligned}$$

viii)

$$\begin{aligned}
(L_{^HZ}\,{}^{CG}g)(^VA,^VB) &= L_{^HZ}\,{}^{CG}g(^VA,^VB) -^{CG} g(L_{^HZ}{}^VA,^VB) -^{CG} g(^VA, L_{^HZ}{}^VB) \\
&= {}^HZ(\frac{1}{\alpha}^V (G(A,B) + G(A,t)G(B,t)))^{CG}g(^V(\nabla_Z A),^VB) \\
&\quad -^{CG}g(^VA,^V(\nabla_Z B) \\
&= {}^V(\nabla_Z \frac{1}{\alpha})^V (G(A,B) + G(A,t)G(B,t)) \\
&\quad +\frac{1}{\alpha}^V (\nabla_Z (G(A,B) + G(A,t)G(B,t))) \\
&\quad -\frac{1}{\alpha}^V (G((\nabla_Z A),B) + G((\nabla_Z A),t)G(B,t)) \\
&\quad -\frac{1}{\alpha}^V (G(A,(\nabla_Z B)) + G(A,t)G((\nabla_Z B),t)) \\
&= {}^V(\nabla_Z \frac{1}{\alpha})^V (G(A,B) + G(A,t)G(B,t)) + \frac{1}{\alpha}^V ((\nabla_Z G)(A,B)) \\
&\quad +\frac{1}{\alpha}^V (\nabla_Z (G(A,t)G(B,t))) - \frac{1}{\alpha}^V (G((\nabla_Z A),t)G(B,t)) \\
&\quad -\frac{1}{\alpha}^V (G(A,t)G((\nabla_Z B),t))
\end{aligned}$$

$\square$

References

[1] Akyol, M. A., Sarı, R. and Aksoy, E., Semi-invariant -Riemannian submersions from almost contact metric manifolds, Int. J. Geom. Methods Mod. Phys. 14, 175007 4 (2017) DOI: http://dx.doi.org/10.1142/S0219887817500748.

[2] Akyol, M. A., Conformal anti-invariant submersions from cosymplectic manifolds, Hacet. J. Math. Stat. 46(2017), no.2, 177-192.

[3] Çakmak, A. and Tarakcı, Ö., Surfaces at a constant distance from the edge of regression on a surface of revolution in . Applied Mathematical Sciences, 10(2016), no.15, 707-719.

[4] Çakmak, A., Karacan, M.K., Kiziltug, S. and Yoon, D.W., Translatıon surfaces in the 3-dımensıonal Galılean space satısfyıng . Bull. Korean Math. Soc. https://doi.org/10.4134/BKMS.b160442.

[5] Çayır, H. and Akdağ, K., Some notes on almost paracomplex structures associated with the diagonal lifts and operators on cotangent bundle, New Trends in Mathematical Sciences, 4(2016), no.4, 42-50.

[6] Çayır, H. and Köseoğlu, G., Lie Derivatives of Almost Contact Structure and Almost Paracontact Structure With Respect to X^C and X^V on Tangent Bundle $T(M)$, New Trends in Mathematical Sciences, 4(2016), no.1, 153-159.

[7] Cengiz, N. and Salimov, A. A., Complete lifts of derivations to tensor bundles, Bol. Soc. Mat. Mexicana (3) 8(2002), no.1, 75–82.

[8] Gancarzewicz, J. and Rahmani, N., Relevent horizontal des connexions linearies au fibre vectoriel associe avec le fibre principal des repres lineaires, Annales Polinici Math., 48(1988), 281-289.

[9] Gezer, A. and Altunbas, M., On the $(1,1)-$ tensor bundle with Cheeger-Grommol type metric, Proc. Indian Acad. Sci.(Math Sci.) 125(2015), no.4, 569-576.

[10] Gündüzalp, Y., Slant submersions from almost paracontact Riemannian manifolds, product Riemannian manifolds, Kuwait Journal of Science, 42(2015), no.1, 17-29.

[11] Gündüzalp, Y., Semi-slant submersions from almost product Riemannian manifolds, DEMONSTRATIO MATHEMATICA, 49(2016), no.4.

[12] Khan, M. N. I., and Jun, J.B., Lorentzian Almost r-para-contact Structure in Tangent Bundle, Journal of the Chungcheong Mathematical Society, 27(2014), no.1, 29-34.

[13] Kobayashi, S. and Nomizu, K., Foundations of Differential Geometry-Volume I, John Wiley & Sons, Inc, New York, 1963.

[14] Lai, K. F. and Mok, K. P., On the differential geometry of the $(1,1)-$ tensor bundle, Tensor (New Series), 63(2002), no.1, 15-27.

[15] Ledger, A. J. and Yano, K., Almost complex structures on the tensor bundles, J. Diff. Geom., 1(1967), 355-368.

[16] Salimov, A.A., Tensor Operators and Their applications, Nova Science Publ., New York, 2013.

[17] Salimov, A. and Gezer, A., On the geometry of the (1,1) -tensor bundle with Sasaki type metric, Chin. Ann. Math. Ser. B 32(2011), no.3, 369–386.

[18] Yano, K. and Ishihara, S., Tangent and Cotangent Bundles, Marcel Dekker, New York, 1973.

Department of Mathematics, Faculty of Arts and Sciences,, Giresun University, 28100, Giresun, Turkey.

E-mail address: `hasim.cayir@giresun.edu.tr`

Department of Computer Science, College of Computer, Qassim University,, Buraidah-51452, P.O. Box 6688, Saudi Arabia.

E-mail address: `m.nazrul@qu.edu.sa`