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Solvability of a Second-Order Rational System of Difference Equations

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Abstract

In this paper, we represent the admissible solutions of the system of second-order rational difference equations given below in terms of Lucas and Fibonacci sequences:

$$x_{n+1} = \frac{L_{m+2}+L_{m+1}y_{n-1}}{L_{m+3}+L_{m+2}y_{n-1}}, \quad y_{n+1} = \frac{L_{m+2}+L_{m+1}z_{n-1}}{L_{m+3}+L_{m+2}z_{n-1}},$$
$$z_{n+1} = \frac{L_{m+2}+L_{m+1}w_{n-1}}{L_{m+3}+L_{m+2}w_{n-1}}, \quad w_{n+1} = \frac{L_{m+2}+L_{m+1}x_{n-1}}{L_{m+3}+L_{m+2}x_{n-1}}.$$

where $n \in \mathbb{N}_0$, $\{L_m\}_{m=-\infty}^{+\infty}$ is Lucas sequence and the initial conditions x_{-1}, x_0, y_{-1}, y_0, z_{-1}, z_0, w_{-1}, w_0 are arbitrary real numbers such that $v_{-i} \neq -\dfrac{L_{m+3}}{L_{m+2}}$, where $v_{-i} = x_{-i}, y_{-i}, z_{-i}, w_{-i}$, $i = 0, 1$ and $m \in \mathbb{Z}$.

1. Introduction and preliminaries

Recently, there has been a growing interest in the study of finding closed-form solutions of difference equations and systems of difference equations. Some of the forms of solutions of these equations are representable via well-known integer sequences such as Fibonacci numbers [1, 2], Horadam numbers [3], Lucas numbers [4, 5], and Padovan numbers [6]. For more on Fibonacci and Lucas numbers, one can see [7, 8], for more on difference equations and systems of difference equations solvable in closed form, one can see [9]-[24].

The Lucas sequence has the same recursive relationship as the Fibonacci sequence, where each term is the sum of the two previous terms, and defined as follows:

$$L_{n+1} = L_n + L_{n-1}, \quad n \geq 1, \tag{1.1}$$

but with different initial values , $L_0 = 2$, $L_1 = 1$. The solution of Equation (1.1) is given by the formula

$$L_n = \alpha^n + \beta^n,$$

where

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \textit{and} \quad \beta = \frac{1-\sqrt{5}}{2}.$$

The formula of terms with negative indices in the Lucas sequence is

$$L_{-n} = (-1)^n L_n.$$

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In [25], the authors represented the general solution of the following difference equation

$$x_{n+1} = \frac{1}{1+x_n}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

in terms of the initial value x_0 and Fibonacci sequence. It was proved by induction that, every well-defined solution of equation (1.2) can be written in the following form

$$x_n = \frac{F_n + F_{n-1}x_0}{F_{n+1} + F_n x_0}, \quad n \in \mathbb{N}_0,$$

where $\{F_n\}_{n=0}^{\infty}$ is Fibonacci sequence. They also proved that, every well-defined solution of the equation

$$x_{n+1} = \frac{1}{-1+x_n}, \quad n \in \mathbb{N}_0, \tag{1.3}$$

can be written in the following form

$$x_n = \frac{F_{-n} + F_{-(n-1)}x_0}{F_{-(n+1)} + F_{-n}x_0}, \quad n \in \mathbb{N}_0,$$

where the terms of the Fibonacci sequence with negative indices are calculated by the formula

$$F_{-n} = F_{-n+2} - F_{-n+1}, \quad n \in \mathbb{N}_0,$$

where $F_0 = 0$ and $F_1 = 1$.

Khelifa et al. [5] give some theoretical explanations related to the representation of the general solution to the system of three higher-order rational difference equations

$$x_{n+1} = \frac{1+2y_{n-k}}{3+y_{n-k}}, \quad y_{n+1} = \frac{1+2z_{n-k}}{3+z_{n-k}}, \quad z_{n+1} = \frac{1+2x_{n-k}}{3+x_{n-k}},$$

where $n,k \in \mathbb{N}_0$, giving its solution in terms of Fibonacci and Lucas sequences.

Recently in Khelifa et al. [4], the following higher-order rational difference equations

$$x^{(1)}_{n+1} = \frac{1+2x^{(2)}_{n-k}}{3+x^{(2)}_{n-k}}, \; x^{(2)}_{n+1} = \frac{1+2x^{(3)}_{n-k}}{3+x^{(3)}_{n-k}}, \cdots, \; x^{(2p+1)}_{n+1} = \frac{1+2x^{(1)}_{n-k}}{3+x^{(1)}_{n-k}},$$

in terms of Fibonacci and Lucas sequences, where the initial values $x^{(i)}_{-k}$, $x^{(i)}_{-k+1}, \cdots, x^{(i)}_{-1}$ and $x^{(i)}_0$, $i = 1,2,\cdots,2p+1$ are real numbers such that, the denominator does not equal zero in each equation. Some theoretical explanations related to the representation of the general solution are also given.

Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \cdots, x_{n-k}), \; n = 0,1,\cdots. \tag{1.4}$$

The Good set to Equation (1.4) is the set of all initial points $(x_0, x_{-1}, ..., x_{-k})$ for which the corresponding solution $\{x_n\}_{n=-k}^{\infty}$ is well-defined or admissible solution.

Here, we list a set of identities concerning the Fibonacci and Lucas sequences that may be used in the paper [7, 8].

For $s,m,r,\theta \in \mathbb{N}$, we have

1. $F_m = F_{s+1}F_{m-s} + F_sF_{m-(s+1)}$,
2. $L_m = F_{s+1}L_{m-s} + F_sL_{m-(s+1)}$,
3. $F_sL_{m+3} + F_{s-1}L_{m+2} = L_{s+m+2}$,
4. $L_rL_{(\theta-1)r+1} + L_{r-1}L_{(\theta-1)r} = 5F_{\theta r}$,
5. $L_{r+1}L_{(\theta-1)r} + L_rL_{(\theta-1)r-1} = 5F_{\theta r}$,
6. $L_{r+1}L_{(\theta-1)r+1} + L_rL_{(\theta-1)r} = 5F_{\theta r+1}$,
7. $L_rL_{(\theta-1)r} + L_{r-1}L_{(\theta-1)r-1} = 5F_{\theta r-1}$,
8. $L_{\theta(m+2)-1} + L_{\theta(m+2)+1} = 5F_{\theta(m+2)}$.

Now, consider the system of second-order rational difference equations

$$\begin{aligned} x_{n+1} &= \frac{L_{m+2} + L_{m+1}y_{n-1}}{L_{m+3} + L_{m+2}y_{n-1}}, \quad y_{n+1} = \frac{L_{m+2} + L_{m+1}z_{n-1}}{L_{m+3} + L_{m+2}z_{n-1}}, \\ z_{n+1} &= \frac{L_{m+2} + L_{m+1}w_{n-1}}{L_{m+3} + L_{m+2}w_{n-1}}, \quad w_{n+1} = \frac{L_{m+2} + L_{m+1}x_{n-1}}{L_{m+3} + L_{m+2}x_{n-1}}, \end{aligned} \tag{1.5}$$

where $\{L_m\}_{m=-\infty}^{+\infty}$ is Lucas sequence and the initial conditions x_{-1}, x_0, y_{-1}, y_0, z_{-1},z_0 and w_{-1},w_0 are arbitrary real numbers such that $v_{-i} \neq -\frac{L_{m+3}}{L_{m+2}}$, where $v_{-i} = x_{-i}, y_{-i}, z_{-i}, w_{-i}$, $i = 0,1$ and $m \in \mathbb{Z}$.

In this paper, we shall represent the admissible solutions of the system (1.5) in terms of Fibonacci and Lucas sequences.

2. Solvability of system (1.5)

In this section, we investigate the solvability of the system (1.5).

From (1.5), we can write for $t = 0,1$

$$x_{2(n+1)-t} = \frac{L_{m+2} + L_{m+1} y_{2n-t}}{L_{m+3} + L_{m+2} y_{2n-t}}, \quad y_{2(n+1)-t} = \frac{L_{m+2} + L_{m+1} z_{2n-t}}{L_{m+3} + L_{m+2} z_{2n-t}},$$
$$z_{2(n+1)-t} = \frac{L_{m+2} + L_{m+1} w_{2n-t}}{L_{m+3} + L_{m+2} w_{2n-t}}. \quad w_{2(n+1)-t} = \frac{L_{m+2} + L_{m+1} x_{2n-t}}{L_{m+3} + L_{m+2} x_{2n-t}}.$$

Let

$$x'_n = x_{2n-t},\ y'_n = y_{2n-t},\ z'_n = z_{2n-t},\ w'_n = w_{2n-t}, \tag{2.1}$$

where $t = 0,1$.

Then, the system (1.5) becomes

$$\begin{aligned} x'_{n+1} &= \frac{L_{m+2} + L_{m+1} y'_n}{L_{m+3} + L_{m+2} y'_n}, \quad y'_{n+1} = \frac{L_{m+2} + L_{m+1} z'_n}{L_{m+3} + L_{m+2} z'_n}, \\ z'_{n+1} &= \frac{L_{m+2} + L_{m+1} w'_n}{L_{m+3} + L_{m+2} w'_n}, \quad w'_{n+1} = \frac{L_{m+2} + L_{m+1} x'_n}{L_{m+3} + L_{m+2} x'_n}. \end{aligned} \tag{2.2}$$

If we use the second recurrence relation in (2.2) in the first, we obtain

$$x'_{n+1} = \frac{F_{2m+4} + F_{2m+3} z'_{n-1}}{F_{2m+5} + F_{2m+4} z'_{n-1}}, \quad n \geq 1.$$

The substitution of z'_{n-1} into x'_{n+1}, leads to

$$x'_{n+1} = \frac{L_{3m+6} + L_{3m+5} w'_{n-2}}{L_{3m+7} + L_{3m+6} w'_{n-2}}, \quad n \geq 2.$$

Finally, after substituting with w'_{n-2} into x'_{n+1}, we get

$$x'_{n+1} = \frac{F_{4m+8} + F_{4m+7} x'_{n-3}}{F_{4m+9} + F_{4m+8} x'_{n-3}}, \quad n \geq 3.$$

Therefore, the system (2.2) can be written in the following form:

$$x'_{n+1} = \frac{F_{4m+8} + F_{4m+7} x'_{n-3}}{F_{4m+9} + F_{4m+8} x'_{n-3}}, \quad n \geq 3. \tag{2.3}$$

Let us introduce the notation

$$x_n^{\prime(j)} = x'_{4n+j}, \quad n \in \mathbb{N}_0, \tag{2.4}$$

where $j \in \{0,1,2,3\}$.

Using this notation, Equation (2.3) can be written as

$$x_{n+1}^{\prime(j)} = \frac{F_{4m+8} + F_{4m+7} x_n^{\prime(j)}}{F_{4m+9} + F_{4m+8} x_n^{\prime(j)}}, \quad j \in \{0,1,2,3\} \text{ and } n \geq 3. \tag{2.5}$$

Now consider the equation

$$u_{n+1} = \frac{F_{4m+8} + F_{4m+7} u_n}{F_{4m+9} + F_{4m+8} u_n}, \quad n \geq 3. \tag{2.6}$$

The solution of Equation (2.6) is (can be found in [20])

$$u_n = \frac{F_{4n(m+2)} + F_{4n(m+2)-1}u_0}{F_{4n(m+2)+1} + F_{4n(m+2)}u_0},\ n \in \mathbb{N}_0,$$

where $(F_m)_{n=-\infty}^{+\infty}$ is Fibonacci sequence.

Then the solution of Equation (2.5) is given by

$$x_n'^{(j)} = \frac{F_{4n(m+2)} + F_{4n(m+2)-1}x_0'^{(j)}}{F_{4n(m+2)+1} + F_{4n(m+2)}x_0'^{(j)}},\quad j \in \{0,1,2,3\} \text{ and } n \in \mathbb{N}_0.$$

Therefore, the solution of Equation (2.5) can be written as

$$x'_{4n+j} = \frac{F_{4n(m+2)} + F_{4n(m+2)-1}x'_j}{F_{4n(m+2)+1} + F_{4n(m+2)}x'_j},\quad j \in \{0,1,2,3\} \text{ and } n \in \mathbb{N}_0.$$

Theorem 2.1. *Let $(x'_n, y'_n, z'_n, w'_n)_{n\geq 0}$ be an admissible solution of the system (2.2). Then we get*

$$\begin{aligned}
x'_{4n} &= \frac{F_{4n(m+2)} + F_{4n(m+2)-1}x'_0}{F_{4n(m+2)+1} + F_{4n(m+2)}x'_0}, & z'_{4n} &= \frac{F_{4n(m+2)} + F_{4n(m+2)-1}z'_0}{F_{4n(m+2)+1} + F_{4n(m+2)}z'_0},\\
x'_{4n+1} &= \frac{L_{4n(m+2)+(m+2)} + L_{4n(m+2)+(m+1)}y'_0}{L_{4n(m+2)+(m+3)} + L_{4n(m+2)+(m+2)}y'_0}, & z'_{4n+1} &= \frac{L_{4n(m+2)+(m+2)} + L_{4n(m+2)+(m+1)}w'_0}{L_{4n(m+2)+(m+3)} + L_{4n(m+2)+(m+2)}w'_0},\\
x'_{4n+2} &= \frac{F_{4n(m+2)+(2m+4)} + F_{4n(m+2)+(2m+3)}z'_0}{F_{4n(m+2)+(2m+5)} + F_{4n(m+2)+(2m+4)}z'_0}, & z'_{4n+2} &= \frac{F_{4n(m+2)+(2m+4)} + F_{4n(m+2)+(2m+3)}x'_0}{F_{4n(m+2)+(2m+5)} + F_{4n(m+2)+(2m+4)}x'_0},\\
x'_{4n+3} &= \frac{L_{4n(m+2)+(3m+6)} + L_{4n(m+2)+(3m+5)}w'_0}{L_{4n(m+2)+(3m+7)} + L_{4n(m+2)+(3m+6)}w'_0}, & z'_{4n+3} &= \frac{L_{4n(m+2)+(3m+6)} + L_{4n(m+2)+(3m+5)}y'_0}{L_{4n(m+2)+(3m+7)} + L_{4n(m+2)+(3m+6)}y'_0},\\
y'_{4n} &= \frac{F_{4n(m+2)} + F_{4n(m+2)-1}y'_0}{F_{4n(m+2)+1} + F_{4n(m+2)}y'_0}, & w'_{4n} &= \frac{F_{4n(m+2)} + F_{4n(m+2)-1}w'_0}{F_{4n(m+2)+1} + F_{4n(m+2)}w'_0},\\
y'_{4n+1} &= \frac{L_{4n(m+2)+(m+2)} + L_{4n(m+2)+(m+1)}z'_0}{L_{4n(m+2)+(m+3)} + L_{4n(m+2)+(m+2)}z'_0}, & w'_{4n+1} &= \frac{L_{4n(m+2)+(m+2)} + L_{4n(m+2)+(m+1)}x'_0}{L_{4n(m+2)+(m+3)} + L_{4n(m+2)+(m+2)}x'_0},\\
y'_{4n+2} &= \frac{F_{4n(m+2)+(2m+4)} + F_{4n(m+2)+(2m+3)}w'_0}{F_{4n(m+2)+(2m+5)} + F_{4n(m+2)+(2m+4)}w'_0}, & w'_{4n+2} &= \frac{F_{4n(m+2)+(2m+4)} + F_{4n(m+2)+(2m+3)}y'_0}{F_{4n(m+2)+(2m+5)} + F_{4n(m+2)+(2m+4)}y'_0},\\
y'_{4n+3} &= \frac{L_{4n(m+2)+(3m+6)} + L_{4n(m+2)+(3m+5)}x'_0}{L_{4n(m+2)+(3m+7)} + L_{4n(m+2)+(3m+6)}x'_0}, & w'_{4n+3} &= \frac{L_{4n(m+2)+(3m+6)} + L_{4n(m+2)+(3m+5)}z'_0}{L_{4n(m+2)+(3m+7)} + L_{4n(m+2)+(3m+6)}z'_0},
\end{aligned} \tag{2.7}$$

where $n \in \mathbb{N}_0$, $(L_m)_{m=-\infty}^{+\infty}$ is Lucas sequence and $(F_m)_{m=-\infty}^{+\infty}$ is Fibonacci sequence.

Proof. Let $(x'_n, y'_n, z'_n, w'_n)_{n\geq 0}$ be a solution to system (2.2). Then, $(x'_n)_{n\geq 0}$ is a solution to Equation (2.5) and so

$$x'_{4n+j} = \frac{F_{4n(m+2)} + F_{4n(m+2)-1}x'_j}{F_{4n(m+2)+1} + F_{4n(m+2)}x'_j},$$

where $m \in \mathbb{Z}$, $j \in \{0,1,2,3\}$. For $j = 0$, we have

$$x'_{4n} = \frac{F_{4n(m+2)} + F_{4n(m+2)-1}x'_0}{F_{4n(m+2)+1} + F_{4n(m+2)}x'_0}.$$

We also have

$$x'_{4n+1} = \frac{F_{4n(m+2)} + F_{4n(m+2)-1}x'_1}{F_{4n(m+2)+1} + F_{4n(m+2)}x'_1},$$

where $x'_1 = \dfrac{L_{m+2} + L_{m+1}y'_0}{L_{m+3} + L_{m+2}y'_0}$.

Using identity (2), we get

$$x'_{4n+1} = \frac{L_{4n(m+2)+(m+2)} + L_{4n(m+2)+(m+1)}y'_0}{L_{4n(m+2)+(m+3)} + L_{4n(m+2)+(m+2)}y'_0}.$$

Similarly,

$$x'_{4n+2}=\frac{F_{4n(m+2)}+F_{4n(m+2)-1}x'_2}{F_{4n(m+2)+1}+F_{4n(m+2)}x'_2},$$

where $x'_2=\dfrac{F_{2m+4}+F_{2m+3}z'_0}{F_{2m+5}+F_{2m+4}z'_0}$.

Using identity (1), we get

$$x'_{4n+2}=\frac{F_{4n(m+2)+2m+4}+F_{4n(m+2)+2m+3)}z'_0}{F_{4n(m+2)+2m+5}+F_{4n(m+2)+2m+4}z'_0}.$$

Finally, for $j=3$, we have

$$x'_{4n+3}=\frac{F_{4n(m+2)}+F_{4n(m+2)-1}x'_3}{F_{4n(m+2)+1}+F_{4n(m+2)}x'_3},$$

where $x'_1=\dfrac{L_{3m+6}+L_{3m+5}w'_0}{L_{3m+7}+L_{3m+6}w'_0}$.

Again using identity (2), we get

$$x'_{4n+3}=\frac{L_{4n(m+2)+3m+6}+L_{4n(m+2)+3m+5}w'_0}{L_{4n(m+2)+3m+7}+L_{4n(m+2)+3m+6}w'_0}.$$

Then

$$\begin{aligned}
x'_{4n} &= \frac{F_{4n(m+2)}+F_{4n(m+2)-1}x'_0}{F_{4n(m+2)+1}+F_{4n(m+2)}x'_0},\\
x'_{4n+1} &= \frac{L_{4n(m+2)+(m+2)}+L_{4n(m+2)+(m+1)}y'_0}{L_{4n(m+2)+(m+3)}+L_{4n(m+2)+(m+2)}y'_0},\\
x'_{4n+2} &= \frac{F_{4n(m+2)+(2m+4)}+F_{4n(m+2)+(2m+3)}z'_0}{F_{4n(m+2)+(2m+5)}+F_{4n(m+2)+(2m+4)}z'_0},\\
x'_{4n+3} &= \frac{L_{4n(m+2)+(3m+6)}+L_{4n(m+2)+(3m+5)}w'_0}{L_{4n(m+2)+(3m+7)}+L_{4n(m+2)+(3m+6)}w'_0}.
\end{aligned}$$

In the same way, after some calculations and using the fact that

$$y'_n=\frac{L_{m+2}+L_{m+1}z'_{n-1}}{L_{m+3}+L_{m+2}z'_{n-1}},\quad z'_n=\frac{L_{m+2}+L_{m+1}w'_{n-1}}{L_{m+3}+L_{m+2}w'_{n-1}},w'_n=\frac{L_{m+2}+L_{m+1}x'_{n-1}}{L_{m+3}+L_{m+2}x'_{n-1}}.$$

we find

$$\begin{aligned}
y'_{4n} &= \frac{F_{4n(m+2)}+F_{4n(m+2)-1}y'_0}{F_{4n(m+2)+1}+F_{4n(m+2)}y'_0}, & z'_{4n} &= \frac{F_{4n(m+2)}+F_{4n(m+2)-1}z'_0}{F_{4n(m+2)+1}+F_{4n(m+2)}z'_0},\\
y'_{4n+1} &= \frac{L_{4n(m+2)+(m+2)}+L_{4n(m+2)+(m+1)}z'_0}{L_{4n(m+2)+(m+3)}+L_{4n(m+2)+(m+2)}z'_0}, & z'_{4n+1} &= \frac{L_{4n(m+2)+(m+2)}+L_{4n(m+2)+(m+1)}w'_0}{L_{4n(m+2)+(m+3)}+L_{4n(m+2)+(m+2)}w'_0},\\
y'_{4n+2} &= \frac{F_{4n(m+2)+(2m+4)}+F_{4n(m+2)+(2m+3)}w'_0}{F_{4n(m+2)+(2m+5)}+F_{4n(m+2)+(2m+4)}w'_0}, & z'_{4n+2} &= \frac{F_{4n(m+2)+(2m+4)}+F_{4n(m+2)+(2m+3)}x'_0}{F_{4n(m+2)+(2m+5)}+F_{4n(m+2)+(2m+4)}x'_0},\\
y'_{4n+3} &= \frac{L_{4n(m+2)+(3m+6)}+L_{4n(m+2)+(3m+5)}x'_0}{L_{4n(m+2)+(3m+7)}+L_{4n(m+2)+(3m+6)}x'_0}, & z'_{4n+3} &= \frac{L_{4n(m+2)+(3m+6)}+L_{4n(m+2)+(3m+5)}y'_0}{L_{4n(m+2)+(3m+7)}+L_{4n(m+2)+(3m+6)}y'_0},
\end{aligned}$$

$$\begin{aligned}
w'_{4n} &= \frac{F_{4n(m+2)}+F_{4n(m+2)-1}w'_0}{F_{4n(m+2)+1}+F_{4n(m+2)}w'_0},\\
w'_{4n+1} &= \frac{L_{4n(m+2)+(m+2)}+L_{4n(m+2)+(m+1)}x'_0}{L_{4n(m+2)+(m+3)}+L_{4n(m+2)+(m+2)}x'_0},\\
w'_{4n+2} &= \frac{F_{4n(m+2)+(2m+4)}+F_{4n(m+2)+(2m+3)}y'_0}{F_{4n(m+2)+(2m+5)}+F_{4n(m+2)+(2m+4)}y'_0},\\
w'_{4n+3} &= \frac{L_{4n(m+2)+(3m+6)}+L_{4n(m+2)+(3m+5)}z'_0}{L_{4n(m+2)+(3m+7)}+L_{4n(m+2)+(3m+6)}z'_0}.
\end{aligned}$$

□

The following theorem is our main result that shows the solvability of the system (1.5).

Theorem 2.2. *Let* $\{x_n, y_n, z_n, w_n\}_{n\geq -1}$ *be an admissible solution of system (1.5). Then for* $n \in \mathbb{N}$, *we get*

$$x_{8n-1} = \frac{F_{4n(m+2)} + F_{4n(m+2)-1}x_{-1}}{F_{4n(m+2)+1} + F_{4n(m+2)}x_{-1}}, \quad x_{8n+3} = \frac{F_{4n(m+2)+(2m+4)} + F_{4n(m+2)+(2m+3)}z_{-1}}{F_{4n(m+2)+(2m+5)} + F_{4n(m+2)+(2m+4)}z_{-1}},$$

$$x_{8n} = \frac{F_{4n(m+2)} + F_{4n(m+2)-1}x_0}{F_{4n(m+2)+1} + F_{4n(m+2)}x_0}, \quad x_{8n+4} = \frac{F_{4n(m+2)+(2m+4)} + F_{4n(m+2)+(2m+3)}z_0}{F_{4n(m+2)+(2m+5)} + F_{4n(m+2)+(2m+4)}z_0},$$

$$x_{8n+1} = \frac{L_{4n(m+2)+(m+2)} + L_{4n(m+2)+(m+1)}y_{-1}}{L_{4n(m+2)+(m+3)} + L_{4n(m+2)+(m+2)}y_{-1}}, \quad x_{8n+5} = \frac{L_{4n(m+2)+(3m+6)} + L_{4n(m+2)+(3m+5)}w_{-1}}{L_{4n(m+2)+(3m+7)} + L_{4n(m+2)+(3m+6)}w_{-1}},$$

$$x_{8n+2} = \frac{L_{4n(m+2)+(m+2)} + L_{4n(m+2)+(m+1)}y_0}{L_{4n(m+2)+(m+3)} + L_{4n(m+2)+(m+2)}y_0}, \quad x_{8n+6} = \frac{L_{4n(m+2)+(3m+6)} + L_{4n(m+2)+(3m+5)}w_0}{L_{4n(m+2)+(3m+7)} + L_{4n(m+2)+(3m+6)}w_0},$$

$$y_{8n-1} = \frac{F_{4n(m+2)} + F_{4n(m+2)-1}y_{-1}}{F_{4n(m+2)+1} + F_{4n(m+2)}y_{-1}}, \quad y_{8n+3} = \frac{F_{4n(m+2)+(2m+4)} + F_{4n(m+2)+(2m+3)}w_{-1}}{F_{4n(m+2)+(2m+5)} + F_{4n(m+2)+(2m+4)}w_{-1}},$$

$$y_{8n} = \frac{F_{4n(m+2)} + F_{4n(m+2)-1}y_0}{F_{4n(m+2)+1} + F_{4n(m+2)}y_0}, \quad y_{8n+4} = \frac{F_{4n(m+2)+(2m+4)} + F_{4n(m+2)+(2m+3)}w_0}{F_{4n(m+2)+(2m+5)} + F_{4n(m+2)+(2m+4)}w_0},$$

$$y_{8n+1} = \frac{L_{4n(m+2)+(m+2)} + L_{4n(m+2)+(m+1)}z_{-1}}{L_{4n(m+2)+(m+3)} + L_{4n(m+2)+(m+2)}z_{-1}}, \quad y_{8n+5} = \frac{L_{4n(m+2)+(3m+6)} + L_{4n(m+2)+(3m+5)}x_{-1}}{L_{4n(m+2)+(3m+7)} + L_{4n(m+2)+(3m+6)}x_{-1}},$$

$$y_{8n+2} = \frac{L_{4n(m+2)+(m+2)} + L_{4n(m+2)+(m+1)}z_0}{L_{4n(m+2)+(m+3)} + L_{4n(m+2)+(m+2)}z_0}, \quad y_{8n+6} = \frac{L_{4n(m+2)+(3m+6)} + L_{4n(m+2)+(3m+5)}x_0}{L_{4n(m+2)+(3m+7)} + L_{4n(m+2)+(3m+6)}x_0},$$

$$z_{8n-1} = \frac{F_{4n(m+2)} + F_{4n(m+2)-1}z_{-1}}{F_{4n(m+2)+1} + F_{4n(m+2)}z_{-1}}, \quad z_{8n+3} = \frac{F_{4n(m+2)+(2m+4)} + F_{4n(m+2)+(2m+3)}x_{-1}}{F_{4n(m+2)+(2m+5)} + F_{4n(m+2)+(2m+4)}x_{-1}},$$

$$z_{8n} = \frac{F_{4n(m+2)} + F_{4n(m+2)-1}z_0}{F_{4n(m+2)+1} + F_{4n(m+2)}z_0}, \quad z_{8n+4} = \frac{F_{4n(m+2)+(2m+4)} + F_{4n(m+2)+(2m+3)}x_0}{F_{4n(m+2)+(2m+5)} + F_{4n(m+2)+(2m+4)}x_0},$$

$$z_{8n+1} = \frac{L_{4n(m+2)+(m+2)} + L_{4n(m+2)+(m+1)}w_{-1}}{L_{4n(m+2)+(m+3)} + L_{4n(m+2)+(m+2)}w_{-1}}, \quad z_{8n+5} = \frac{L_{4n(m+2)+(3m+6)} + L_{4n(m+2)+(3m+5)}y_{-1}}{L_{4n(m+2)+(3m+7)} + L_{4n(m+2)+(3m+6)}y_{-1}},$$

$$z_{8n+2} = \frac{L_{4n(m+2)+(m+2)} + L_{4n(m+2)+(m+1)}w_0}{L_{4n(m+2)+(m+3)} + L_{4n(m+2)+(m+2)}w_0}, \quad z_{8n+6} = \frac{L_{4n(m+2)+(3m+6)} + L_{4n(m+2)+(3m+5)}y_0}{L_{4n(m+2)+(3m+7)} + L_{4n(m+2)+(3m+6)}y_0},$$

and

$$w_{8n-1} = \frac{F_{4n(m+2)} + F_{4n(m+2)-1}w_{-1}}{F_{4n(m+2)+1} + F_{4n(m+2)}w_{-1}}, \quad w_{8n+3} = \frac{F_{4n(m+2)+(2m+4)} + F_{4n(m+2)+(2m+3)}y_{-1}}{F_{4n(m+2)+(2m+5)} + F_{4n(m+2)+(2m+4)}y_{-1}},$$

$$w_{8n} = \frac{F_{4n(m+2)} + F_{4n(m+2)-1}w_0}{F_{4n(m+2)+1} + F_{4n(m+2)}w_0}, \quad w_{8n+4} = \frac{F_{4n(m+2)+(2m+4)} + F_{4n(m+2)+(2m+3)}y_0}{F_{4n(m+2)+(2m+5)} + F_{4n(m+2)+(2m+4)}y_0},$$

$$w_{8n+1} = \frac{L_{4n(m+2)+(m+2)} + L_{4n(m+2)+(m+1)}x_{-1}}{L_{4n(m+2)+(m+3)} + L_{4n(m+2)+(m+2)}x_{-1}}, \quad w_{8n+5} = \frac{L_{4n(m+2)+(3m+6)} + L_{4n(m+2)+(3m+5)}z_{-1}}{L_{4n(m+2)+(3m+7)} + L_{4n(m+2)+(3m+6)}z_{-1}},$$

$$w_{8n+2} = \frac{L_{4n(m+2)+(m+2)} + L_{4n(m+2)+(m+1)}x_0}{L_{4n(m+2)+(m+3)} + L_{4n(m+2)+(m+2)}x_0}, \quad w_{8n+6} = \frac{L_{4n(m+2)+(3m+6)} + L_{4n(m+2)+(3m+5)}z_0}{L_{4n(m+2)+(3m+7)} + L_{4n(m+2)+(3m+6)}z_0},$$

where $(L_m)_{m=-\infty}^{+\infty}$ *is the Lucas sequence,* $(F_m)_{m=-\infty}^{+\infty}$ *is the Fibonacci sequence.*

Proof. We have

$$x'_n = x_{2n-t},\ y'_n = y_{2n-t},\ z'_n = z_{2n-t},\ w'_n = w_{2n-t},\ t = 0, 1.$$

Then for $t = 0, 1$, we have

$$x'_{4n} = x_{8n-t},\ y'_{4n} = y_{8n-t},\ z'_{4n} = z_{8n-t},\ w'_{4n} = w_{8n-t},$$

and

$$x'_0 = x_{-t},\ y'_0 = y_{-t},\ z'_0 = z_{-t},\ w'_0 = w_{-t}.$$

Using Theorem (2.1), we can write for $t = 0,1$

$$x_{8n-t} = \frac{F_{4n(m+2)} + F_{4n(m+2)-1}x_{-t}}{F_{4n(m+2)+1} + F_{4n(m+2)}x_{-t}},$$
$$y_{8n-t} = \frac{F_{4n(m+2)} + F_{4n(m+2)-1}y_{-t}}{F_{4n(m+2)+1} + F_{4n(m+2)}y_{-t}},$$
$$z_{8n-t} = \frac{F_{4n(m+2)} + F_{4n(m+2)-1}z_{-t}}{F_{4n(m+2)+1} + F_{4n(m+2)}z_{-t}},$$
$$w_{8n-t} = \frac{F_{4n(m+2)} + F_{4n(m+2)-1}w_{-t}}{F_{4n(m+2)+1} + F_{4n(m+2)}w_{-t}}.$$

Also, for $t = 0,1$, we have

$$x'_{4n+1} = x_{8n+2-t},\ y'_{4n+1} = y_{8n+2-t},\ z'_{4n+1} = z_{8n+2-t},\ w'_{4n+1} = w_{8n+2-t}.$$

Using Theorem (2.1), we get for $t = 0,1$

$$x_{8n+2-t} = \frac{L_{4n(m+2)+(m+2)} + L_{4n(m+2)+(m+1)}y_{-t}}{L_{4n(m+2)+(m+3)} + L_{4n(m+2)+(m+2)}y_{-t}},$$
$$y_{8n+2-t} = \frac{L_{4n(m+2)+(m+2)} + L_{4n(m+2)+(m+1)}z_{-t}}{L_{4n(m+2)+(m+3)} + L_{4n(m+2)+(m+2)}z_{-t}},$$
$$z_{8n+2-t} = \frac{L_{4n(m+2)+(m+2)} + L_{4n(m+2)+(m+1)}w_{-t}}{L_{4n(m+2)+(m+3)} + L_{4n(m+2)+(m+2)}w_{-t}},$$
$$w_{8n+2-t} = \frac{L_{4n(m+2)+(m+2)} + L_{4n(m+2)+(m+1)}x_{-t}}{L_{4n(m+2)+(m+3)} + L_{4n(m+2)+(m+2)}x_{-t}}.$$

In the same way, we get for $t = 0,1$

$$x_{8n+4-t} = \frac{F_{4n(m+2)+(2m+4)} + F_{4n(m+2)+(2m+3)}z_{-t}}{F_{4n(m+2)+(2m+5)} + F_{4n(m+2)+(2m+4)}z_{-t}},$$
$$y_{8n+4-t} = \frac{F_{4n(m+2)+(2m+4)} + F_{4n(m+2)+(2m+3)}w_{-t}}{F_{4n(m+2)+(2m+5)} + F_{4n(m+2)+(2m+4)}w_{-t}},$$
$$z_{8n+4-t} = \frac{F_{4n(m+2)+(2m+4)} + F_{4n(m+2)+(2m+3)}x_{-t}}{F_{4n(m+2)+(2m+5)} + F_{4n(m+2)+(2m+4)}x_{-t}},$$
$$w_{8n+4-t} = \frac{F_{4n(m+2)+(2m+4)} + F_{4n(m+2)+(2m+3)}y_{-t}}{F_{4n(m+2)+(2m+5)} + F_{4n(m+2)+(2m+4)}y_{-t}},$$

and

$$x_{8n+6-t} = \frac{L_{4n(m+2)+(3m+6)} + L_{4n(m+2)+(3m+5)}w_{-t}}{L_{4n(m+2)+(3m+7)} + L_{4n(m+2)+(3m+6)}w_{-t}},$$
$$y_{8n+6-t} = \frac{L_{4n(m+2)+(3m+6)} + L_{4n(m+2)+(3m+5)}x_{-t}}{L_{4n(m+2)+(3m+7)} + L_{4n(m+2)+(3m+6)}x_{-t}},$$
$$z_{8n+6-t} = \frac{L_{4n(m+2)+(3m+6)} + L_{4n(m+2)+(3m+5)}y_{-t}}{L_{4n(m+2)+(3m+7)} + L_{4n(m+2)+(3m+6)}y_{-t}},$$
$$w_{8n+6-t} = \frac{L_{4n(m+2)+(3m+6)} + L_{4n(m+2)+(3m+5)}z_{-t}}{L_{4n(m+2)+(3m+7)} + L_{4n(m+2)+(3m+6)}z_{-t}}.$$

This completes the proof. □

3. Special cases

We end this paper by illustrating the cases $m = -1$ and $m = 0$ in system (1.5).

Case $m = -1$ When $m = -1$ in system (1.5), we obtain the system of difference equations

$$x_{n+1} = \frac{1+2y_{n-1}}{3+y_{n-1}}, \quad y_{n+1} = \frac{1+2z_{n-1}}{3+z_{n-1}}, z_{n+1} = \frac{1+2w_{n-1}}{3+w_{n-1}}, \quad w_{n+1} = \frac{1+2x_{n-1}}{3+x_{n-1}}, \quad n \in \mathbb{N}_0. \tag{3.1}$$

For system (3.1), by applying Theorem (2.2) we get

$$x_{8n-1}=\frac{F_{4n}+F_{4n-1}x_{-1}}{F_{4n+1}+F_{4n}x_{-1}},\qquad x_{8n+3}=\frac{F_{4n+2}+F_{4n+1}z_{-1}}{F_{4n+3}+F_{4n+2}z_{-1}},$$
$$x_{8n}=\frac{F_{4n}+F_{4n-1}x_{0}}{F_{4n+1}+F_{4n}x_{0}},\qquad x_{8n+4}=\frac{F_{4n+2}+F_{4n+1}z_{0}}{F_{4n+3}+F_{4n+2}z_{0}},$$
$$x_{8n+1}=\frac{L_{4n+1}+L_{4n}y_{-1}}{L_{4n+2}+L_{4n+1}y_{-1}},\qquad x_{8n+5}=\frac{L_{4n+3}+L_{4n+2}w_{-1}}{L_{4n+4}+L_{4n+3}w_{-1}},$$
$$x_{8n+2}=\frac{L_{4n+1}+L_{4n}y_{0}}{L_{4n+2}+L_{4n+1}y_{0}},\qquad x_{8n+6}=\frac{L_{4n+3}+L_{4n+2}w_{0}}{L_{4n+4}+L_{4n+3}w_{0}},$$

$$y_{8n-1}=\frac{F_{4n}+F_{4n-1}y_{-1}}{F_{4n+1}+F_{4n}y_{-1}},\qquad y_{8n+3}=\frac{F_{4n+2}+F_{4n+1}w_{-1}}{F_{4n+3}+F_{4n+2}w_{-1}},$$
$$y_{8n}=\frac{F_{4n}+F_{4n-1}y_{0}}{F_{4n+1}+F_{4n}y_{0}},\qquad y_{8n+4}=\frac{F_{4n+2}+F_{4n+1}w_{0}}{F_{4n+3}+F_{4n+2}w_{0}},$$
$$y_{8n+1}=\frac{L_{4n+1}+L_{4n}z_{-1}}{L_{4n+2}+L_{4n+1}z_{-1}},\qquad y_{8n+5}=\frac{L_{4n+3}+L_{4n+2}x_{-1}}{L_{4n+4}+L_{4n+3}x_{-1}},$$
$$y_{8n+2}=\frac{L_{4n+1}+L_{4n}z_{0}}{L_{4n+2}+L_{4n+1}z_{0}},\qquad y_{8n+6}=\frac{L_{4n+3}+L_{4n+2}x_{0}}{L_{4n+4}+L_{4n+3}x_{0}},$$

$$z_{8n-1}=\frac{F_{4n}+F_{4n-1}z_{-1}}{F_{4n+1}+F_{4n}z_{-1}},\qquad z_{8n+3}=\frac{F_{4n+2}+F_{4n+1}x_{-1}}{F_{4n+3}+F_{4n+2}x_{-1}},$$
$$z_{8n}=\frac{F_{4n}+F_{4n-1}z_{0}}{F_{4n+1}+F_{4n}z_{0}},\qquad z_{8n+4}=\frac{F_{4n+2}+F_{4n+1}x_{0}}{F_{4n+3}+F_{4n+2}x_{0}},$$
$$z_{8n+1}=\frac{L_{4n+1}+L_{4n}w_{-1}}{L_{4n+2}+L_{4n+1}w_{-1}},\qquad z_{8n+5}=\frac{L_{4n+3}+L_{4n+2}y_{-1}}{L_{4n+4}+L_{4n+3}y_{-1}},$$
$$z_{8n+2}=\frac{L_{4n+1}+L_{4n}w_{0}}{L_{4n+2}+L_{4n+1}w_{0}},\qquad z_{8n+6}=\frac{L_{4n++3}+L_{4n+2}y_{0}}{L_{4n+4}+L_{4n+3}y_{0}},$$

$$w_{8n-1}=\frac{F_{4n}+F_{4n-1}w_{-1}}{F_{4n+1}+F_{4n}w_{-1}},\qquad w_{8n+3}=\frac{F_{4n+2}+F_{4n+1}y_{-1}}{F_{4n+3}+F_{4n+2}y_{-1}},$$
$$w_{8n}=\frac{F_{4n}+F_{4n-1}w_{0}}{F_{4n+1}+F_{4n}w_{0}},\qquad w_{8n+4}=\frac{F_{4n+2}+F_{4n+1}y_{0}}{F_{4n+3}+F_{4n+2}y_{0}},$$
$$w_{8n+1}=\frac{L_{4n+1}+L_{4n}x_{-1}}{L_{4n+2}+L_{4n+1}x_{-1}},\qquad w_{8n+5}=\frac{L_{4n+3}+L_{4n+2}z_{-1}}{L_{4n+4}+L_{4n+3}z_{-1}},$$
$$w_{8n+2}=\frac{L_{4n+1}+L_{4n}x_{0}}{L_{4n+2}+L_{4n+1}x_{0}},\qquad w_{8n+6}=\frac{L_{4n+3}+L_{4n+2}z_{0}}{L_{4n+4}+L_{4n+3}z_{0}},$$

Case $m=0$ When $m=0$ in system (1.5), we obtain the system of difference equations

$$x_{n+1}=\frac{3+y_{n-1}}{4+3y_{n-1}},\quad y_{n+1}=\frac{3+z_{n-1}}{4+3z_{n-1}},z_{n+1}=\frac{3+w_{n-1}}{4+3w_{n-1}},\quad w_{n+1}=\frac{3+x_{n-1}}{4+3x_{n-1}},\quad n\in\mathbb{N}_0. \tag{3.2}$$

For system (3.2), applying Theorem (2.2) we get

$$x_{8n-1}=\frac{F_{8n}+F_{8n-1}x_{-1}}{F_{8n+1}+F_{8n}x_{-1}},\qquad x_{8n+3}=\frac{F_{8n+4}+F_{8n+3}z_{-1}}{F_{8n+5}+F_{8n+4}z_{-1}},$$
$$x_{8n}=\frac{F_{8n}+F_{8n-1}x_{0}}{F_{8n+1}+F_{8n}x_{0}},\qquad x_{8n+4}=\frac{F_{8n+4}+F_{8n+3}z_{0}}{F_{8n+5}+F_{8n+4}z_{0}},$$
$$x_{8n+1}=\frac{L_{8n+2}+L_{8n+1}y_{-1}}{L_{8n+3}+L_{8n+2}y_{-1}},\qquad x_{8n+5}=\frac{L_{8n+6}+L_{8n+5}w_{-1}}{L_{8n+7}+L_{8n+6}w_{-1}},$$
$$x_{8n+2}=\frac{L_{8n+2}+L_{8n+1}y_{0}}{L_{8n+3}+L_{8n+2}y_{0}},\qquad x_{8n+6}=\frac{L_{8n+6}+L_{8n+5}w_{0}}{L_{8n+7}+L_{8n+6}w_{0}},$$

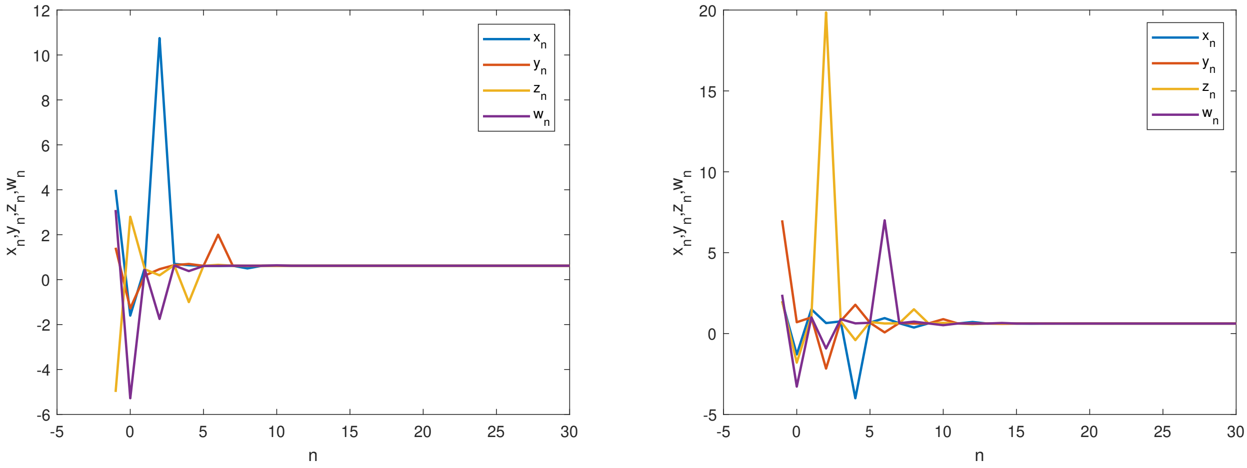

Figure 1: System (3.1) (left) and System (3.2) (right).

$$y_{8n-1}=\frac{F_{8n}+F_{8n-1}y_{-1}}{F_{8n+1}+F_{8n}y_{-1}}, \qquad y_{8n+3}=\frac{F_{8n+4}+F_{8n+3}w_{-1}}{F_{8n+5}+F_{8n+4}w_{-1}},$$
$$y_{8n}=\frac{F_{8n}+F_{8n-1}y_{0}}{F_{8n+1}+F_{8n}y_{0}}, \qquad y_{8n+4}=\frac{F_{8n+4}+F_{8n+3}w_{0}}{F_{8n+5}+F_{8n+4}w_{0}},$$
$$y_{8n+1}=\frac{L_{8n+2}+L_{8n+1}z_{-1}}{L_{8n+3}+L_{8n+2}z_{-1}}, \qquad y_{8n+5}=\frac{L_{8n+6}+L_{8n+5}x_{-1}}{L_{8n+7}+L_{8n+6}x_{-1}},$$
$$y_{8n+2}=\frac{L_{8n+2}+L_{8n+1}z_{0}}{L_{8n+3}+L_{8n+2}z_{0}}, \qquad y_{8n+6}=\frac{L_{8n+6}+L_{8n+5}x_{0}}{L_{8n+7}+L_{8n+6}x_{0}},$$

$$z_{8n-1}=\frac{F_{8n}+F_{8n-1}z_{-1}}{F_{8n+1}+F_{8n}z_{-1}}, \qquad z_{8n+3}=\frac{F_{8n+4}+F_{8n+3}x_{-1}}{F_{8n+5}+F_{8n+4}x_{-1}},$$
$$z_{8n}=\frac{F_{8n}+F_{8n-1}z_{0}}{F_{8n+1}+F_{8n}z_{0}}, \qquad z_{8n+4}=\frac{F_{8n+4}+F_{8n+3}x_{0}}{F_{8n+5}+F_{8n+4}x_{0}},$$
$$z_{8n+1}=\frac{L_{8n+2}+L_{8n+1}w_{-1}}{L_{8n+3}+L_{8n+2}w_{-1}}, \qquad z_{8n+5}=\frac{L_{8n+6}+L_{8n+5}y_{-1}}{L_{8n+7}+L_{8n+6}y_{-1}},$$
$$z_{8n+2}=\frac{L_{8n+2}+L_{8n+1}w_{0}}{L_{8n+3}+L_{8n+2}w_{0}}, \qquad z_{8n+6}=\frac{L_{8n+6}+L_{8n+5}y_{0}}{L_{8n+7}+L_{8n+6}y_{0}},$$

$$w_{8n-1}=\frac{F_{8n}+F_{8n-1}w_{-1}}{F_{8n+1}+F_{8n}w_{-1}}, \qquad w_{8n+3}=\frac{F_{8n+4}+F_{8n+3}y_{-1}}{F_{8n+5}+F_{8n+4}y_{-1}},$$
$$w_{8n}=\frac{F_{8n}+F_{8n-1}w_{0}}{F_{8n+1}+F_{8n}w_{0}}, \qquad w_{8n+4}=\frac{F_{8n+4}+F_{8n+3}y_{0}}{F_{8n+5}+F_{8n+4}y_{0}},$$
$$w_{8n+1}=\frac{L_{8n+2}+L_{8n+1}x_{-1}}{L_{8n+3}+L_{8n+2}x_{-1}}, \qquad w_{8n+5}=\frac{L_{8n+6}+L_{8n+5}z_{-1}}{L_{8n+7}+L_{8n+6}z_{-1}},$$
$$w_{8n+2}=\frac{L_{8n+2}+L_{8n+1}x_{0}}{L_{8n+3}+L_{8n+2}x_{0}}, \qquad w_{8n+6}=\frac{L_{8n+6}+L_{8n+5}z_{0}}{L_{8n+7}+L_{8n+6}z_{0}},$$

Example 3.1. *Fig.1. (left) represents system (3.1) with* $x_{-1}=2$, $x_0=-1.29$, $y_{-1}=7$, $y_0=0.7$, $z_{-1}=2$, $z_0=-1.8$, $w_{-1}=2.4$, $w_0=-3.28$

Example 3.2. *Fig.1. (right) represents system (3.2) with* $x_{-1}=4$, $x_0=-1.6$, $y_{-1}=1.42$, $y_0=-1.28$, $z_{-1}=-5$, $z_0=2.8$, $w_{-1}=3.1$, $w_0=-5.28$.

4. Conclusion

In this paper, we showed that the system of difference equations

$$x_{n+1} = \frac{L_{m+2} + L_{m+1}y_{n-1}}{L_{m+3} + L_{m+2}y_{n-1}}, \quad y_{n+1} = \frac{L_{m+2} + L_{m+1}z_{n-1}}{L_{m+3} + L_{m+2}z_{n-1}},$$

$$z_{n+1} = \frac{L_{m+2} + L_{m+1}w_{n-1}}{L_{m+3} + L_{m+2}w_{n-1}}, \quad w_{n+1} = \frac{L_{m+2} + L_{m+1}x_{n-1}}{L_{m+3} + L_{m+2}x_{n-1}}.$$

where the coefficients are the well-known Lucas numbers is solvable in closed form.

In fact, its solution is represented in terms of Lucas and Fibonacci numbers.

We also provided two illustrative examples for the case $m = -1$ and $m = 0$.

We conjecture that, the results in this paper can be satisfied to a more general case of the aforementioned system.

Declarations

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