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On the Geometric and Physical Properties of Conformable Derivative

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Abstract

In this article, we explore the advantages geometric and physical implications of the conformable derivative. One of the key benefits of the conformable derivative is its ability to approximate the tangent at points where the classical tangent is not readily available. By employing conformable derivatives, alternative tangents can be created to overcome this limitation. Thanks to these alternative (conformable) tangents, physical interpretation can be made with alternative velocity vectors. Furthermore, the conformable derivative proves to be valuable in situations where the tangent plane cannot be defined. It enables the creation of alternative tangent planes, offering a solution in cases where the traditional approach falls short. Geometrically speaking, the conformable derivative carries significant meaning. It provides insights into the local behavior of a function and its relationship with nearby points. By understanding the conformable derivative, we gain a deeper understanding of how a function evolves and changes within its domain. A several examples are presented in the article to better understand the article and visualize the concepts discussed. These examples are accompanied by visual representations generated using the Mathematica program, aiding in a clearer understanding of the proposed ideas. By combining theoretical explanations, practical examples, and visualizations, this article aims to provide a comprehensive exploration of the advantages and geometric and physical implications of the conformable derivative.

Keywords: *Conformable derivative, Curvatures, Frenet frame, Surface*

AMS Subject Classification (2020): *53A04; 26A33*

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1. Introduction

Curves and surfaces hold great significance within the realm of differential geometry. When studying curves, one of the fundamental concepts is the tangent. The tangent vector of a curve plays a crucial role in analyzing the curve's behavior, such as exploring its Frenet frame or determining its curvatures. The construction of the Frenet

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frame heavily relies on the tangent vector of the curve. Similarly, the curvatures of the curve are computed based on its tangent. In the context of surfaces, tangent vectors and tangent planes assume a similar significance as the tangent vector does for curves. Tangent vectors and tangent planes are essential when investigating various surface concepts. These concepts include the surface's normal, first fundamental form, second fundamental form, and many others. Tangent vectors and tangent planes provide vital information for understanding the geometry and properties of surfaces. However, it becomes challenging to examine points where the concept of a tangent, either for curves or surfaces, does not exist. At such points, where the tangent is undefined, it becomes impossible to apply conventional methods that rely on tangent-based calculations and analyses. These points pose limitations in terms of exploring the local behavior and properties of curves and surfaces. Therefore, the existence and availability of tangent vectors and tangent planes are fundamental for the comprehensive study and analysis of curves and surfaces within the field of differential geometry. They serve as indispensable tools for understanding the geometry and various concepts associated with these mathematical objects.

The concept of the local conformable derivative and integral, initially introduced in 2014 by Khalil et al., has garnered significant attention from scientists and has been the subject of numerous publications. This novel definition incorporates a limit form similar to the classical derivative. Notably, the conformable derivative exhibits essential properties such as fractional linearity, the product rule, the quotient rule, Rolle's theorem, and the mean value theorem [1]. The subsequent development of this theory by Abdeljawad further enriched its applications. He introduced definitions for the left and right conformable derivatives, formulated higher-order conformable integral definitions for $\alpha > 1$, established conformable versions of the Gronwall inequality, chain rule, and partial integration formulas for congruent fractional derivatives. Additionally, power series expansions and Laplace transform techniques were extended to the conformable derivative framework [2]. The conformable derivative has found widespread use in various disciplines, with a particular emphasis on applied sciences [3–5] and physics [6–8]. Its application has proven to be valuable in solving problems and addressing phenomena in these fields. Researchers have leveraged the conformable derivative to gain deeper insights into complex systems, making it a powerful tool for analysis and modeling. The versatility and effectiveness of the conformable derivative have contributed to its growing popularity and adoption across different scientific domains. Its utilization in applied sciences and physics reflects its capability to capture the intricate dynamics and behaviors of real-world phenomena. As the research continues to advance, the conformable derivative is expected to continue playing a pivotal role in expanding our understanding of various disciplines.

The theory of curves can be described as the study of the motion of a point in a plane or space using the techniques of linear algebra and calculus. Considering the adventure of the literature in the last ten years, it is observed that fractional calculus is started to be used for curves and surfaces in differential geometry. T. Yajima and K. Kamasaki are made the first study on this subject by examining surfaces with fractional calculus [9]. Later, T. Yajima et al. are obtained Frenet formulas using fractional derivatives [10]. In another study, K.A. Lazopoulos and A.K. Lazopoulos are studied fractional differentiable manifolds [11]. M.E. Aydın et al. are studied plane curves in equiaffine geometry in fractional order [12]. U. Gozutok et al. are analyzed the basic concepts of curves and Frenet frame in fractional order with the help of conformable local fractional derivative [13]. On the other hand A. Has and B. Yılmaz are investigated some special curves and curve pairs in fractional order with the help of conformable Frenet frame [14, 15]. In addition, electromagnetic fields and magnetic curves are investigated under conformable derivative by A. Has and B. Yılmaz [16–18]. There are many more studies on this topic [19–21].

In this study, our objective is to present a geometric interpretation of conformable derivatives and highlight their advantages. We begin by delving into the concept of tangent vectors for curves. At points where the classical tangent does not exist, we introduce an alternative tangent that is defined using conformable derivatives. This allows us to establish a comprehensive understanding of the curve's behavior in those critical regions. Expanding upon this idea, we extend our investigation to surfaces. Similar to curves, we encounter points where the tangent vectors of the surface are undefined. To overcome this challenge, we employ conformable derivatives to generate alternative tangent vectors. These vectors collectively contribute to the formation of an alternative tangent plane that stretches across the surface at the respective point. Through this approach, we are able to explore the local behavior of surfaces in a way that would not be possible solely with classical tangents. Lastly, we aim to provide a geometric meaning to the conformable derivative. By studying its properties and implications within the context of curves and surfaces, we aim to shed light on its geometrical significance. This deeper understanding will allow us to grasp the underlying geometric principles that govern the behavior of functions and objects under conformable differentiation. Through this study, we strive to elucidate the geometric interpretation of conformable derivatives, showcase their advantages, and establish their relevance in various mathematical contexts.

2. Basics definitions and theorems in conformable calculus and conformable differential geometry

Given $s \mapsto x(s) \in \mathbb{E}^3$, $s \in I \subset \mathbb{R}$, the *conformable derivative* of x at s is defined by [1]

$$D_\alpha(x)(s) = \lim_{\varepsilon \to 0} \frac{x(s + \varepsilon s^{1-\alpha}) - x(s)}{\varepsilon}.$$

Let $Dx(s) = dx(s)/ds$. We then notice

$$D_\alpha x(s) = s^{1-\alpha} dx(s)/ds.$$

Denote by $D_\alpha x(s)$ the α-th order conformable derivative of $x(s)$ for each $s > 0$, $0 < \alpha < 1$.

It can be said that the conformable derivative provides some properties such as linearity, Leibniz rule and chain rule as in the classical derivative as follows

1. $D_\alpha(ax + by)(s) = aD_\alpha(x)(s) + bD_\alpha(y)(s),$ for all $a, b \in \mathbb{R}$,
2. $D_\alpha(s^p) = ps^{p-\alpha}$ for all $p \in \mathbb{R}$,
3. $D_\alpha(\lambda) = 0$, for all constant functions $x(s) = \lambda$,
4. $D_\alpha(xy)(s) = x(s)D_\alpha y(s) + y(s)D_\alpha x(s)$,
5. $D_\alpha(\frac{x}{y})(s) = \frac{x(s)D_\alpha y(s) - y(s)D_\alpha x(s)}{y^2(s)}$,
6. $D_\alpha(y \circ x)(s) = x(s)^{\alpha-1} D_\alpha x(s) D_\alpha y(x(s))$

where x, y be α-differentiable for each $s > 0$ and $0 < \alpha < 1$ [1].

The definition of the conformable integral is given as the inverse operator of the conformable derivative. The conformable integral of the function $x(s)$ is defined by [1]

$$I_\alpha^a f(t) = I_1^a(t^{\alpha-1} f) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx.$$

The effect of conformable analysis on vector-valued functions is investigated, and the limit and derivative of vector-valued functions also are investigated. In the following theorem, the conformable derivative of vector-valued functions is given.

Theorem 2.1. *Let x be a vector-valued function with n variables, and let x be a vector-valued function $x(s_1, ..., s_n) = (x_1(s_1, ..., s_n), ..., x_m(s_1, ..., s_n))$. So x is $\alpha-$differentiable at $t = (t_1, ..., t_n) \in \mathbb{R}$, for all $t_i > 0$ if and only if each x_i is, and [22]*

$$D_\alpha x(t) = (D_\alpha x_1(t), ..., D_\alpha x_m(t)).$$

Definition 2.1. Let $\mathbf{x} = \mathbf{x}(s)$ be a regular unit speed conformable curve in the Euclidean $3-$space where s measures its arc length. Also, let $\mathbf{t} = D^\alpha(\mathbf{x})(s)s^{\alpha-1}$ be its unit tangent vector, $\mathbf{n} = \frac{D^\alpha(\mathbf{t})(s)}{\|D^\alpha(\mathbf{t})(s)\|}$ be its principal normal vector and $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ be its binormal vector. The triple $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ be the conformable Frenet frame of the curve x. Then the conformable Frenet formula of the curve is given by

$$\begin{pmatrix} D^\alpha(\mathbf{t})(s) \\ D^\alpha(\mathbf{n})(s) \\ D^\alpha(\mathbf{b})(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_\alpha(s) & 0 \\ -\kappa_\alpha(s) & 0 & \tau_\alpha(s) \\ 0 & -\tau_\alpha(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{pmatrix} \tag{2.1}$$

where $\kappa_\alpha(s) = \|D^\alpha(\mathbf{t})(s)\|$ and $\tau_\alpha(s) = \langle D^\alpha(\mathbf{n})(s),\ \mathbf{b}\rangle$ are curvature and torsion of x, respectively (see details [13]).

Conclusion 1. *Let $\mathbf{x} = \mathbf{x}(s)$ be a regular unit speed conformable curve in the Euclidean $3-$space where s measures its arc length. The following relation exists between the curvature and torsion of the curve $\mathbf{x}$ and the conformable curvature and torsion [15]*

$$\kappa_\alpha = s^{1-\alpha}\kappa, \tag{2.2}$$
$$\tau_\alpha = s^{1-\alpha}\tau. \tag{2.3}$$

Conclusion 2. *Let $\mathbf{x} = \mathbf{x}(s)$ be a regular unit speed conformable curve where s measures its arc length. As can be seen from equation (2.1), when x is a unit speed curve, the conformable derivative has no effect on the Frenet vectors, so the Frenet vectors do not undergo any change. However, considering equations (2.2) and (2.3), the curvature and torsion of the curve $\mathbf{x}$ has changed under the conformable derivative [15].*

In this section, basic definitions and theorems of $\mathcal{C}_\alpha-$surfaces will be given. The concepts in this section are studied by A. Has and B. Yılmaz [23].

Definition 2.2. A subset $\mathcal{M} \subset \mathbb{R}^3$ is called a $\mathcal{C}_\alpha-$regular surface if for each point $p \in \mathcal{M}$, there exists a neighborhood V of $p \in \mathbb{R}^3$ and a map $\varphi : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ of an open set $U \subset \mathbb{R}^2$ onto V intersection $\mathcal{M}$ such that
i. $\varphi : U \to V \cap \mathcal{M}$ is a homeomorphism,
ii. φ is conformable differentiable
iii. Each map $\varphi : U \to \mathcal{M}$ is a conformable regular patch.

Definition 2.3. Let $\mathcal{M}$ be the $\mathcal{C}_\alpha-$surface is given by the parameterization $\varphi : U \subset \mathbb{R}^2 \to \mathbb{R}^3$. In this case, the vectors $D_u^\alpha \varphi$ and $D_v^\alpha \varphi$ are the $\mathcal{C}_\alpha-$tangent vector of the $\mathcal{C}_\alpha-$surface. That is, they are $\mathcal{C}_\alpha-$tangent to the $\mathcal{C}_\alpha-$surface at point $P \in \mathcal{M}$. Thus, the $\mathcal{C}_\alpha-$plane span by the vectors $D_u^\alpha \varphi$ and $D_v^\alpha \varphi$ is called the $\mathcal{C}_\alpha-$tangent plane of the $\mathcal{C}_\alpha-$surface. Also, the space of $\mathcal{C}_\alpha-$tangent vectors we called $\mathcal{C}_\alpha-$tangent space and is denoted by $T_p^\alpha \mathcal{M}$.

3. Geometric meaning of the local conformable derivative

In this section, the geometric meaning of conformable derivative and its advantages over classical derivative will be explained.

3.1 Why local conformable derivative?

Differential geometry, a field that employs linear algebra and calculus techniques, focuses on the study of curves, surfaces, and high-dimensional manifolds. However, in the past decade, there has been a noticeable emergence of a new trend within differential geometry, where fractional calculus techniques are being utilized. This development has raised questions about the geometric properties associated with fractional derivative and integral operators. It is important to note that non-local fractional derivative operators such as Riemann-Liouville, Caputo, and Riesz do not adhere to the classical Leibniz and chain rules. These rules form the basis of the classical derivative used in differential geometry. Consequently, constructing a framework for differential geometry using non-local fractional derivatives poses significant challenges. Instead, a more advantageous approach is to employ local conformable derivative. Conformable local derivative operator satisfy the Leibniz and chain rules, making them suitable for constructing a differential geometry framework. By utilizing conformable local derivative, researchers can explore the geometric implications of fractional calculus techniques within the context of differential geometry. By embracing local conformable derivative operator, the field of differential geometry can navigate the complexities associated with non-local fractional derivatives and benefit from the inherent advantages provided by local derivative operators that adhere to the classical rules of calculus.

3.2 What is the advantage of the local conformable derivative?

Fractional derivatives, as is known, search for any fractional order derivatives of a function. Accordingly, at a point where there is no integer derivative of a function, its fractional derivative can be found. For example, consider the function $\mathbf{x}(u) = 2\sqrt{u}$. Here $\mathbf{x}'(0)$ does not exist. However, the result $D_{\frac{1}{2}}\mathbf{x}(0) = 1$ can be easily reached. As it can be easily seen from the example, the tangent of the curve $\mathbf{x}$ cannot be mentioned at the point where there is no classical derivative. The absence of the concept of tangent, which is the most basic concept of curves and surfaces, causes a bottleneck in this regard, in other words degeneration. At such points, which lead to degeneracy where there is no derivative, the tangent can be approached by choosing a fractional order derivative very close to first instead of the first-order derivative giving the tangent. Thus, necessary investigations can be made by making a fractional approximation to the tangent at a degenerate point where there is no tangent of the curve or the surface.

Case 1: Let x be a $\mathcal{C}_\alpha$(conformable)-differentiable curve. Let us choose a point u_0 where the traditional derivative of the curve $\mathbf{x}$ does not exist. At such a u_0 degenerate point, the tangent cannot be mentioned, because the derivative of the curve $\mathbf{x}$ does not exist. As it is known, when defining the Frenet frame at any point of the curve, the most important element is the tangent vector of the curve. In this case, the Frenet frame cannot be defined at the degenerate u_0 point. In such a case, with the help of conformable derivative, we can define a frame parallel to the Frenet frame by defining the classical tangent of the curve as an alternative to the $\mathcal{C}_\alpha-$tangent (see details [24]).

Example 3.1. Let $\mathbf{x}: I \subset \mathbb{R} \to \mathbb{E}^3$ be a $\mathcal{C}_\alpha-$curve in $\mathbb{R}^3$ parameterized by

$$\mathbf{x}(u) = \left(2u^{\frac{1}{2}},\ u^{\frac{3}{2}},\ u^{\frac{5}{2}}\right).$$

The tangent of the curve $\mathbf{x}$ obtained with the classical derivative and the tangent obtained with the conformable derivative for $\alpha = \frac{1}{2}$ and $-2 < u < 2$ are as follows, respectively.

$$\begin{aligned} T &= \frac{1}{\sqrt{\frac{1}{u} + \frac{9}{4}u + \frac{25}{4}u^2}} \left(u^{\frac{-1}{2}}, \frac{3}{2}u^{\frac{1}{2}}, \frac{5}{2}u^{\frac{3}{2}}\right), \\ T_{\frac{1}{2}} &= \frac{1}{\sqrt{1 + \frac{9}{4}u^2 + \frac{25}{4}u^3}} \left(1, \frac{3}{2}u, \frac{5}{2}u^2\right). \end{aligned}$$

where T and $T_{\frac{1}{2}}$ are the classical tangent and $\mathcal{C}_\alpha-$tangent of the curve $\mathbf{x}(u)$, respectively. It is clear that for the point $u_0 = 0$ the classical tangent of the $\mathbf{x}$ curve does not exist. In Fig. (1) we present the graph classical tangent and $\mathcal{C}_\alpha-$tangent of the curve $\mathbf{x}(u)$.

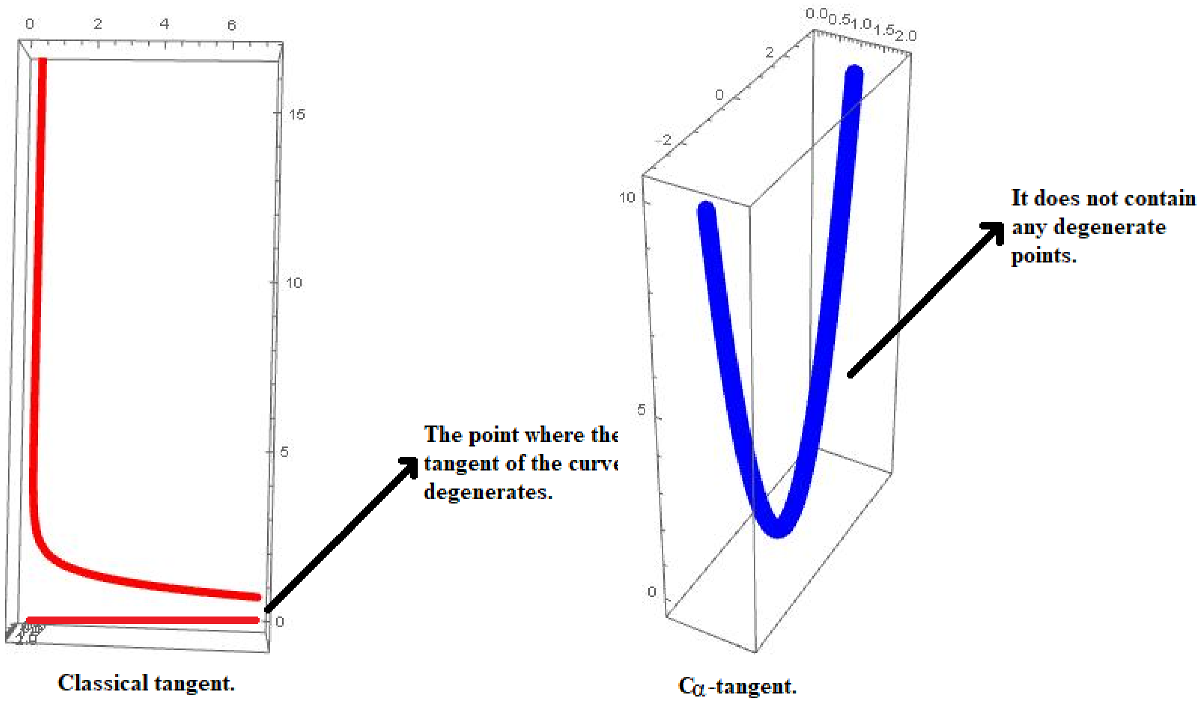

Figure 1. Classical tangent and $\mathcal{C}_\alpha-$tangent of the curve $\mathbf{x}(s)$.

Case 2: Let $\mathcal{M}$ be the $\mathcal{C}_\alpha-$surface is given by the parameterization $\varphi(u, v)$. In order to define the tangent plane of the conformable surface M, the φ_u and φ_v partial derivatives that spanned the tangent plane are needed. In this case, the tangent plane of the surface cannot be mentioned at a point u_0 or v_0 where at least one of the derivatives of φ_u or φ_v does not exist. As it is well known, one of the most important concepts of the surface is the tangent plane. At the point where the tangent plane cannot be defined, it becomes impossible to work on the surface. Since the traditional derivative loses its function at such a degenerate u_0 or v_0 point, necessary studies can be done by obtaining the $\mathcal{C}_\alpha-$tangent plane with the help of $D_u^\alpha \varphi$ and $D_v^\alpha \varphi$ conformable partial derivatives (see details [23]).

Example 3.2. Let $\mathcal{M}$ be a $\mathcal{C}_\alpha-$surface in $\mathbb{R}^3$ parameterized by

$$\varphi(u, v) = \left(\int\int v^{\frac{1}{2}-\alpha} u^{\alpha-1} \cos u du dv, -\int\int v^{\frac{1}{2}-\alpha} u^{\alpha-1} \sin u du dv, \int v^{\frac{1}{2}-\alpha} dv\right).$$

In Fig. 2, the degenerate (there is no derivative) points the classical tangent plane for $v = 0$ and the $\mathcal{C}_\alpha-$-tangent plane for $v = 0$, $\alpha = \frac{1}{2}$ are given.

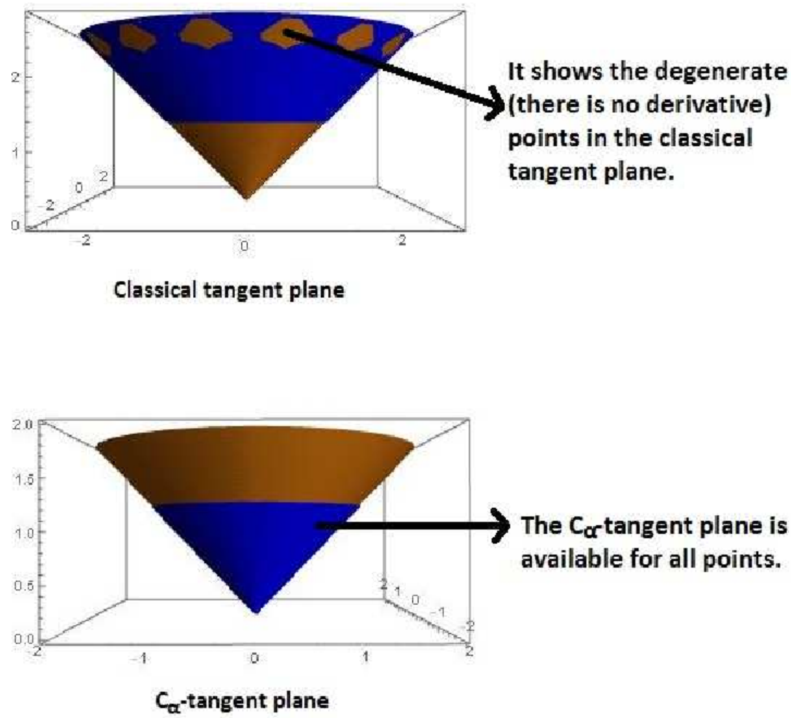

Figure 2. Yellow areas show the surface, and blue areas show tangent planes.

3.3 Geometric meaning of the conformable derivative

The geometric interpretation of the conformable derivative is based on the notion of fractal geometry. In fractal geometry, objects exhibit self-similarity at different scales. The conformable derivative captures this self-similar behavior of a function by considering its local fractional variations. Geometrically, it can be understood as analyzing the "zooming in" behavior of the function at that point, similar to the classical derivative capturing the local linear behavior. Overall, the geometric interpretation of the conformable derivative relates to the self-similarity and scaling properties of functions, enabling us to understand their behavior at different levels of detail and resolution. *More specifically, the conformable derivative can be explained as a measure of how much a straight line and plane bends to form a curve and a surface* (see details [24])

Example 3.3. Let consider the $s \mapsto \mathbf{x}(s) = (s, \int s^{1-\alpha} ds)$, $\mathcal{C}_\alpha-$line passing through the point $P = (0, 0)$ and whose direction is $v = (s^{1-\alpha}, s^{1-\alpha})$.
In Fig. (3) we present the graph of the conformable line for different α values.

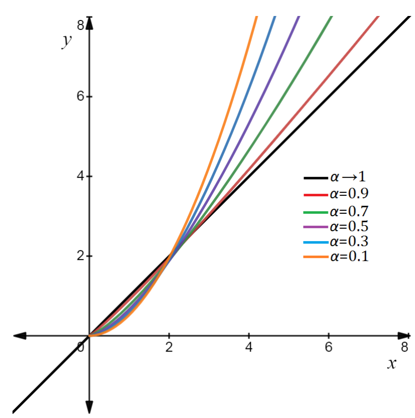

Figure 3. Transformation from line to curve.

Remark 3.1. In differential geometry, a classical line bends depending on the α values with the effect of conformable derivative. Thus, with specially selected α values, how much the line deviates from the plane can be measured.

Example 3.4. Let X be a representation point of the $\mathcal{C}_\alpha-$plane that contains the point $P = (0,0,0)$ and whose normal is $v = (2^{1-\alpha}, -3^{1-\alpha}, 0)$. If X representative point is chosen as follows

$$\begin{aligned} \mathbf{x}_1(s) &= \int x^{1-\alpha} dx, \\ \mathbf{x}_2(s) &= \int y^{1-\alpha} dy, \\ \mathbf{x}_3(s) &= 0 \end{aligned}$$

we get the conformable plane. In Fig. (4) we present the graph of the conformable plane for different α values.

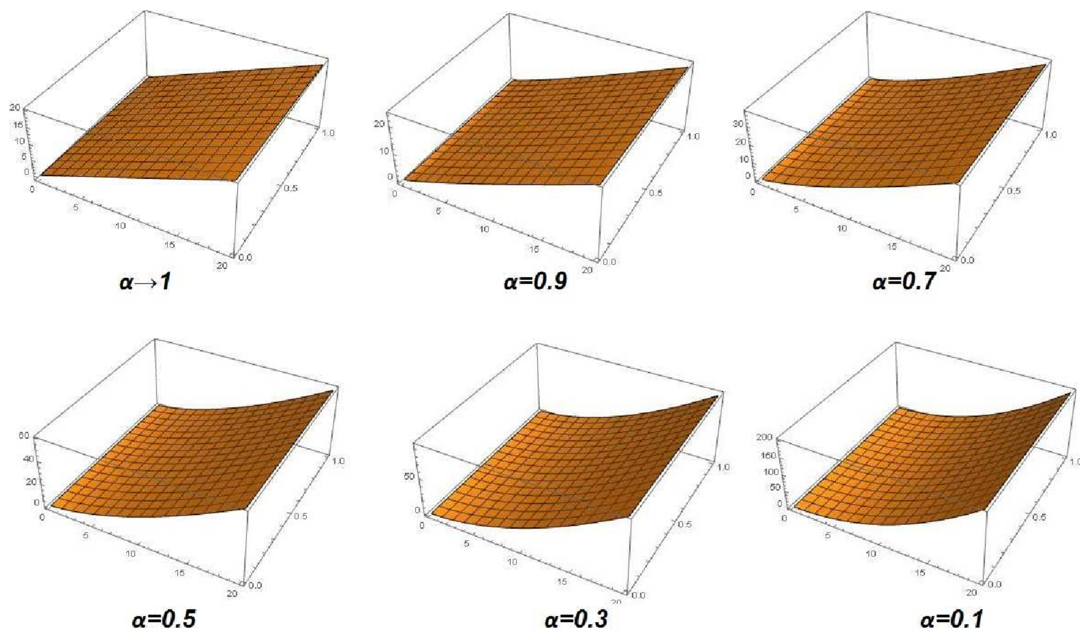

Figure 4. Transformation from plane to surface.

Remark 3.2. In differential geometry, a classical plane bends and transforms into a surface, depending on the α values, with the effect of the concerted derivative. Thus, with specially selected α values, the measure of separation of a surface from the plane can be obtained.

3.4 Physical meaning of the conformable derivative

The velocity vector plays a crucial role in describing the motion of an object over time. It represents how the object's position changes as time progresses. The concept of velocity is defined as the ratio of the change in position of an object to the change in time, and its direction indicates the object's direction of motion.

The relationship between the tangent vector and the velocity vector can be explained as follows: When an object is in motion, its velocity vector is aligned with the tangent direction of the path along which the object moves. In other words, the velocity vector is parallel to the slope (direction) of the object's path and, therefore, points in the direction of the tangent vector. If the object moves in a straight line, the velocity vector and the tangent vector align in the same direction. However, when the object follows a curvilinear path, the velocity vector constantly adjusts in parallel with the slope of the path, hence always directed towards the tangent vector.

In certain instances, the tangent vector may be undefined at certain points along the path. Consequently, at these points, the velocity vector will also be undefined. This situation arises when the classical derivative is not defined. At such points, discussing a physical interpretation becomes challenging since the velocity vector's meaning is lost.

Nevertheless, the conformable derivative comes to the rescue by addressing this undefinedness and allowing for the notion of "conformable velocity" to be introduced. As a result, the conformable derivative provides an advantage in terms of physical interpretation, enabling us to understand the object's behavior even at points where the classical derivative fails.

In conclusion, the velocity vector is vital in characterizing object motion, and it aligns with the tangent vector of the object's path during movement. In cases where the tangent vector is undefined, the velocity vector also becomes undefined, hindering a physical interpretation. However, the conformable derivative offers a solution, eliminating this undefinedness and facilitating a meaningful interpretation through the concept of conformable velocity.

Example 3.5. In Subsection 3.2, an example is presented where the velocity vector is not defined at the point $u_0 = 0$, making it difficult to establish a physical interpretation. However, a solution is found using the conformable derivative, allowing the creation of an alternative (conformable) velocity vector at the point $u_0 = 0$ for $\alpha = \frac{1}{2}$ and enabling a meaningful physical interpretation. The conformable derivative is a mathematical tool used to define derivatives of non-integer order. In this context, it helps overcome the limitation of traditional derivatives, which are not defined for non-integer values. By introducing the conformable derivative with $\alpha = \frac{1}{2}$, it becomes possible to extend the concept of the velocity vector to points like $u_0 = 0$, where traditional derivatives fail. With the introduction of the conformable derivative and considering $\alpha = \frac{1}{2}$, a new velocity vector can be constructed at the point $u_0 = 0$. This new velocity vector provides valuable insights into the physical interpretation of the system, even at previously undefined points. It allows us to understand the behavior of the system at $u_0 = 0$ in a way that was not possible before, offering new perspectives and understanding of the underlying dynamics. In summary, the use of the conformable derivative with $\alpha = \frac{1}{2}$ provides a powerful mathematical tool that enables the definition of the velocity vector at points where it was previously undefined. This breakthrough allows for a more comprehensive and meaningful physical interpretation of the system, enriching our understanding of its behavior and characteristics.

In Fig. 5, the classical velocity vector of the $\mathbf{x}(u) = 2\sqrt{u}$ equation, where $u_0 = 0$, and the compatible velocity vector for $\alpha = \frac{1}{2}$ are illustrated, respectively. The Fig. 5 demonstrates an intriguing contrast between the two velocity vectors. When considering the classical velocity vector, where $u_0 = 0$, it is evident that there is a limitation in terms of physical interpretation. This is because, at this particular point, the classical derivative fails to provide meaningful information about the body's motion. The classical derivative, which relies on integer values for differentiation, encounters issues when dealing with non-integer values like $u_0 = 0$. As a result, it becomes impossible to establish a clear physical interpretation for the body's velocity at this specific point. However, the situation takes a different turn with the introduction of the compatible velocity vector for $\alpha = \frac{1}{2}$. With the help of the conformable derivative, which extends the concept of differentiation to non-integer values, the compatible velocity vector becomes accessible and well-defined along the entire real axis. This remarkable advantage of the conformable derivative enables a continuous interpretation of the body's velocity, even at points where the classical derivative fails. As a consequence, the compatible velocity vector not only provides a consistent interpretation throughout the entire real axis but also removes the ambiguity associated with the classical velocity vector when $u_0 = 0$. The conformable derivative allows us to overcome the limitations of traditional derivatives, granting a more comprehensive understanding of the body's motion and behavior.

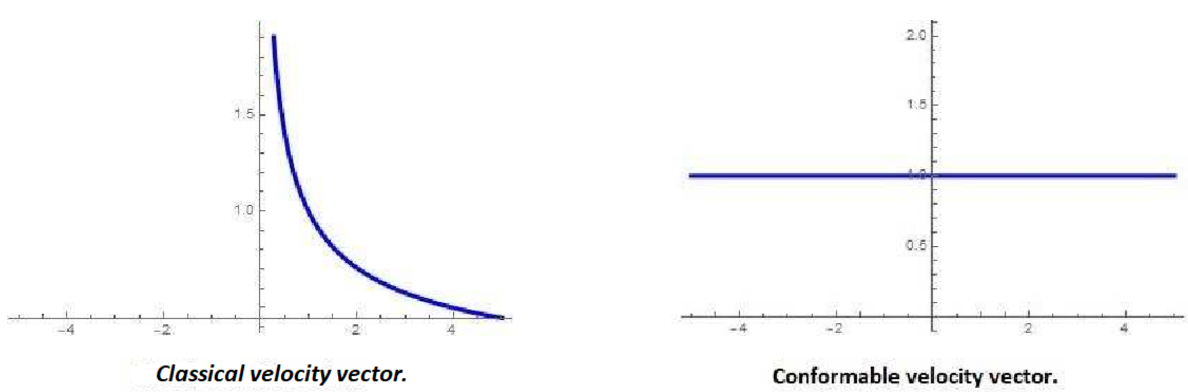

Figure 5. Classical and conformable velocity vectors, respectively.

Conclusion

As is known, the relationship between the conformable derivative and the classical derivative is given by $D_\alpha x(s) = s^{1-\alpha} dx(s)/ds$. When examining the effect of the conformable derivative on lines and planes, the term $s^{1-\alpha}$ causes the line to bend, transforming it into a curve, and the plane to bend, transforming it into a surface. We can refer to the expression $s^{1-\alpha}$ as the bending measure, as it quantifies the amount of bending based on the specific values assigned to α. Geometrically, the conformable derivative can be interpreted as a measure of bending. It captures the degree to which the curve or surface deviates from its original straight form. The bending measure provides valuable insight into the geometric properties and behavior of the objects under consideration. Furthermore, as observed in *Case 1* and *Case 2*, the conformable derivative demonstrates significant advantages in the realm of differential geometry. It enables a deeper understanding and analysis of the geometrical aspects associated with curves, surfaces, and their transformations. The conformable derivative opens up new avenues and perspectives for exploring the intricate connections between bending, geometry, and the underlying mathematical principles. Through these observations, it becomes apparent that the conformable derivative plays a crucial role in differential geometry, offering powerful tools for investigating and comprehending the bending phenomena exhibited by curves and surfaces.

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