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Integral Inequalities of Steffensen Type for Some Different Classes of Functions

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Abstract

In this paper, we have obtained some new integral inequalities of Steffensen type for P-function and η-convex function with the help of identities proved by Mitrinovic et al..

Keywords: Convex function, η-convex function, P-function, Steffensen inequality, Hermite-Hadamard inequality.

1. Introduction

Many function classes have been defined in the historical process of mathematics and one of these function classes is the class of convex functions. This function class has offered new application areas to mathematicians. The definition of this function class, which allows many new results to be obtained with the studies carried out on it and therefore attracts the attention of mathematicians, is as follows.

Definition 1.1 *The function* $f:[a,b]\subseteq \mathbb{R}\to\mathbb{R}$, *is said to be convex if the following inequality holds*

$$f(\lambda x+(1-\lambda)y)\le \lambda f(x)+(1-\lambda)f(y) \quad (1.1)$$

for all $x,y\in[a,b]$ and $\lambda\in[0,1]$. We say that f is concave if $(-f)$ is convex.

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One of the most important results obtained for convex functions is the inequality given below which is known as Hermite Hadamard inequality in the literature.

Assume that $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is a convex function defined on the interval I of $\mathbb{R}$ where $a<b$. The following statement holds

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+f(b)}{2}. \quad (1.2)$$

Both inequalities hold in the reversed direction if f is concave.

One of the functions defined in the class of convex functions is the η-convex function. In (Gordji et al. 2015), Gordji et al. introduced the idea of η-convex functions as generalization of ordinary convex functions and gave the following definition for η-convexity of functions.

Definition 1.2 *A function* $f:[a,b]\to\mathbb{R}$ *is said to be η-convex (or convex with respect to η) if the inequality*

$$f(\lambda x+(1-\lambda)y)\le f(y)+\lambda\eta(f(x),f(y))$$

holds for all $x,y\in[a,b]$, $\lambda\in[0,1]$ and η is defined by $\eta: f([a,b])\times f([a,b])\to\mathbb{R}$.

In the above definition if we set $\eta(x,y)=x-y$, then we can directly obtain the classical definition of a convex function. To see more results and details on η-convex functions see (Delavar and Dragomir 2017, Gordji et al. 2015, Gordji et al. 2016).

Another function defined in the different class of functions is the P function and its definition is given below.

Definition 1.3 (Dragomir et al. 1995) *A function* $f : I \subset \mathbb{R} \to \mathbb{R}$ *is* P *function or that* f *belongs to the class of* $P(I)$, *if it is nonnegative and for* $a, b \in I$ *and* $\lambda \in [0,1]$ *satisfies the following inequality,*

$$f(\lambda a + (1-\lambda)b) \le f(a) + f(b).$$

The Hermite-Hadamard inequality obtained for P functions is as follows.

Theorem 1.1 (Dragomir et al. 1995, Dragomir and Pearce 200) *Let* $I = [a,b]$. *Let P(I) be class of* P *functions defined on* I *and* $f \in P(I)$ *be integrable function, then the following inequality of Hermite-Hadamard type holds*

$$f\left(\frac{a+b}{2}\right) \le \frac{2}{b-a}\int_a^b f(x)dx \le 2\big(f(a)+f(b)\big). \quad (1.3)$$

In order to study certain inequalities between mean values, Steffensen (Steffensen 1918) has proved to following inequality (see also (Mitrinovic 1993, p.311)):

Theorem 1.2 *Let* f *and* g *be two integrable functions defined on* (a,b). f *is decreasing and for each* $t \in (a,b)$, $0 \le g(t) \le 1$. *Then, the following inequality*

$$\int_{b-\lambda}^{b} f(t)dt \le \int_a^b f(t)g(t)dt \le \int_a^{a+\lambda} f(t)dt \quad (1.4)$$

holds, where $\lambda = \int_a^b g(t)dt$.

Some minor generalization of Steffensen's inequality in the (1.4) was considered by Hayashi (Hayashi 1919), using the substituting $g(t)/A$ for $g(t)$, where A is positive constant. For other result involving Steffensen's type inequality, see (Hayashi 1919, Mitrinovic et al. 1993).

In (Mitrinovic et al. 1993), Mitrinovic et al. proved the following equality:

Lemma 1.1 *Let* $f, g : [a,b] \subset \mathbb{R}^+ \to \mathbb{R}$ *be integrable such that* $0 \le g(t) \le 1$, *for all* $t \in [a,b]$ *and* $\int_a^b g(t) f'(t)dt$ *exists. Then we have the following representation*

$$\int_a^{a+\lambda} f(x)dx - \int_a^b f(x)g(x)dx \quad (1.5)$$

$$= -\int_a^{a+\lambda}\left(\int_a^x (1-g(t))dt\right) f'(x)dx - \int_{a+\lambda}^b \left(\int_x^b g(t)dt\right) f'(x)dx,$$

and

$$\int_a^b f(x)g(x)dx - \int_{b-\lambda}^b f(x)dx \quad (1.6)$$

$$= -\int_a^{b-\lambda}\left(\int_a^x g(t)dt\right) f'(x)dx - \int_{b-\lambda}^b \left(\int_x^b (1-g(t))dt\right) f'(x)dx,$$

where $\lambda := \int_a^b g(t)dt$.

In (Alomari et al. 2017), using this identity, Alomari introduced some new Steffensen type inequalities for s-convex functions.

The aim of this paper is to obtain new integral inequalities of Steffensen type for η-convex functions and P function.

2. *Steffensen's type inequalities for η-convex function*

Theorem 2.1 *Let* $f, g : [a,b] \subset \mathbb{R}^+ \to \mathbb{R}$ *be integrable such that* $0 \le g(t) \le 1$, *for all* $t \in [a,b]$ *such that* $\int_a^b g(t) f'(t)dt$ *exists. If* f *is absolutely continuous on* $[a,b]$ *such that* $|f'|$ *is* $\eta-$*convex function on* $[a,b]$, *then we have*

$$\left|\int_a^{a+\lambda} f(x)dx - \int_a^b f(x)g(x)dx\right|$$

$$\le \frac{\lambda^2}{6}\left[3|f'(a)| + 2\eta(|f'(a+\lambda)|, |f'(a)|)\right]$$

$$+\frac{(b-a-\lambda)^2}{6}\left[3|f'(a+\lambda)| + \eta(|f'(b)|, |f'(a+\lambda)|)\right] \quad (2.1)$$

and

$$\left|\int_a^b f(x)g(x)dx - \int_{b-\lambda}^b f(x)dx\right|$$

$$\le \frac{(b-a-\lambda)^2}{6}\left[3|f'(a)| + \frac{2(b-a-\lambda)\eta(|f'(a+\lambda)|, |f'(a)|)}{\lambda}\right]$$

$$+\frac{\lambda^2}{6}\left[3|f'(a+\lambda)| + \frac{(3b-3a-5\lambda)\eta(|f'(b)|, |f'(a+\lambda)|)}{b-a-\lambda}\right] \quad (2.2)$$

where $\lambda := \int_a^b g(t)dt$.

Proof. Using Lemma 1.1 and since $|f'|$ is η-convex function, we have

$$\left|\int_a^{a+\lambda} f(x)dx - \int_a^b f(x)g(x)dx\right|$$

$$\leq \left|\int_a^{a+\lambda}\left(\int_a^x (1-g(t))dt\right)f'(x)dx\right| + \left|\int_{a+\lambda}^b\left(\int_x^b g(t)dt\right)f'(x)dx\right|$$

$$\leq \int_a^{a+\lambda}\left|\int_a^x (1-g(t))dt\right| |f'(x)|dx + \int_{a+\lambda}^b\left|\int_x^b g(t)dt\right| |f'(x)|dx$$

$$= \int_a^{a+\lambda}\left|\int_a^x (1-g(t))dt\right|\left|f'\left(\frac{x-a}{\lambda}(a+\lambda)+\frac{a+\lambda-x}{\lambda}a\right)\right|dx$$

$$+ \int_{a+\lambda}^b\left|\int_x^b g(t)dt\right|\left|f'\left(\frac{x-a-\lambda}{b-a-\lambda}b+\frac{b-x}{b-a-\lambda}(a+\lambda)\right)\right|dx$$

$$\leq \int_a^{a+\lambda}\left|\int_a^x (1-g(t))dt\right|\left[|f'(a)|+\frac{x-a}{\lambda}\eta(|f'(a+\lambda)|,|f'(a)|)\right]dx$$

$$+ \int_{a+\lambda}^b\left|\int_x^b g(t)dt\right|\left[|f'(a+\lambda)|+\frac{x-a-\lambda}{b-a-\lambda}\eta(|f'(b)|,|f'(a+\lambda)|)\right]dx$$

$$= \int_a^{a+\lambda}\left|\int_a^x (1-g(t))dt\right| |f'(a)|dx$$

$$+ \int_a^{a+\lambda}\left|\int_a^x (1-g(t))dt\right|\frac{x-a}{\lambda}\eta(|f'(a+\lambda)|,|f'(a)|)dx$$

$$+ \int_{a+\lambda}^b\left|\int_x^b g(t)dt\right| |f'(a+\lambda)|dx$$

$$+ \int_{a+\lambda}^b\left|\int_x^b g(t)dt\right|\frac{x-a-\lambda}{b-a-\lambda}\eta(|f'(b)|,|f'(a+\lambda)|)dx$$

$$\leq |f'(a)|\int_a^{a+\lambda}(x-a)dx + \frac{\eta(|f'(a+\lambda)|,|f'(a)|)}{\lambda}\int_a^{a+\lambda}(x-a)^2dx$$

$$+ |f'(a+\lambda)|\int_{a+\lambda}^b(b-x)dx + \frac{\eta(|f'(b)|,|f'(a+\lambda)|)}{b-a-\lambda}\int_{a+\lambda}^b(b-x)(x-a-\lambda)dx$$

$$= \frac{\lambda^2}{2}|f'(a)| + \frac{\lambda^3}{3}\frac{\eta(|f'(a+\lambda)|,|f'(a)|)}{\lambda}$$

$$+ \frac{(b-a-\lambda)^2}{2}|f'(a+\lambda)| + \frac{(b-a-\lambda)^3}{6}\frac{\eta(|f'(b)|,|f'(a+\lambda)|)}{b-a-\lambda}$$

$$= \frac{\lambda^2}{6}[3|f'(a)|+2\eta(|f'(a+\lambda)|,|f'(a)|)]$$

$$+ \frac{(b-a-\lambda)^2}{6}[3|f'(a+\lambda)|+\eta(|f'(b)|,|f'(a+\lambda)|)]$$

and so we proved inequality (2.1). Then similarly

$$\left|\int_a^b f(x)g(x)dx - \int_{b-\lambda}^b f(x)dx\right|$$

$$\leq \left|\int_a^{b-\lambda}\left(\int_a^x g(t)dt\right)f'(x)dx\right| + \left|\int_{b-\lambda}^b\left(\int_x^b (1-g(t))dt\right)f'(x)dx\right|$$

$$\leq \int_a^{b-\lambda}\left|\int_a^x g(t)dt\right| |f'(x)|dx + \int_{b-\lambda}^b\left|\int_x^b (1-g(t))dt\right| |f'(x)|dx$$

$$= \int_a^{b-\lambda}\left|\int_a^x g(t)dt\right|\left|f'\left(\frac{x-a}{\lambda}(a+\lambda)+\frac{a+\lambda-x}{\lambda}a\right)\right|dx$$

$$+ \int_{b-\lambda}^b\left|\int_x^b (1-g(t))dt\right|\left|f'\left(\frac{x-a-\lambda}{b-a-\lambda}b+\frac{b-x}{b-a-\lambda}(a+\lambda)\right)\right|dx$$

$$\leq \int_a^{b-\lambda}\left|\int_a^x g(t)dt\right|\left[|f'(a)|+\frac{x-a}{\lambda}\eta(|f'(a+\lambda)|,|f'(a)|)\right]dx$$

$$+ \int_{b-\lambda}^b\left|\int_x^b (1-g(t))dt\right|\left[|f'(a+\lambda)|+\frac{x-a-\lambda}{b-a-\lambda}\eta(|f'(b)|,|f'(a+\lambda)|)\right]dx$$

$$= |f'(a)|\int_a^{b-\lambda}\left|\int_a^x g(t)dt\right|dx + \int_a^{b-\lambda}\left|\int_a^x g(t)dt\right|\frac{x-a}{\lambda}\eta(|f'(a+\lambda)|,|f'(a)|)dx$$

$$+ |f'(a+\lambda)|\int_{b-\lambda}^b\left|\int_x^b g(t)dt\right|dx$$

$$+ \int_{b-\lambda}^b\left|\int_x^b (1-g(t))dt\right|\frac{x-a-\lambda}{b-a-\lambda}\eta(|f'(b)|,|f'(a+\lambda)|)dx$$

$$\leq |f'(a)|\int_a^{b-\lambda}(x-a)dx+\frac{\eta(|f'(a+\lambda)|,|f'(a)|)}{b-a-\lambda}\int_a^{b-\lambda}(x-a)^2dx$$

$$+|f'(a+\lambda)|\int_{b-\lambda}^{b}(b-x)dx+\frac{\eta(|f'(b)|,|f'(a+\lambda)|)}{b-a-\lambda}\int_{b-\lambda}^{b}(b-x)(x-a-\lambda)dx$$

$$=\frac{(b-a-\lambda)^2}{2}|f'(a)|+\frac{(b-a-\lambda)^3}{3}\frac{\eta(|f'(a+\lambda)|,|f'(a)|)}{\lambda}$$

$$+|f'(a+\lambda)|\frac{\lambda^2}{2}+\frac{\lambda^2(3b-3a-5\lambda)}{6}\frac{\eta(|f'(b)|,|f'(a+\lambda)|)}{b-a-\lambda}$$

and so we proved inequality (2.2).

Theorem 2.2 *Let* $f,g:[a,b]\subset \mathbb{R}^+\to\mathbb{R}$ *be integrable such that* $0\leq g(t)\leq 1$, *for all* $t\in[a,b]$ *such that* $\int_a^b g(t)f'(t)dt$ *exists. If* f *is absolutely continuous on* $[a,b]$ *such that* $|f'|$ *is* $\eta-$*convex function on* $[a,b]$, *then we have*

$$\left|\int_a^{a+\lambda}f(x)dx-\int_a^b f(x)g(x)dx\right|$$

$$\leq\int_{a+\lambda}^{b}g(t)dt\left[\lambda\left(|f'(a+\lambda)|+\frac{\eta(|f'(a)|,|f'(a+\lambda)|)}{2}\right)\right.$$

$$\left.+(b-a-\lambda)\left(|f'(b)|+\frac{\eta(|f'(a+\lambda)|,|f'(b)|)}{2}\right)\right]$$

$$\leq(b-a-\lambda)\left[\lambda\left(|f'(a+\lambda)|+\frac{\eta(|f'(a)|,|f'(a+\lambda)|)}{2}\right)\right.$$

$$\left.+(b-a-\lambda)\left(|f'(b)|+\frac{\eta(|f'(a+\lambda)|,|f'(b)|)}{2}\right)\right] \tag{2.3}$$

and

$$\left|\int_a^b f(x)g(x)dx-\int_{b-\lambda}^{b}f(x)dx\right|$$

$$\leq\int_{b-\lambda}^{b}g(t)dt\left[(b-a-\lambda)\left(|f'(b-\lambda)|+\frac{\eta(|f'(a)|,|f'(b-\lambda)|)}{2}\right)\right.$$

$$\left.+\lambda\left(|f'(b)|+\frac{\eta(|f'(b-\lambda)|,|f'(b)|)}{2}\right)\right]$$

$$\leq(b-a-\lambda)\left[(b-a-\lambda)\left(|f'(b-\lambda)|+\frac{\eta(|f'(a)|,|f'(b-\lambda)|)}{2}\right)\right.$$

$$\left.+\lambda\left(|f'(b)|+\frac{\eta(|f'(b-\lambda)|,|f'(b)|)}{2}\right)\right]$$

where $\lambda:=\int_a^b g(t)dt$.

Proof. From Lemma 1.1, we have

$$\left|\int_a^{a+\lambda}f(x)dx-\int_a^b f(x)g(x)dx\right|\leq\sup_{x\in[a,a+\lambda]}\left[\int_a^x(1-g(t))dt\right]\int_a^{a+\lambda}|f'(x)|dx$$

$$+\sup_{x\in[a+\lambda,b]}\left[\int_x^b g(t)dt\right]\int_{a+\lambda}^{b}|f'(x)|dx.$$

Since $|f'|$ is $\eta-$convex on $[a,b]$, we get

$$\int_a^{a+\lambda}|f'(x)|dx\leq\lambda\left(|f'(a+\lambda)|+\frac{\eta(|f'(a)|,|f'(a+\lambda)|)}{2}\right),$$

and

$$\int_{a+\lambda}^{b}|f'(x)|dx\leq(b-a-\lambda)\left(|f'(b)|+\frac{\eta(|f'(a+\lambda)|,|f'(b)|)}{2}\right).$$

Therefore, we have

$$\left|\int_a^{a+\lambda}f(x)dx-\int_a^b f(x)g(x)dx\right|$$

$$\leq\lambda\left(|f'(a+\lambda)|+\frac{\eta(|f'(a)|,|f'(a+\lambda)|)}{2}\right)\left[\int_a^{a+\lambda}(1-g(t))dt\right]$$

$$+(b-a-\lambda)\left(|f'(b)|+\frac{\eta(|f'(a+\lambda)|,|f'(b)|)}{2}\right)\left[\int_{a+\lambda}^{b}g(t)dt\right]$$

$$\leq\max\left\{\int_a^{a+\lambda}(1-g(t))dt,\int_{a+\lambda}^{b}g(t)dt\right\}\left[\lambda\left(|f'(a+\lambda)|+\frac{\eta(|f'(a)|,|f'(a+\lambda)|)}{2}\right)\right.$$

$$\left.+(b-a-\lambda)\left(|f'(b)|+\frac{\eta(|f'(a+\lambda)|,|f'(b)|)}{2}\right)\right]$$

$$=\int_{a+\lambda}^{b} g(t)dt\left[\lambda\left(|f'(a+\lambda)|+\frac{\eta(|f'(a)|,|f'(a+\lambda)|)}{2}\right)\right.$$

$$\left.+(b-a-\lambda)\left(|f'(b)|+\frac{\eta(|f'(a+\lambda)|,|f'(b)|)}{2}\right)\right]$$

which proves the first inequality (2.3). The second inequality in (2.3) follows directly, since $0\le g(t)\le 1$ for all $t\in[a,b]$, then

$$0\le\int_{a+\lambda}^{b} g(t)dt\le b-a-\lambda.$$

Similarly

$$\left|\int_a^b f(x)g(x)dx-\int_{b-\lambda}^{b} f(x)dx\right|\le \sup_{x\in[a,b-\lambda]}\left[\int_a^x g(t)dt\right]\int_a^{b-\lambda}|f'(x)|dx$$

$$+\sup_{x\in[b-\lambda,b]}\left[\int_x^b (1-g(t))dt\right]\int_{b-\lambda}^{b}|f'(x)|dx.$$

Since $|f'|$ is $\eta-$convex on $[a,b]$, we get

$$\int_a^{b-\lambda}|f'(x)|dx\le(b-a-\lambda)\left(|f'(b-\lambda)|+\frac{\eta(|f'(a)|,|f'(b-\lambda)|)}{2}\right),$$

and

$$\int_{b-\lambda}^{b}|f'(x)|dx\le\lambda\left(|f'(b)|+\frac{\eta(|f'(b-\lambda)|,|f'(b)|)}{2}\right).$$

Therefore, we have

$$\left|\int_a^b f(x)g(x)dx-\int_{b-\lambda}^{b} f(x)dx\right|$$

$$\le(b-a-\lambda)\left(|f'(b-\lambda)|+\frac{\eta(|f'(a)|,|f'(b-\lambda)|)}{2}\right)\left[\int_a^{b-\lambda} g(t)dt\right]$$

$$+\lambda\left(|f'(b)|+\frac{\eta(|f'(b-\lambda)|,|f'(b)|)}{2}\right)\left[\int_{b-\lambda}^{b}(1-g(t))dt\right]$$

$$\le\max\left\{\int_a^{b-\lambda} g(t)dt,\int_{b-\lambda}^{b}(1-g(t))dt\right\}\left[(b-a-\lambda)\left(|f'(b-\lambda)|+\frac{\eta(|f'(a)|,|f'(b-\lambda)|)}{2}\right)\right.$$

$$\left.+\lambda\left(|f'(b)|+\frac{\eta(|f'(b-\lambda)|,|f'(b)|)}{2}\right)\right]$$

$$=\int_{b-\lambda}^{b} g(t)dt\left[(b-a-\lambda)\left(|f'(b-\lambda)|+\frac{\eta(|f'(a)|,|f'(b-\lambda)|)}{2}\right)\right.$$

$$\left.+\lambda\left(|f'(b)|+\frac{\eta(|f'(b-\lambda)|,|f'(b)|)}{2}\right)\right]$$

which proves the first inequality (2.4). The second inequality in (2.4) follows directly, since $0\le g(t)\le 1$ for all $t\in[a,b]$, then

$$0\le\int_a^{b-\lambda} g(t)dt\le b-a-\lambda.$$

Theorem 2.3 *Let* $f,g:[a,b]\subset\mathbb{R}^+\to\mathbb{R}$ *be integrable such that* $0\le g(t)\le 1$*, for all* $t\in[a,b]$ *such that* $\int_a^b g(t)f'(t)dt$ *exists. If* f *is absolutely continuous on* $[a,b]$ *such that* $|f'|$ *is* $\eta-$*convex function on* $[a,b]$ *and* $q>1$*, then we have*

$$\left|\int_a^{a+\lambda} f(x)dx-\int_a^b f(x)g(x)dx\right|$$

$$\le\frac{\lambda^2}{(p+1)^{\frac{1}{p}}}\left[\left(|f'(a+\lambda)|^q+\frac{\eta(|f'(a)|^q,|f'(a+\lambda)|^q}{2}\right)^{\frac{1}{q}}\right.$$

$$\left.+\frac{(b-a-\lambda)^2}{(p+1)^{\frac{1}{p}}}\left(|f'(b)|^q+\frac{\eta(|f'(a+\lambda)|^q,|f'(b)|^q)}{2}\right)^{\frac{1}{q}}\right] \tag{2.5}$$

and

$$\left|\int_a^b f(x)g(x)dx-\int_{b-\lambda}^{b} f(x)dx\right|$$

$$\le\frac{(b-a-\lambda)^2}{(p+1)^{\frac{1}{p}}}\left[\left(|f'(b-\lambda)|^q+\frac{\eta(|f'(a)|^q,|f'(b-\lambda)|^q}{2}\right)^{\frac{1}{q}}\right.$$

$$\left.+\frac{\lambda^2}{(p+1)^{\frac{1}{p}}}\left(|f'(b)|^q+\frac{\eta(|f'(b-\lambda)|^q,|f'(b)|^q)}{2}\right)^{\frac{1}{q}}\right] \tag{2.6}$$

where $\lambda:=\int_a^b g(t)dt$.

Proof. From Lemma 1.1, $\eta-$convexity of $|f'|^q$ and using the Hölder inequality for $q>1$, and $p=\dfrac{q}{q-1}$, we obtain

$$\left|\int_a^{a+\lambda} f(x)dx-\int_a^b f(x)g(x)dx\right|$$

$$\le\int_a^{a+\lambda}\left|\int_a^x(1-g(t))dt\right||f'(x)|dx+\int_{a+\lambda}^{b}\left|\int_x^b g(t)dt\right||f'(x)|dx$$

$$\le\left(\int_a^{a+\lambda}\left|\int_a^x(1-g(t))dt\right|^p dx\right)^{\frac{1}{p}}\left(\int_a^{a+\lambda}|f'(x)|^q dx\right)^{\frac{1}{q}}$$

$$+\left(\int_{a+\lambda}^{b}\left|\int_{x}^{b}g(t)dt\right|^{p}dx\right)^{\frac{1}{p}}\left(\int_{a+\lambda}^{b}|f'(x)|^{q}dx\right)^{\frac{1}{q}}:=M \tag{2.7}$$

where p is the conjugate of q.

Since $|f'|^q$ is $\eta-$convex on $[a,b]$, we have

$$\int_{a}^{a+\lambda}|f'(x)|^{q}dx\le\lambda\left(|f'(a+\lambda)|^{q}+\frac{\eta(|f'(a)|^{q},|f'(a+\lambda)|^{q})}{2}\right),$$

and

$$\int_{a+\lambda}^{b}|f'(x)|^{q}dx\le(b-a-\lambda)\left(|f'(b)|^{q}+\frac{\eta(|f'(a+\lambda)|^{q},|f'(b)|^{q})}{2}\right)$$

which gives by (2.7)

$$M\le\left(\int_{a}^{a+\lambda}(x-a)^{p}dx\right)^{\frac{1}{p}}\lambda^{\frac{1}{q}}\left[|f'(a+\lambda)|^{q}+\frac{\eta(|f'(a)|^{q},|f'(a+\lambda)|^{q})}{2}\right]^{\frac{1}{q}}$$

$$+\left(\int_{a+\lambda}^{b}(b-x)^{p}dx\right)^{\frac{1}{p}}(b-a-\lambda)^{\frac{1}{q}}\left[|f'(b)|^{q}+\frac{\eta(|f'(a+\lambda)|^{q},|f'(b)|^{q})}{2}\right]^{\frac{1}{q}}$$

$$=\frac{\lambda^{2}}{(p+1)^{\frac{1}{p}}}\left[|f'(a+\lambda)|^{q}+\frac{\eta(|f'(a)|^{q},|f'(a+\lambda)|^{q})}{2}\right]^{\frac{1}{q}}$$

$$+\frac{(b-a-\lambda)^{2}}{(p+1)^{\frac{1}{p}}}\left[|f'(b)|^{q}+\frac{\eta(|f'(a+\lambda)|^{q},|f'(b)|^{q})}{2}\right]^{\frac{1}{q}}$$

giving the inequality (2.5). Similarly

$$\left|\int_{a}^{b}f(x)g(x)dx-\int_{b-\lambda}^{b}f(x)dx\right| \tag{2.8}$$

$$\le\int_{a}^{b-\lambda}\left|\int_{a}^{x}g(t)dt\right||f'(x)|dx+\int_{b-\lambda}^{b}\left|\int_{x}^{b}(1-g(t))dt\right||f'(x)|dx$$

$$\le\left(\int_{a}^{b-\lambda}\left|\int_{a}^{x}g(t)dt\right|^{p}dx\right)^{\frac{1}{p}}\left(\int_{a}^{b-\lambda}|f'(x)|^{q}dx\right)^{\frac{1}{q}}$$

$$+\left(\int_{b-\lambda}^{b}\left|\int_{x}^{b}(1-g(t))dt\right|^{p}dx\right)^{\frac{1}{p}}\left(\int_{b-\lambda}^{b}|f'(x)|^{q}dx\right)^{\frac{1}{q}}:=N$$

where p is the conjugate of q.

Since $|f'|^q$ is $\eta-$convex on $[a,b]$, we have

$$\int_{a}^{b-\lambda}|f'(x)|^{q}dx\le(b-a-\lambda)\left(|f'(b-\lambda)|^{q}+\frac{\eta(|f'(a)|^{q},|f'(b-\lambda)|^{q})}{2}\right),$$

and

$$\int_{b-\lambda}^{b}|f'(x)|^{q}dx\le\lambda\left(|f'(b)|^{q}+\frac{\eta(|f'(b-\lambda)|^{q},|f'(b)|^{q})}{2}\right)$$

which gives by (2.8)

$$N\le\left(\int_{a}^{b-\lambda}(x-a)^{p}dx\right)^{\frac{1}{p}}(b-a-\lambda)^{\frac{1}{q}}\left[|f'(b-\lambda)|^{q}+\frac{\eta(|f'(a)|^{q},|f'(b-\lambda)|^{q})}{2}\right]^{\frac{1}{q}}$$

$$+\left(\int_{b-\lambda}^{b}(b-x)^{p}dx\right)^{\frac{1}{p}}\lambda^{\frac{1}{q}}\left[|f'(b)|^{q}+\frac{\eta(|f'(b-\lambda)|^{q},|f'(b)|^{q})}{2}\right]^{\frac{1}{q}}$$

$$=\frac{(b-a-\lambda)^{2}}{(p+1)^{\frac{1}{p}}}\left[|f'(b-\lambda)|^{q}+\frac{\eta(|f'(a)|^{q},|f'(b-\lambda)|^{q})}{2}\right]^{\frac{1}{q}}$$

$$+\frac{\lambda^{2}}{(p+1)^{\frac{1}{p}}}\left[|f'(b)|^{q}+\frac{\eta(|f'(b-\lambda)|^{q},|f'(b)|^{q})}{2}\right]^{\frac{1}{q}}$$

giving the inequality (2.6).

3. *Steffensen's type inequalities for P- function*

Theorem 3.1 *Let $f,g:[a,b]\subset\mathbb{R}^{+}\to\mathbb{R}$ be integrable such that $0\le g(t)\le 1$, for all $t\in[a,b]$ such that $\int_{a}^{b}g(t)f'(t)dt$ exists. If f is absolutely continuous on $[a,b]$ such that $|f'|$ is P function on $[a,b]$, then we have*

$$\left|\int_{a}^{a+\lambda}f(x)dx-\int_{a}^{b}f(x)g(x)dx\right| \tag{3.1}$$

$$\le\frac{\lambda^{2}}{2}\left[|f'(a)|+|f'(a+\lambda)|\right]+\frac{(b-a-\lambda)^{2}}{2}\left[|f'(b)|+|f'(a+\lambda)|\right]$$

and

$$\left|\int_{a}^{b}f(x)g(x)dx-\int_{b-\lambda}^{b}f(x)dx\right| \tag{3.2}$$

$$\le\frac{(b-a-\lambda)^{2}}{2}\left[|f'(a)|+|f'(a+\lambda)|\right]+\frac{\lambda^{2}}{2}\left[|f'(b)|+|f'(a+\lambda)|\right]$$

where $\lambda:=\int_{a}^{b}g(t)dt$.

Proof. Using Lemma 1.1 and since $|f'|$ is P function, we have

$$\left|\int_{a}^{a+\lambda}f(x)dx-\int_{a}^{b}f(x)g(x)dx\right|$$

$$\le\left|\int_{a}^{a+\lambda}\left(\int_{a}^{x}(1-g(t))dt\right)f'(x)dx\right|+\left|\int_{a+\lambda}^{b}\left(\int_{x}^{b}g(t)dt\right)f'(x)dx\right|$$

$$\le\int_{a}^{a+\lambda}\left|\int_{a}^{x}(1-g(t))dt\right||f'(x)|dx+\int_{a+\lambda}^{b}\left|\int_{x}^{b}g(t)dt\right||f'(x)|dx$$

$$= \int_a^{a+\lambda}\left|\int_a^x (1-g(t))dt\right|\left|f'\left(\frac{x-a}{\lambda}(a+\lambda)+\frac{a+\lambda-x}{\lambda}a\right)\right|dx$$

$$+\int_{a+\lambda}^b\left|\int_x^b g(t)dt\right|\left|f'\left(\frac{x-a-\lambda}{b-a-\lambda}b+\frac{b-x}{b-a-\lambda}(a+\lambda)\right)\right|dx$$

$$\le \int_a^{a+\lambda}\left|\int_a^x (1-g(t))dt\right|\left[|f'(a+\lambda)|+|f'(a)|\right]dx$$

$$+\int_{a+\lambda}^b\left|\int_x^b g(t)dt\right|\left[|f'(b)|+|f'(a+\lambda)|\right]dx$$

$$=\int_a^{a+\lambda}\left|\int_a^x (1-g(t))dt\right||f'(a+\lambda)|dx+\int_a^{a+\lambda}\left|\int_a^x (1-g(t))dt\right||f'(a)|dx$$

$$+\int_{a+\lambda}^b\left|\int_x^b g(t)dt\right||f'(b)|dx+\int_{a+\lambda}^b\left|\int_x^b g(t)dt\right||f'(a+\lambda)|dx$$

$$\le \int_a^{a+\lambda}(x-a)|f'(a+\lambda)|dx+\int_a^{a+\lambda}(x-a)|f'(a)|dx$$

$$+\int_{a+\lambda}^b(b-x)|f'(b)|dx+\int_{a+\lambda}^b(b-x)|f'(a+\lambda)|dx$$

$$=|f'(a+\lambda)|\int_a^{a+\lambda}(x-a)dx+|f'(a)|\int_a^{a+\lambda}(x-a)dx$$

$$+|f'(b)|\int_{a+\lambda}^b(b-x)dx+|f'(a+\lambda)|\int_{a+\lambda}^b(b-x)dx$$

$$=\frac{\lambda^2}{2}[|f'(a+\lambda)|+|f'(a)|]+\frac{(b-a-\lambda)^2}{2}[|f'(b)|+|f'(a+\lambda)|]$$

and so we proved inequality (3.1). Then similarly

$$\left|\int_a^b f(x)g(x)dx-\int_{b-\lambda}^b f(x)dx\right|$$

$$\le\left|\int_a^{b-\lambda}\left(\int_a^x g(t)dt\right)f'(x)dx\right|+\left|\int_{b-\lambda}^b\left(\int_x^b(1-g(t))dt\right)f'(x)dx\right|$$

$$\le\int_a^{b-\lambda}\left|\int_a^x g(t)dt\right||f'(x)|dx+\int_{b-\lambda}^b\left|\int_x^b(1-g(t))dt\right||f'(x)|dx$$

$$=\int_a^{b-\lambda}\left|\int_a^x g(t)dt\right|\left|f'\left(\frac{x-a}{\lambda}(a+\lambda)+\frac{a+\lambda-x}{\lambda}a\right)\right|dx$$

$$+\int_{b-\lambda}^b\left|\int_x^b(1-g(t))dt\right|\left|f'\left(\frac{x-a-\lambda}{b-a-\lambda}b+\frac{b-x}{b-a-\lambda}(a+\lambda)\right)\right|dx$$

$$\le\int_a^{b-\lambda}\left|\int_a^x g(t)dt\right|\left[|f'(a+\lambda)|+|f'(a)|\right]dx$$

$$+\int_{b-\lambda}^b\left|\int_x^b(1-g(t))dt\right|\left[|f'(b)|+|f'(a+\lambda)|\right]dx$$

$$=|f'(a+\lambda)|\int_a^{b-\lambda}\left|\int_a^x g(t)dt\right|dx+|f'(a)|\int_a^{b-\lambda}\left|\int_a^x g(t)dt\right|dx$$

$$+|f'(b)|\int_{b-\lambda}^b\left|\int_x^b g(t)dt\right|dx+|f'(a+\lambda)|\int_{b-\lambda}^b\left|\int_x^b(1-g(t))dt\right|dx$$

$$\le|f'(a+\lambda)|\int_a^{b-\lambda}(x-a)dx+|f'(a)|\int_a^{b-\lambda}(x-a)dx$$

$$+|f'(b)|\int_{b-\lambda}^b(b-x)dx+|f'(a+\lambda)|\int_{b-\lambda}^b(b-x)dx$$

$$=\frac{(b-a-\lambda)^2}{2}[|f'(a+\lambda)|+|f'(a)|]+\frac{\lambda^2}{2}[|f'(b)|+|f'(a+\lambda)|]$$

and so we proved inequality (3.2).

Theorem 3.2 *Let* $f,g:[a,b]\subset\mathbb{R}^+\to\mathbb{R}$ *be integrable such that* $0\le g(t)\le 1$*, for all* $t\in[a,b]$ *such that* $\int_a^b g(t)f'(t)dt$ *exists. If* f *is absolutely continuous on* $[a,b]$ *such that* $|f'|$ *is* P *function on* $[a,b]$*, then the following inequalities holds*

$$\left|\int_a^{a+\lambda}f(x)dx-\int_a^b f(x)g(x)dx\right|$$

$$\le\left(\int_{a+\lambda}^b g(t)dt\right)\left[\lambda\left(|f'(a)|+|f'(a+\lambda)|\right)+(b-a-\lambda)\left(|f'(a+\lambda)+|f'(b)|\right)\right]$$

$$\le(b-a-\lambda)\left[\lambda\left(|f'(a)|+|f'(a+\lambda)|\right)+(b-a-\lambda)\left(|f'(a+\lambda)+|f'(b)|\right)\right]$$

and

$$\left|\int_a^b f(x)g(x)dx-\int_{b-\lambda}^b f(x)dx\right|$$

$$\le\left(\int_a^{b-\lambda}g(t)dt\right)\left[(b-a-\lambda)\left(|f'(b-\lambda)|+|f'(a)|\right)+\lambda\left(|f'(b)|+|f'(b-\lambda)|\right)\right]$$

$$\le\lambda\left[(b-a-\lambda)\left(|f'(b-\lambda)|+|f'(a)|\right)+\lambda\left(|f'(b)|+|f'(b-\lambda)|\right)\right]$$

where $\lambda:=\int_a^b g(t)dt$.

Proof. From Lemma 1.1, we can write

$$\left|\int_a^{a+\lambda} f(x)dx - \int_a^b f(x)g(x)dx\right| \le \sup_{x\in[a,a+\lambda]}\left[\int_a^x (1-g(t))dt\right]\int_a^{a+\lambda} |f'(x)|\,dx$$

$$+\sup_{x\in[a+\lambda,b]}\left[\int_x^b g(t)dt\right]\int_{a+\lambda}^b |f'(x)|\,dx.$$

Since $|f'|$ is P function on $[a,b]$ and using (1.3), we have

$$\int_a^{a+\lambda} |f'(x)|\,dx \le \lambda\left(|f'(a)|+|f'(a+\lambda)|\right),$$

and

$$\int_{a+\lambda}^b |f'(x)|\,dx \le (b-a-\lambda)\left(|f'(a+\lambda)+|f'(b)|\right).$$

Therefore, we have

$$\left|\int_a^{a+\lambda} f(x)dx - \int_a^b f(x)g(x)dx\right|$$

$$\le \lambda\left(|f'(a)|+|f'(a+\lambda)|\right)\left[\int_a^{a+\lambda}(1-g(t))dt\right]$$

$$+(b-a-\lambda)\left(|f'(a+\lambda)+|f'(b)|\right)\left[\int_{a+\lambda}^b g(t)dt\right]$$

$$\le \max\left\{\int_a^{a+\lambda}(1-g(t))dt, \int_{a+\lambda}^b g(t)dt\right\}\left[\lambda\left(|f'(a)|+|f'(a+\lambda)|\right)\right.$$

$$\left.+(b-a-\lambda)\left(|f'(a+\lambda)+|f'(b)|\right)\right]$$

$$=\left(\int_{a+\lambda}^b g(t)dt\right)\left[\lambda\left(|f'(a)|+|f'(a+\lambda)|\right)+(b-a-\lambda)\left(|f'(a+\lambda)+|f'(b)|\right)\right].$$

Since $0\le g(t)\le 1$ for all $t\in[a,b]$, we can write

$$0\le \int_{a+\lambda}^b g(t)dt \le b-a-\lambda.$$

So, we obtained

$$\left|\int_a^{a+\lambda} f(x)dx - \int_a^b f(x)g(x)dx\right|$$

$$\le\left(\int_{a+\lambda}^b g(t)dt\right)\left[\lambda\left(|f'(a)|+|f'(a+\lambda)|\right)+(b-a-\lambda)\left(|f'(a+\lambda)+|f'(b)|\right)\right]$$

$$\le (b-a-\lambda)\left[\lambda\left(|f'(a)|+|f'(a+\lambda)|\right)+(b-a-\lambda)\left(|f'(a+\lambda)+|f'(b)|\right)\right].$$

Similarly

$$\left|\int_a^b f(x)g(x)dx - \int_{b-\lambda}^b f(x)dx\right| \le \sup_{x\in[a,b-\lambda]}\left[\int_a^x g(t)dt\right]\int_a^{b-\lambda} |f'(x)|\,dx$$

$$+\sup_{x\in[b-\lambda,b]}\left[\int_x^b (1-g(t))dt\right]\int_{b-\lambda}^b |f'(x)|\,dx.$$

Since $|f'|$ is P function on $[a,b]$ and using (1.3), we get

$$\int_a^{b-\lambda} |f'(x)|\,dx \le (b-a-\lambda)\left(|f'(b-\lambda)|+|f'(a)|\right),$$

and

$$\int_{b-\lambda}^b |f'(x)|\,dx \le \lambda\left(|f'(b)|+|f'(b-\lambda)|\right).$$

Therefore, we have

$$\left|\int_a^b f(x)g(x)dx - \int_{b-\lambda}^b f(x)dx\right|$$

$$\le (b-a-\lambda)\left(|f'(b-\lambda)|+|f'(a)|\right)\left[\int_a^{b-\lambda} g(t)dt\right]$$

$$+\lambda\left(|f'(b)|+|f'(b-\lambda)|\right)\left[\int_{b-\lambda}^b (1-g(t))dt\right]$$

$$\le \max\left\{\int_a^{b-\lambda} g(t)dt, \int_{b-\lambda}^b (1-g(t))dt\right\}\left[(b-a-\lambda)\left(|f'(b-\lambda)|+|f'(a)|\right)\right.$$

$$\left.+\lambda\left(|f'(b)|+|f'(b-\lambda)|\right)\right]$$

$$=\left(\int_{b-\lambda}^b (1-g(t))dt\right)\left[(b-a-\lambda)\left(|f'(b-\lambda)|+|f'(a)|\right)+\lambda\left(|f'(b)|+|f'(b-\lambda)|\right)\right]$$

Since $0\le g(t)\le 1$ for all $t\in[a,b]$, we can write

$$0\le \int_{b-\lambda}^b (1-g(t))dt \le \lambda.$$

So, we obtained

$$\left|\int_a^b f(x)g(x)dx - \int_{b-\lambda}^b f(x)dx\right|$$

$$\le\left(\int_a^{b-\lambda} g(t)dt\right)\left[(b-a-\lambda)\left(|f'(b-\lambda)|+|f'(a)|\right)+\lambda\left(|f'(b)|+|f'(b-\lambda)|\right)\right]$$

$$\le \lambda\left[(b-a-\lambda)\left(|f'(b-\lambda)|+|f'(a)|\right)+\lambda\left(|f'(b)|+|f'(b-\lambda)|\right)\right]$$

and the proof is completed.

4. Conclusion

In the present paper, we prove some Steffensen's type inequalities by utilizing P-function and η-convex function. The new bounds can establish by used different classes of convex functions instead of this convex functions by researches interested in the subject.

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