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HERMITE-HADAMARD TYPE FRACTIONAL INTEGRAL INEQUALITIES FOR TWICE DIFFERENTIABLE GENERALIZED (s, m, φ)-PREINVEX FUNCTIONS

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Abstract. In the present paper, a new class of generalized (s, m, φ)-preinvex function is introduced and some new integral inequalities for the left-hand side of Gauss-Jacobi type quadrature formula involving generalized (s, m, φ)-preinvex functions along with beta function are given. Moreover, some generalizations of Hermite-Hadamard type inequalities for generalized (s, m, φ)-preinvex functions that are twice differentiable via Riemann-Liouville fractional integrals are established. At the end, some applications to special means are given.

1. Introduction

The following notation are used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$ and I° to denote the interior of I. For any subset $K \subseteq \mathbb{R}^n, K^\circ$ is used to denote the interior of K. $\mathbb{R}^n$ is used to denote a n-dimensional vector space. The nonnegative real numbers are denoted by $\mathbb{R}_\circ = [0, +\infty)$. The set of integrable functions on the interval $[a, b]$ is denoted by $L_1[a, b]$.

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1.1. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on I and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

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Fractional calculus ([8]) and the references cited therein, was introduced at the end of the nineteenth century by Liouville and Riemann, the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics.

Definition 1.1. Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad b > x,$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-u} u^{\alpha-1} du$. Here $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$.
In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Due to the wide application of fractional integrals, some authors extended to study fractional Hermite-Hadamard type inequalities for functions of different classes ([2],[8],[11]-[13],[15],[16]) and the references cited therein.

Now, let us recall some definitions of various convex functions.

Definition 1.2. ([4]) A nonnegative function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}_\circ$ is said to be P-function or P-convex, if

$$f(tx + (1-t)y) \leq f(x) + f(y), \quad \forall x, y \in I,\ t \in [0,1].$$

Definition 1.3. ([6]) A function $f : \mathbb{R}_\circ \longrightarrow \mathbb{R}$ is said to be s-convex in the second sense, if

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y), \tag{1.2}$$

for all $x, y \geq 0,\ \lambda \in [0,1]$ and $s \in (0,1]$.

It is clear that a 1-convex function must be convex on $\mathbb{R}_\circ$ as usual. The s-convex functions in the second sense have been investigated in ([6]).

Definition 1.4. ([1]) A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$, if $x + t\eta(y,x) \in K$ for every $x, y \in K$ and $t \in [0,1]$.

Notice that every convex set is invex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not necessarily true ([1],[17]).

Definition 1.5. ([10]) The function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect η, if for every $x, y \in K$ and $t \in [0,1]$, we have that

$$f\left(x + t\eta(y,x)\right) \leq (1-t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not true.

The Gauss-Jacobi type quadrature formula has the following

$$\int_a^b (x-a)^p (b-x)^q f(x)dx = \sum_{k=0}^{+\infty} B_{m,k} f(\gamma_k) + R^{\star}_m |f|, \tag{1.3}$$

for certain $B_{m,k}, \gamma_k$ and rest $R^{\star}_m|f|$ ([14]).
Recently, Liu ([7]) obtained several integral inequalities for the left-hand side of (1.3) under the Definition 1.2 of P-function.
Also in ([9]), Özdemir et al. established several integral inequalities concerning the left-hand side of (1.3) via some kinds of convexity.

Motivated by these results, in Section 2, the notion of generalized (s, m, φ)-preinvex function is introduced and some new integral inequalities for the left-hand side of (1.3) involving generalized (s, m, φ)-preinvex functions along with beta function are given. In Section 3, some generalizations of Hermite-Hadamard type inequalities for generalized (s, m, φ)-preinvex functions that are twice differentiable via fractional integrals are given. In Section 4, some applications to special means are given. These general inequalities give us some new estimates for Hermite-Hadamard type fractional integral inequalities.

2. New integral inequalities

Definition 2.1. ([5]) A set $K \subseteq \mathbb{R}^n$ is said to be m-invex with respect to the mapping $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}^n$ for some fixed $m \in (0,1]$, if $mx + t\eta(y, mx) \in K$ holds for each $x, y \in K$ and any $t \in [0,1]$.

Remark 2.1. In Definition 2.1, under certain conditions, the mapping $\eta(y, mx)$ could reduce to $\eta(y, x)$. For example when $m = 1$, then the m-invex set degenerates an invex set on K.

We next give new definition, to be referred as generalized (s, m, φ)-preinvex function.

Definition 2.2. Let $K \subseteq \mathbb{R}$ be an open m-invex set with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ and $\varphi : I \longrightarrow K$ a continuous function. For $f : K \longrightarrow \mathbb{R}$, $\forall\, s \in (0,1]$ and some fixed $m \in (0,1]$, if

$$f\left(m\varphi(x) + \lambda\eta(\varphi(y), \varphi(x), m)\right) \leq m(1-\lambda)^s f(\varphi(x)) + \lambda^s f(\varphi(y)) \tag{2.1}$$

is valid $\forall\, x, y \in I, \lambda \in [0,1]$, then we say that $f(x)$ is a generalized (s, m, φ)-preinvex function with respect to η.

Remark 2.2. In Definition 2.2, it is worthwhile to note that the class of generalized (s, m, φ)-preinvex function is a generalization of the class of s-convex in the second sense function given in Definition 1.3.

In this section, in order to prove our main results regarding some new integral inequalities involving generalized (s, m, φ)-preinvex functions along with beta function, we need the following new lemma:

Lemma 2.1. *Let $\varphi : I \longrightarrow K$ be a continuous function. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow \mathbb{R}$ is a continuous function on K° with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for $\eta(\varphi(b), \varphi(a), m) > 0$. Then for any fixed $m \in (0,1]$ and $p, q > 0$, we have*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx$$
$$= \eta^{p+q+1}(\varphi(b), \varphi(a), m) \int_0^1 t^p (1-t)^q f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))dt.$$

Proof. It is easy to observe that

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$= \eta(\varphi(b),\varphi(a),m)\int_0^1 (m\varphi(a)+t\eta(\varphi(b),\varphi(a),m)-m\varphi(a))^p$$
$$\times(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-m\varphi(a)-t\eta(\varphi(b),\varphi(a),m))^q$$
$$\times f(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))dt$$
$$= \eta^{p+q+1}(\varphi(b),\varphi(a),m)\int_0^1 t^p(1-t)^q f(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))dt.$$

□

The following definition will be used in the sequel.

Definition 2.3. The Euler beta function is defined for $x, y > 0$ as

$$\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Theorem 2.1. *Let $\varphi : I \longrightarrow K$ be a continuous function. Assume that $f : K = [m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)] \longrightarrow \mathbb{R}$ is a continuous function on K° where $\eta(\varphi(b),\varphi(a),m) > 0$. Let $k > 1$ and $s \in (0,1]$. If $|f|^{\frac{k}{k-1}}$ is a generalized (s,m,φ)-preinvex function on an open m-invex set K with respect to $\eta : K\times K\times(0,1] \longrightarrow \mathbb{R}$ for some fixed $m \in (0,1]$, then for any fixed $p, q > 0$,*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$
$$\le \frac{\eta^{p+q+1}(\varphi(b),\varphi(a),m)}{(s+1)^{\frac{k-1}{k}}}\Big[\beta(kp+1,kq+1)\Big]^{\frac{1}{k}}$$
$$\times\Big(m|f(\varphi(a))|^{\frac{k}{k-1}}+|f(\varphi(b))|^{\frac{k}{k-1}}\Big)^{\frac{k-1}{k}}.$$

Proof. Since $|f|^{\frac{k}{k-1}}$ is a generalized (s,m,φ)-preinvex function on K, combining with Lemma 2.1, Definition 2.3, Hölder inequality and property of the modulus, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$
$$\le |\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\left[\int_0^1 t^{kp}(1-t)^{kq}dt\right]^{\frac{1}{k}}$$
$$\times\left[\int_0^1 \big|f(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))\big|^{\frac{k}{k-1}}dt\right]^{\frac{k-1}{k}}$$
$$\le \eta^{p+q+1}(\varphi(b),\varphi(a),m)\Big[\beta(kp+1,kq+1)\Big]^{\frac{1}{k}}$$
$$\times\left[\int_0^1 \Big(m(1-t)^s|f(\varphi(a))|^{\frac{k}{k-1}}+t^s|f(\varphi(b))|^{\frac{k}{k-1}}\Big)dt\right]^{\frac{k-1}{k}}$$

$$= \frac{\eta^{p+q+1}(\varphi(b),\varphi(a),m)}{(s+1)^{\frac{k-1}{k}}}\Big[\beta(kp+1,kq+1)\Big]^{\frac{1}{k}}$$
$$\times\Big(m|f(\varphi(a))|^{\frac{k}{k-1}}+|f(\varphi(b))|^{\frac{k}{k-1}}\Big)^{\frac{k-1}{k}}.$$

□

Theorem 2.2. *Let $\varphi : I \longrightarrow K$ be a continuous function. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow \mathbb{R}$ is a continuous function on K° where $\eta(\varphi(b), \varphi(a), m) > 0$. Let $l \geq 1$ and $s \in (0, 1]$. If $|f|^l$ is a generalized (s, m, φ)-preinvex function on an open m-invex set K with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}$ for some fixed $m \in (0, 1]$, then for any fixed $p, q > 0$,*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$
$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\Big[\beta(p+1,q+1)\Big]^{\frac{l-1}{l}}$$
$$\times\Big[m|f(\varphi(a))|^l\beta\left(p+1,q+s+1\right)+|f(\varphi(b))|^l\beta\left(p+s+1,q+1\right)\Big]^{\frac{1}{l}}.$$

Proof. Since $|f|^l$ is a generalized (s, m, φ)-preinvex function on K, combining with Lemma 2.1, Definition 2.3, the well-known power mean inequality and property of the modulus, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$
$$= \eta^{p+q+1}(\varphi(b),\varphi(a),m)$$
$$\times\int_0^1\Big[t^p(1-t)^q\Big]^{\frac{l-1}{l}}\Big[t^p(1-t)^q\Big]^{\frac{1}{l}}f(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))dt$$
$$\leq |\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\left[\int_0^1 t^p(1-t)^q dt\right]^{\frac{l-1}{l}}$$
$$\times\left[\int_0^1 t^p(1-t)^q\big|f(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))\big|^l dt\right]^{\frac{1}{l}}$$
$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\Big[\beta(p+1,q+1)\Big]^{\frac{l-1}{l}}$$
$$\times\left[\int_0^1 t^p(1-t)^q\left(m(1-t)^s|f(\varphi(a))|^l+t^s|f(\varphi(b))|^l\right)dt\right]^{\frac{1}{l}}$$
$$= \eta^{p+q+1}(\varphi(b),\varphi(a),m)\Big[\beta(p+1,q+1)\Big]^{\frac{l-1}{l}}$$
$$\times\Big[m|f(\varphi(a))|^l\beta\left(p+1,q+s+1\right)+|f(\varphi(b))|^l\beta\left(p+s+1,q+1\right)\Big]^{\frac{1}{l}}.$$

□

3. Hermite-Hadamard type fractional integral inequalities

In this section, in order to prove our main results regarding some generalizations of Hermite-Hadamard type inequalities for generalized (s, m, φ)-preinvex functions via fractional integrals, we need the following new fractional integral identity:

Lemma 3.1. *Let $\varphi : I \longrightarrow A$ be a continuous function. Suppose $A \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : A \times A \times (0,1] \longrightarrow \mathbb{R}$ for some fixed $m \in (0,1]$ where $\eta(\varphi(b), \varphi(a), m) > 0$. Assume that $f : A \longrightarrow \mathbb{R}$ be a twice differentiable function on A° and $f'' \in L_1[m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)]$. Then for $\alpha > 0$, we have*

$$\frac{\eta(\varphi(x),\varphi(a),m)^{\alpha+1} f'(m\varphi(a)+\eta(\varphi(x),\varphi(a),m)) + \eta(\varphi(x),\varphi(b),m)^{\alpha+1} f'(m\varphi(b))}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}$$

$$-\frac{\eta(\varphi(x),\varphi(a),m)^{\alpha} f(m\varphi(a)+\eta(\varphi(x),\varphi(a),m)) - \eta(\varphi(x),\varphi(b),m)^{\alpha} f(m\varphi(b))}{\eta(\varphi(b),\varphi(a),m)}$$

$$-\frac{\Gamma(\alpha+1)}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\left[J^{\alpha}_{(m\varphi(b))+} f(m\varphi(b)+\eta(\varphi(x),\varphi(b),m)) - J^{\alpha}_{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))-} f(m\varphi(a))\right]$$

$$= \frac{\eta^{\alpha+2}(\varphi(x),\varphi(a),m)}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)} \int_0^1 t^{\alpha+1} f''(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))dt$$

$$-\frac{\eta^{\alpha+2}(\varphi(x),\varphi(b),m)}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)} \int_0^1 (1-t)^{\alpha+1} f''(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))dt, \tag{3.1}$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-u}u^{\alpha-1}du$ is the Euler gamma function.

Proof. A simple proof of the equality can be done by performing an integration by parts in the integrals from the right side and changing the variable. The details are left to the interested reader. □

Let denote

$$I_{f,\eta,\varphi}(x;\alpha,m,a,b)$$

$$= \frac{\eta^{\alpha+2}(\varphi(x),\varphi(a),m)}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)} \int_0^1 t^{\alpha+1} f''(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))dt$$

$$-\frac{\eta^{\alpha+2}(\varphi(x),\varphi(b),m)}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)} \int_0^1 (1-t)^{\alpha+1} f''(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))dt. \tag{3.2}$$

Using Lemma 3.1 and the relation (3.2), the following results can be obtained for the corresponding version for power of the absolute value of the second derivative.

Theorem 3.1. *Let $\varphi : I \longrightarrow A$ be a continuous function. Suppose $A \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : A \times A \times (0,1] \longrightarrow \mathbb{R}$, $\forall s \in (0,1]$ and for some fixed $m \in (0,1]$ where $\eta(\varphi(b), \varphi(a), m) > 0$. Assume that $f : A \longrightarrow \mathbb{R}$ be a twice differentiable function on A°. If $|f''|^q$ is a generalized (s, m, φ)-preinvex function on $[m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)]$, $q > 1$, $p^{-1} + q^{-1} = 1$, then for $\alpha > 0$, we have*

$$\left|I_{f,\eta,\varphi}(x;\alpha,m,a,b)\right| \leq \frac{1}{(\alpha+1)(s+1)^{1/q}(p(\alpha+1)+1)^{1/p}} \frac{1}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\left\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}\left[|f''(\varphi(x))|^q+m|f''(\varphi(a))|^q\right]^{\frac{1}{q}}\right.$$

$$\left.+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}\left[|f''(\varphi(x))|^q+m|f''(\varphi(b))|^q\right]^{\frac{1}{q}}\right\}. \tag{3.3}$$

Proof. Suppose that $q>1$. Using Lemma 3.1, generalized (s,m,φ)-preinvexity of $|f''|^q$, Hölder inequality and property of the modulus, we have

$$\left|I_{f,\eta,\varphi}(x;\alpha,m,a,b)\right|$$

$$\leq\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}\int_0^1 t^{\alpha+1}|f''(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))|dt$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}\int_0^1 |1-t|^{\alpha+1}|f''(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))|dt$$

$$\leq\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 t^{p(\alpha+1)}dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 |f''(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 (1-t)^{p(\alpha+1)}dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 |f''(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))|^q dt\right)^{\frac{1}{q}}$$

$$\leq\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 t^{p(\alpha+1)})dt\right)^{\frac{1}{p}}$$

$$\times\left[\int_0^1\left(t^s|f''(\varphi(x))|^q+m(1-t)^s|f''(\varphi(a))|^q\right)dt\right]^{\frac{1}{q}}$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 (1-t)^{p(\alpha+1)}dt\right)^{\frac{1}{p}}$$

$$\times\left[\int_0^1\left(t^s|f''(\varphi(x))|^q+m(1-t)^s|f''(\varphi(b))|^q\right)dt\right]^{\frac{1}{q}}$$

$$=\frac{1}{(\alpha+1)(s+1)^{1/q}(p(\alpha+1)+1)^{1/p}}\frac{1}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\left\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}\left[|f''(\varphi(x))|^q+m|f''(\varphi(a))|^q\right]^{\frac{1}{q}}\right.$$

$$\left.+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}\left[|f''(\varphi(x))|^q+m|f''(\varphi(b))|^q\right]^{\frac{1}{q}}\right\}.$$

□

Theorem 3.2. *Let $\varphi : I \longrightarrow A$ be a continuous function. Suppose $A \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : A\times A\times(0,1] \longrightarrow \mathbb{R}$, $\forall s \in (0,1]$ and for some fixed $m \in (0,1]$ where $\eta(\varphi(b),\varphi(a),m) > 0$. Assume that $f : A \longrightarrow \mathbb{R}$ be a twice differentiable function on A°. If $|f''|^q$ is a generalized (s,m,φ)-preinvex function on $[m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]$, $q \geq 1$, then for $\alpha > 0$, we have*

$$\begin{aligned}\left|I_{f,\eta,\varphi}(x;\alpha,m,a,b)\right| \leq \frac{1}{(\alpha+1)(\alpha+2)^{1-\frac{1}{q}}}\frac{1}{\eta(\varphi(b),\varphi(a),m)} \\ \times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}\left[m|f''(\varphi(a))|^q\beta(\alpha+2,s+1)+\frac{|f''(\varphi(x))|^q}{\alpha+s+2}\right]^{\frac{1}{q}} \\ +|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}\left[\frac{m|f''(\varphi(b))|^q}{\alpha+s+2}+|f''(\varphi(x))|^q\beta(s+1,\alpha+2)\right]^{\frac{1}{q}}\Bigg\}.\end{aligned} \tag{3.4}$$

Proof. Suppose that $q \geq 1$. Using Lemma 3.1, Definition 2.3, generalized (s,m,φ)-preinvexity of $|f''|^q$, the well-known power mean inequality and property of the modulus, we have

$$\begin{aligned}&\left|I_{f,\eta,\varphi}(x;\alpha,m,a,b)\right| \\ &\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}\int_0^1 t^{\alpha+1}|f''(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))|dt \\ &+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}\int_0^1 |1-t|^{\alpha+1}|f''(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))|dt \\ &\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 t^{\alpha+1}dt\right)^{1-\frac{1}{q}} \\ &\times\left(\int_0^1 t^{\alpha+1}|f''(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))|^q dt\right)^{\frac{1}{q}} \\ &+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 (1-t)^{\alpha+1}dt\right)^{1-\frac{1}{q}} \\ &\times\left(\int_0^1 (1-t)^{\alpha+1}|f''(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))|^q dt\right)^{\frac{1}{q}} \\ &\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 t^{\alpha+1}dt\right)^{1-\frac{1}{q}} \\ &\times\left[\int_0^1 t^{\alpha+1}\Big(t^s|f''(\varphi(x))|^q+m(1-t)^s|f''(\varphi(a))|^q\Big)dt\right]^{\frac{1}{q}} \\ &+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 (1-t)^{\alpha+1}dt\right)^{1-\frac{1}{q}} \\ &\times\left[\int_0^1 (1-t)^{\alpha+1}\Big(t^s|f''(\varphi(x))|^q+m(1-t)^s|f''(\varphi(b))|^q\Big)dt\right]^{\frac{1}{q}} \\ &= \frac{1}{(\alpha+1)(\alpha+2)^{1-\frac{1}{q}}}\frac{1}{\eta(\varphi(b),\varphi(a),m)}\end{aligned}$$

$$\times\left\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}\left[m|f''(\varphi(a))|^q\beta(\alpha+2,s+1)+\frac{|f''(\varphi(x))|^q}{\alpha+s+2}\right]^{\frac{1}{q}}\right.$$
$$\left.+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}\left[\frac{m|f''(\varphi(b))|^q}{\alpha+s+2}+|f''(\varphi(x))|^q\beta(s+1,\alpha+2)\right]^{\frac{1}{q}}\right\}.$$

□

4. Applications to special means

In the following we give certain generalizations of some notions for a positive valued function of a positive variable.

Definition 4.1. ([3]) A function $M:\mathbb{R}_+^2\longrightarrow\mathbb{R}_+$, is called a Mean function if it has the following properties:

(1) Homogeneity: $M(ax,ay)=aM(x,y)$, for all $a>0$,
(2) Symmetry: $M(x,y)=M(y,x)$,
(3) Reflexivity: $M(x,x)=x$,
(4) Monotonicity: If $x\le x'$ and $y\le y'$, then $M(x,y)\le M(x',y')$,
(5) Internality: $\min\{x,y\}\le M(x,y)\le\max\{x,y\}$.

We consider some means for arbitrary positive real numbers α,β ($\alpha\neq\beta$).

(1) The arithmetic mean:
$$A:=A(\alpha,\beta)=\frac{\alpha+\beta}{2}$$
(2) The geometric mean:
$$G:=G(\alpha,\beta)=\sqrt{\alpha\beta}$$
(3) The harmonic mean:
$$H:=H(\alpha,\beta)=\frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}}$$
(4) The power mean:
$$P_r:=P_r(\alpha,\beta)=\left(\frac{\alpha^r+\beta^r}{2}\right)^{\frac{1}{r}},\quad r\ge 1.$$
(5) The identric mean:
$$I:=I(\alpha,\beta)=\begin{cases}\frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right), & \alpha\neq\beta;\\ \alpha, & \alpha=\beta.\end{cases}$$
(6) The logarithmic mean:
$$L:=L(\alpha,\beta)=\frac{\beta-\alpha}{\ln(\beta)-\ln(\alpha)}.$$
(7) The generalized log-mean:
$$L_p:=L_p(\alpha,\beta)=\left[\frac{\beta^{p+1}-\alpha^{p+1}}{(p+1)(\beta-\alpha)}\right]^{\frac{1}{p}};\quad p\in\mathbb{R}\setminus\{-1,0\}.$$

(8) The weighted p-power mean:

$$M_p\begin{pmatrix} \alpha_1, & \alpha_2, & \cdots & ,\alpha_n \\ u_1, & u_2, & \cdots & ,u_n \end{pmatrix} = \left(\sum_{i=1}^{n} \alpha_i u_i^p\right)^{\frac{1}{p}}$$

where $0 \le \alpha_i \le 1$, $u_i > 0\,(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^{n} \alpha_i = 1$.

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \le G \le L \le I \le A$. Now, let a and b be positive real numbers such that $a < b$. Consider the function $M := M(\varphi(a), \varphi(b)) : [\varphi(a), \varphi(a) + \eta(\varphi(b), \varphi(a))] \times [\varphi(a), \varphi(a) + \eta(\varphi(b), \varphi(a))] \longrightarrow \mathbb{R}_+$, which is one of the above mentioned means and $\varphi : I \longrightarrow A$ be a continuous function, therefore one can obtain various inequalities using the results of Section 3 for these means as follows. Replace $\eta(\varphi(y), \varphi(x), m)$ with $\eta(\varphi(y), \varphi(x))$ and setting $\eta(\varphi(a), \varphi(b)) = M(\varphi(a), \varphi(b))$ for value $m = 1$ in (3.3) and (3.4), one can obtain the following interesting inequalities involving means:

$$\begin{aligned}
&\left|I_{f,M(\cdot,\cdot),\varphi}(x; \alpha, 1, a, b)\right| \\
&= \Bigg| \frac{M^{\alpha+1}(\varphi(a), \varphi(x)) f'(\varphi(a) + M(\varphi(a), \varphi(x))) + M^{\alpha+1}(\varphi(b), \varphi(x)) f'(\varphi(b))}{(\alpha+1) M(\varphi(a), \varphi(b))} \\
&\quad - \frac{M^{\alpha}(\varphi(a), \varphi(x)) f(\varphi(a) + M(\varphi(a), \varphi(x))) - M^{\alpha}(\varphi(b), \varphi(x)) f(\varphi(b))}{M(\varphi(a), \varphi(b))} \\
&\quad - \frac{\Gamma(\alpha+1)}{M(\varphi(a), \varphi(b))} \\
&\quad \times \left[J^{\alpha}_{\varphi(b)+} f(\varphi(b) + M(\varphi(b), \varphi(x))) - J^{\alpha}_{(\varphi(a)+M(\varphi(a),\varphi(x)))-} f(\varphi(a)) \right] \Bigg| \\
&\le \frac{1}{(\alpha+1)(s+1)^{1/q}(p(\alpha+1)+1)^{1/p}} \frac{1}{M(\varphi(a), \varphi(b))} \\
&\quad \times \Bigg\{ M^{\alpha+2}(\varphi(a), \varphi(x)) \left[|f''(\varphi(a))|^q + |f''(\varphi(x))|^q \right]^{\frac{1}{q}} \\
&\quad + M^{\alpha+2}(\varphi(b), \varphi(x)) \left[|f''(\varphi(b))|^q + |f''(\varphi(x))|^q \right]^{\frac{1}{q}} \Bigg\},
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
&\left|I_{f,M(\cdot,\cdot),\varphi}(x; \alpha, 1, a, b)\right| \le \frac{1}{(\alpha+1)(\alpha+2)^{1-\frac{1}{q}}} \frac{1}{M(\varphi(a), \varphi(b))} \\
&\quad \times \Bigg\{ M^{\alpha+2}(\varphi(a), \varphi(x)) \left[|f''(\varphi(a))|^q \beta(\alpha+2, s+1) + \frac{|f''(\varphi(x))|^q}{\alpha+s+2} \right]^{\frac{1}{q}} \\
&\quad + M^{\alpha+2}(\varphi(b), \varphi(x)) \left[\frac{|f''(\varphi(b))|^q}{\alpha+s+2} + |f''(\varphi(x))|^q \beta(s+1, \alpha+2) \right]^{\frac{1}{q}} \Bigg\}.
\end{aligned} \tag{4.2}$$

Letting $M(\varphi(a), \varphi(b)) = A, G, H, P_r, I, L, L_p, M_p$ in (4.1) and (4.2), we get inequalities involving means for a particular choice of a twice differentiable generalized $(s, 1, \varphi)$-preinvex functions f. The details are left to the interested reader.

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