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HESITANT FUZZY SETS ON UP-ALGEBRAS

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Abstract. In this paper, hesitant fuzzy sets on UP-algebras are introduced, proved some results and discussed the generalizations of some concepts of hesitant fuzzy sets. Further, we discuss the relation between characteristic hesitant fuzzy sets and UP-subalgebras (resp. UP-filters, UP-ideals and strongly UP-ideals).

1. Introduction and Preliminaries

Among many algebraic structures, algebras of logic form important class of algebras. Examples of these are including BCK-algebras [5], BCI-algebras [6], BCH-algebras [3], KU-algebras [12], SU-algebras [11], UP-algebras [4] and others. They are strongly connected with logic. For example, BCI-algebras introduced by Iséki [6] in 1966 have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming. BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iséki [5, 6] in 1966 and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras.

A fuzzy subset f of a set S is a function from S to a closed interval $[0, 1]$. The concept of a fuzzy subset of a set was first considered by Zadeh [17] in 1965. The fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere.

In 2009 - 2010, Torra and Narukawa [16, 15] introduced the notion of hesitant fuzzy sets, that is a function from a reference set to a power set of the unit interval. The notion of hesitant fuzzy sets is the other generalization of the notion fuzzy sets. The hesitant fuzzy set theories developed by Torra and others have found many applications in the domain of mathematics and elsewhere.

After the introduction of the notion of hesitant fuzzy sets by Torra and Narukawa [16, 15], several researches were conducted on the generalizations of the notion of

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hesitant fuzzy sets and application to many logical algebras such as: In 2012, Zhu, Xu and Xia [18] introduced the notion of dual hesitant fuzzy sets which is a new extension of fuzzy sets. In 2014, Jun, Ahn and Muhiuddin [8] introduced the notions of hesitant fuzzy soft subalgebras and (closed) hesitant fuzzy soft ideals in BCK/BCI-algebras. Jun and Song [9] introduced the notions of (Boolean, prime, ultra, good) hesitant fuzzy filters and hesitant fuzzy MV-filters of MTL-algebras. In 2015, Jun and Song [10] introduced the notions of hesitant fuzzy prefilters (resp. filters) and positive implicative hesitant fuzzy prefilters (resp. filters) of EQ-algebras. In 2016, Jun and Ahn [7] introduced the notions of hesitant fuzzy subalgebras and hesitant fuzzy ideals of BCK/BCI-algebras. Rezaei and Saeid [13] introduced the notion of hesitant fuzzy (implicative) filters on BE-algebras.

The notions of hesitant fuzzy subalgebras, hesitant fuzzy filters and hesitant fuzzy ideals play an important role in studying the many logical algebras. In this paper, we introduced the notion of hesitant fuzzy sets which is a new extension of fuzzy sets on UP-algebras and the notions of hesitant fuzzy UP-subalgebras, hesitant fuzzy UP-filters, hesitant fuzzy UP-ideals and hesitant fuzzy strongly UP-ideals of UP-algebras and proved some results. Further, we discussed the relation between characteristic hesitant fuzzy sets and UP-subalgebras (resp. UP-filters, UP-ideals and strongly UP-ideals). The aim of this paper is to extend this research to hesitant fuzzy sets on UP-algebras with the connection of the results obtaining in [14].

Before we begin our study, we will introduce the definition of a UP-algebra.

Definition 1.1. [4] An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra*, where A is a nonempty set, $\cdot$ is a binary operation on A, and 0 is a fixed element of A (i.e., a nullary operation) if it satisfies the following axioms: for any $x, y, z \in A$,
(UP-1): $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0$,
(UP-2): $0 \cdot x = x$,
(UP-3): $x \cdot 0 = 0$, and
(UP-4): $x \cdot y = y \cdot x = 0$ implies $x = y$.

From [4], we know that the notion of UP-algebras is a generalization of KU-algebras.

Example 1.1. [4] Let X be a universal set. Define a binary operation $\cdot$ on the power set of X by putting $A \cdot B = B \cap A' = A' \cap B = B - A$ for all $A, B \in \mathcal{P}(X)$. Then $(\mathcal{P}(X), \cdot, \emptyset)$ is a UP-algebra and we shall call it the *power UP-algebra of type 1*.

Example 1.2. [4] Let X be a universal set. Define a binary operation $*$ on the power set of X by putting $A * B = B \cup A' = A' \cup B$ for all $A, B \in \mathcal{P}(X)$. Then $(\mathcal{P}(X), *, X)$ is a UP-algebra and we shall call it the *power UP-algebra of type 2*.

Example 1.3. [4] Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	0	0
2	0	1	0	3
3	0	1	2	0

Then $(A, \cdot, 0)$ is a UP-algebra.

In what follows, let A and B denote UP-algebras unless otherwise specified. The following proposition is very important for the study of UP-algebras.

Proposition 1.1. [4] *In a UP-algebra A, the following properties hold: for any $x, y, z \in A$,*

(1) $x \cdot x = 0$,
(2) $x \cdot y = 0$ *and* $y \cdot z = 0$ *imply* $x \cdot z = 0$,
(3) $x \cdot y = 0$ *implies* $(z \cdot x) \cdot (z \cdot y) = 0$,
(4) $x \cdot y = 0$ *implies* $(y \cdot z) \cdot (x \cdot z) = 0$,
(5) $x \cdot (y \cdot x) = 0$,
(6) $(y \cdot x) \cdot x = 0$ *if and only if* $x = y \cdot x$, *and*
(7) $x \cdot (y \cdot y) = 0$.

Definition 1.2. [4] A subset S of A is called a *UP-subalgebra* of A if the constant 0 of A is in S, and $(S, \cdot, 0)$ itself forms a UP-algebra.

Proposition 1.2. [4] *A nonempty subset S of a UP-algebra $A = (A, \cdot, 0)$ is a UP-subalgebra of A if and only if S is closed under the $\cdot$ multiplication on A.*

Definition 1.3. [4] A subset B of A is called a *UP-ideal* of A if it satisfies the following properties:

(1) the constant 0 of A is in B, and
(2) for any $x, y, z \in A, x \cdot (y \cdot z) \in B$ and $y \in B$ imply $x \cdot z \in B$.

Definition 1.4. [14] A subset F of A is called a *UP-filter* of A if it satisfies the following properties:

(1) the constant 0 of A is in F, and
(2) for any $x, y \in A, x \cdot y \in F$ and $x \in F$ imply $y \in F$.

Definition 1.5. [2] A subset C of A is called a *strongly UP-ideal* of A if it satisfies the following properties:

(1) the constant 0 of A is in C, and
(2) for any $x, y, z \in A, (z \cdot y) \cdot (z \cdot x) \in C$ and $y \in C$ imply $x \in C$.

From [2], we know that the notion of fuzzy UP-subalgebras is a generalization of fuzzy UP-filters, the notion of fuzzy UP-filters is a generalization of fuzzy UP-ideals, and the notion of fuzzy UP-ideals is a generalization of fuzzy strongly UP-ideals.

Definition 1.6. [14] A nonempty subset B of A is called a *prime subset* of A if it satisfies the following property: for any $x, y \in A$,

$$x \cdot y \in B \text{ implies } x \in B \text{ or } y \in B.$$

Definition 1.7. [14] A UP-subalgebra (resp. UP-filter, UP-ideal, strongly UP-ideal) B of A is called a *prime UP-subalgebra* (resp. *prime UP-filter*, *prime UP-ideal*, *prime strongly UP-ideal*) of A if B is a prime subset of A.

Theorem 1.1. [2] *Let S be a subset of A. Then the following statements are equivalent:*

(1) S is a prime UP-subalgebra (resp. prime UP-filter, prime UP-ideal, prime strongly UP-ideal) of A,
(2) $S = A$, and
(3) S is a strongly UP-ideal of A.

Definition 1.8. [15] Let X be a reference set. A *hesitant fuzzy set* (HFS) on X is defined in term of a function h that when applied to X return a subset of $[0,1]$, that is, $\mathrm{h}\colon X \to \mathcal{P}([0,1])$.

If $Y \subseteq X$, the *characteristic hesitant fuzzy set* h_Y on X is a function of X into $\mathcal{P}([0,1])$ defined as follows:

$$\mathrm{h}_Y(x) = \begin{cases} [0,1] & \text{if } x \in Y, \\ \emptyset & \text{if } x \notin Y. \end{cases}$$

By the definition of characteristic hesitant fuzzy sets, h_Y is a function of X into $\{\emptyset, [0,1]\} \subset \mathcal{P}([0,1])$. Hence, h_Y is a hesitant fuzzy set on X.

Definition 1.9. Let h be a hesitant fuzzy set on A. The hesitant fuzzy set $\overline{\mathrm{h}}$ defined by $\overline{\mathrm{h}}(x) = [0,1] - \mathrm{h}(x)$ for all $x \in A$ is said to be the *complement* of h on A.

Remark 1.1. For all hesitant fuzzy set h on A, we have $\mathrm{h} = \overline{\overline{\mathrm{h}}}$.

Definition 1.10. A hesitant fuzzy set h on A is called a *hesitant fuzzy UP-subalgebra* (HFUPS) of A if it satisfies the following property: for any $x, y \in A$,

$$\mathrm{h}(x \cdot y) \supseteq \mathrm{h}(x) \cap \mathrm{h}(y).$$

By Proposition 1.1 (1), we have $\mathrm{h}(0) = \mathrm{h}(x \cdot x) \supseteq \mathrm{h}(x) \cap \mathrm{h}(x) = \mathrm{h}(x)$ for all $x \in A$.

By using Microsoft Excel, we can verify all examples.

Example 1.4. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	0	0	3
3	0	0	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a hesitant fuzzy set h on A as follows:

$$\mathrm{h}(0) = \{0.5, 0.6, 0.7\}, \mathrm{h}(1) = \{0.5\}, \mathrm{h}(2) = \{0.6\}, \text{ and } \mathrm{h}(3) = \emptyset.$$

Using this data, we can show that h is a hesitant fuzzy UP-subalgebra of A.

Definition 1.11. A hesitant fuzzy set h on A is called a *hesitant fuzzy UP-filter* (HFUPF) of A if it satisfies the following properties: for any $x, y \in A$,

(1) $\mathrm{h}(0) \supseteq \mathrm{h}(x)$, and
(2) $\mathrm{h}(y) \supseteq \mathrm{h}(x \cdot y) \cap \mathrm{h}(x)$.

Example 1.5. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	3	3
2	0	1	0	0
3	0	1	2	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a hesitant fuzzy set h on A as follows:

$$\mathrm{h}(0) = [0,1], \mathrm{h}(1) = \{0.7\}, \mathrm{h}(2) = \emptyset, \text{ and } \mathrm{h}(3) = \{0.2, 0.3\}.$$

Using this data, we can show that h is a hesitant fuzzy UP-filter of A.

Definition 1.12. A hesitant fuzzy set h on A is called a *hesitant fuzzy UP-ideal* (HFUPI) of A if it satisfies the following properties: for any $x, y, z \in A$,

(1) $\mathrm{h}(0) \supseteq \mathrm{h}(x)$, and
(2) $\mathrm{h}(x \cdot z) \supseteq \mathrm{h}(x \cdot (y \cdot z)) \cap \mathrm{h}(y)$.

Example 1.6. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	0	0	3
3	0	0	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a hesitant fuzzy set h on A as follows:

$$\mathrm{h}(0) = \{0.4, 0.5, 0.7\}, \mathrm{h}(1) = \{0.4, 0.5\}, \mathrm{h}(2) = \{0.5\}, \text{ and } \mathrm{h}(3) = \emptyset.$$

Using this data, we can show that h is a hesitant fuzzy UP-ideal of A.

Definition 1.13. A hesitant fuzzy set h on A is called a *hesitant fuzzy strongly UP-ideal* (HFSUPS) of A if it satisfies the following properties: for any $x, y, z \in A$,

(1) $\mathrm{h}(0) \supseteq \mathrm{h}(x)$, and
(2) $\mathrm{h}(x) \supseteq \mathrm{h}((z \cdot y) \cdot (z \cdot x)) \cap \mathrm{h}(y)$.

Example 1.7. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	0	0
2	0	1	0	0
3	0	1	2	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a hesitant fuzzy set h on A as follows:

$$\mathrm{h}(0) = \{0.6, 0.9\}, \mathrm{h}(1) = \{0.6, 0.9\}, \mathrm{h}(2) = \{0.6, 0.9\}, \text{ and } \mathrm{h}(3) = \{0.6, 0.9\}.$$

Using this data, we can show that h is a hesitant fuzzy strongly UP-ideal of A.

By Definition 1.10, 1.11, 1.12, and 1.13, we will discuss the generalizations of these concepts.

Theorem 1.2. *Every hesitant fuzzy strongly UP-ideal of A is a hesitant fuzzy UP-ideal of A.*

Proof. Assume that h is a hesitant fuzzy strongly UP-ideal of A. Then for all $x, y, z \in A, \mathrm{h}(0) \supseteq \mathrm{h}(x)$, and

$$\begin{aligned}
\mathrm{h}(x \cdot z) &\supseteq \mathrm{h}((z \cdot y) \cdot (z \cdot (x \cdot z))) \cap \mathrm{h}(y) \\
\text{(By Proposition 1.1 (5))} \quad &= \mathrm{h}((z \cdot y) \cdot 0) \cap \mathrm{h}(y) \\
\text{(By UP-3)} \quad &= \mathrm{h}(0) \cap \mathrm{h}(y) \\
&= \mathrm{h}(y) \\
&\supseteq \mathrm{h}(x \cdot (y \cdot z)) \cap \mathrm{h}(y).
\end{aligned}$$

Hence, h is a hesitant fuzzy UP-ideal of A. □

Example 1.8. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	0	0	3
3	0	0	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a hesitant fuzzy set h on A as follows:

$$\mathrm{h}(0) = \{0.4, 0.5, 0.7\}, \mathrm{h}(1) = \{0.4, 0.5\}, \mathrm{h}(2) = \{0.5\}, \text{ and } \mathrm{h}(3) = \emptyset.$$

Using this data, we can show that h is a hesitant fuzzy UP-ideal of A. Since $\mathrm{h}(1) = \{0.4, 0.5\} \not\supseteq \{0.4, 0.5, 0.7\} = \mathrm{h}((2 \cdot 0) \cdot (2 \cdot 1)) \cap \mathrm{h}(0)$, we have h is not a hesitant fuzzy strongly UP-ideal of A.

Theorem 1.3. *Every hesitant fuzzy UP-ideal of A is a hesitant fuzzy UP-filter of A.*

Proof. Assume that h is a hesitant fuzzy UP-ideal of A. Then for all $x, y \in A, \mathrm{h}(0) \supseteq \mathrm{h}(x)$, and

$$\begin{aligned}
&\text{(By UP-2)} & \mathrm{h}(y) &= \mathrm{h}(0 \cdot y) \\
&& &\supseteq \mathrm{h}(0 \cdot (x \cdot y)) \cap \mathrm{h}(x) \\
&\text{(By UP-2)} & &= \mathrm{h}(x \cdot y) \cap \mathrm{h}(x).
\end{aligned}$$

Hence, h is a hesitant fuzzy UP-filter of A. □

Example 1.9. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	3	3
2	0	1	0	0
3	0	1	2	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a hesitant fuzzy set h on A as follows:

$$\mathrm{h}(0) = [0, 1], \mathrm{h}(1) = \{0.7\}, \mathrm{h}(2) = \emptyset, \text{ and } \mathrm{h}(3) = \{0.2, 0.5\}.$$

Using this data, we can show that h is a hesitant fuzzy UP-filter of A. Since $\mathrm{h}(3 \cdot 2) = \emptyset \not\supseteq \{0.7\} = \mathrm{h}(3 \cdot (1 \cdot 2)) \cap \mathrm{h}(1)$, we have h is not a hesitant fuzzy UP-ideal of A.

Theorem 1.4. *Every hesitant fuzzy UP-filter of A is a hesitant fuzzy UP-subalgebra of A.*

Proof. Assume that h is a hesitant fuzzy UP-filter of A. Then for all $x, y \in A$,

$$\begin{aligned}
&& \mathrm{h}(x \cdot y) &\supseteq \mathrm{h}(y \cdot (x \cdot y)) \cap \mathrm{h}(y) \\
&\text{(By Proposition 1.1 (5))} & &= \mathrm{h}(0) \cap \mathrm{h}(y) \\
&& &= \mathrm{h}(y) \\
&& &\supseteq \mathrm{h}(x) \cap \mathrm{h}(y).
\end{aligned}$$

Hence, h is a hesitant fuzzy UP-subalgebra of A. □

Example 1.10. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	0	0	3
3	0	0	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a hesitant fuzzy set h on A as follows:

$$\mathrm{h}(0) = \{0.5, 0.6, 0.9\}, \mathrm{h}(1) = \{0.5\}, \mathrm{h}(2) = \{0.6\}, \text{ and } \mathrm{h}(3) = \emptyset.$$

Using this data, we can show that h is a hesitant fuzzy UP-subalgebra of A. Since $\mathrm{h}(1) = \{0.5\} \not\supseteq \{0.6\} = \mathrm{h}(2 \cdot 1) \cap \mathrm{h}(2)$, we have h is not a hesitant fuzzy UP-filter of A.

By Theorem 1.2, 1.3, and 1.4 and Example 1.8, 1.9, and 1.10, we have that the notion of hesitant fuzzy UP-ideals of UP-algebras is the generalization of the notion of hesitant fuzzy strongly UP-ideals, the notion of hesitant fuzzy UP-filters of UP-algebras is the generalization of the notion of hesitant fuzzy UP-ideals, and the notion of hesitant fuzzy UP-subalgebras of UP-algebras is the generalization of the notion of hesitant fuzzy UP-filters. Then we have the diagram of hesitant fuzzy sets below.

$$\boxed{\text{HFSUPIs}} \longrightarrow \boxed{\text{HFUPIs}} \longrightarrow \boxed{\text{HFUPFs}} \longrightarrow \boxed{\text{HFUPSs}}$$

2. Characteristic Hesitant Fuzzy Sets

In this section, we discuss the relation between characteristic hesitant fuzzy sets and UP-subalgebras (resp. UP-filters, UP-ideals and strongly UP-ideals), and study the concept of prime and weakly prime of subsets and of hesitant fuzzy sets of a UP-algebra.

Theorem 2.1. *A hesitant fuzzy set* h *on A is a hesitant fuzzy strongly UP-ideal of A if and only if it is a constant hesitant fuzzy set on A.*

Proof. Assume that h is a hesitant fuzzy strongly UP-ideal of A. Then $\mathrm{h}(0) \supseteq \mathrm{h}(x)$ and $\mathrm{h}(x) \supseteq \mathrm{h}((z \cdot y) \cdot (z \cdot x)) \cap \mathrm{h}(y)$ for all $x, y, z \in A$. For any $x \in A$, we choose $z = x$ and $y = 0$. Then

$$\begin{aligned}
\mathrm{h}(x) &\supseteq \mathrm{h}((x \cdot 0) \cdot (x \cdot x)) \cap \mathrm{h}(0) \\
\text{(By UP-3 and Proposition 1.1 (1))} \quad &= \mathrm{h}(0 \cdot 0) \cap \mathrm{h}(0) \\
\text{(By UP-2)} \quad &= \mathrm{h}(0) \cap \mathrm{h}(0) \\
&= \mathrm{h}(0) \\
&\supseteq \mathrm{h}(x),
\end{aligned}$$

so $\mathrm{h}(0) = \mathrm{h}(x)$. Hence, h is a constant hesitant fuzzy set on A.

Conversely, assume that h is a constant hesitant fuzzy set on A. Then, for all $x \in A, \mathrm{h}(0) = \mathrm{h}(x)$, so $\mathrm{h}(0) \supseteq \mathrm{h}(x)$. For all $x, y, z \in A, \mathrm{h}(x) = \mathrm{h}((z \cdot y) \cdot (z \cdot x)) = \mathrm{h}(y)$, so $\mathrm{h}(x) = \mathrm{h}((z \cdot y) \cdot (z \cdot x)) \cap \mathrm{h}(y)$. Thus $\mathrm{h}(x) \supseteq \mathrm{h}((z \cdot y) \cdot (z \cdot x)) \cap \mathrm{h}(y)$. Hence, h is a hesitant fuzzy strongly UP-ideal of A. □

Theorem 2.2. *A nonempty subset S of A is a UP-subalgebra of A if and only if the characteristic hesitant fuzzy set h_S is a hesitant fuzzy UP-subalgebra of A.*

Proof. Assume that S is a UP-subalgebra of A. Let $x, y \in A$.

Case 1: $x, y \in S$. Then $\mathrm{h}_S(x) = [0,1]$ and $\mathrm{h}_S(y) = [0,1]$. Thus $\mathrm{h}_S(x) \cap \mathrm{h}_S(y) = [0,1]$. Since S is a UP-subalgebra of A, we have $x \cdot y \in S$ and so $\mathrm{h}_S(x \cdot y) = [0,1]$. Therefore, $\mathrm{h}_S(x \cdot y) = [0,1] \supseteq [0,1] = \mathrm{h}_S(x) \cap \mathrm{h}_S(y)$.

Case 2: $x \in S$ and $y \notin S$. Then $\mathrm{h}_S(x) = [0,1]$ and $\mathrm{h}_S(y) = \emptyset$. Thus $\mathrm{h}_S(x) \cap \mathrm{h}_S(y) = \emptyset$. Therefore, $\mathrm{h}_S(x \cdot y) \supseteq \emptyset = \mathrm{h}_S(x) \cap \mathrm{h}_S(y)$.

Case 3: $x \notin S$ and $y \in S$. Then $\mathrm{h}_S(x) = \emptyset$ and $\mathrm{h}_S(y) = [0,1]$. Thus $\mathrm{h}_S(x) \cap \mathrm{h}_S(y) = \emptyset$. Therefore, $\mathrm{h}_S(x \cdot y) \supseteq \emptyset = \mathrm{h}_S(x) \cap \mathrm{h}_S(y)$.

Case 4: $x \notin S$ and $y \notin S$. Then $\mathrm{h}_S(x) = \emptyset$ and $\mathrm{h}_S(y) = \emptyset$. Thus $\mathrm{h}_S(x) \cap \mathrm{h}_S(y) = \emptyset$. Therefore, $\mathrm{h}_S(x \cdot y) \supseteq \emptyset = \mathrm{h}_S(x) \cap \mathrm{h}_S(y)$.

Hence, h_S is a hesitant fuzzy UP-subalgebra of A.

Conversely, assume that h_S is a hesitant fuzzy UP-subalgebra of A. Let $x, y \in S$. Then $\mathrm{h}_S(x) = [0,1]$ and $\mathrm{h}_S(y) = [0,1]$. Thus $\mathrm{h}_S(x \cdot y) \supseteq \mathrm{h}_S(x) \cap \mathrm{h}_S(y) = [0,1]$, so $\mathrm{h}_S(x \cdot y) = [0,1]$. Hence, $x \cdot y \in S$ and so S is a UP-subalgebra of A. □

Lemma 2.1. *The constant* 0 *of* A *is in a nonempty subset* B *of* A *if and only if* $\mathrm{h}_B(0) \supseteq \mathrm{h}_B(x)$ *for all* $x \in A$.

Proof. If $0 \in B$, then $\mathrm{h}_B(0) = [0,1]$. Thus $\mathrm{h}_B(0) = [0,1] \supseteq \mathrm{h}_B(x)$ for all $x \in A$.

Conversely, assume that $\mathrm{h}_B(0) \supseteq \mathrm{h}_B(x)$ for all $x \in A$. Since B is a nonempty subset of A, we have $a \in B$ for some $a \in A$. Then $\mathrm{h}_B(0) \supseteq \mathrm{h}_B(a) = [0,1]$, so $\mathrm{h}_B(0) = [0,1]$. Hence, $0 \in B$. □

Theorem 2.3. *A nonempty subset* F *of* A *is a UP-filter of* A *if and only if the characteristic hesitant fuzzy set* h_F *is a hesitant fuzzy UP-filter of* A.

Proof. Assume that F is a UP-filter of A. Since $0 \in F$, it follows from Lemma 2.1 that $\mathrm{h}_F(0) \supseteq \mathrm{h}_F(x)$ for all $x \in A$. Next, let $x, y \in A$.

Case 1: $x, y \in F$. Then $\mathrm{h}_F(x) = [0,1]$ and $\mathrm{h}_F(y) = [0,1]$. Therefore, $\mathrm{h}_F(y) = [0,1] \supseteq \mathrm{h}_F(x \cdot y) = \mathrm{h}_F(x \cdot y) \cap \mathrm{h}_F(x)$.

Case 2: $x \notin F$ and $y \in F$. Then $\mathrm{h}_F(x) = \emptyset$ and $\mathrm{h}_F(y) = [0,1]$. Thus $\mathrm{h}_F(y) = [0,1] \supseteq \emptyset = \mathrm{h}_F(x \cdot y) \cap \mathrm{h}_F(x)$.

Case 3: $x \in F$ and $y \notin F$. Then $\mathrm{h}_F(x) = [0,1]$ and $\mathrm{h}_F(y) = \emptyset$. Since F is a UP-filter of A, we have $x \cdot y \notin F$ or $x \notin F$. But $x \in F$, so $x \cdot y \notin F$. Then $\mathrm{h}_F(x \cdot y) = \emptyset$. Thus $\mathrm{h}_F(y) = \emptyset \supseteq \emptyset = \mathrm{h}_F(x \cdot y) \cap \mathrm{h}_F(x)$.

Case 4: $x \notin F$ and $y \notin F$. Then $\mathrm{h}_F(x) = \emptyset$ and $\mathrm{h}_F(y) = \emptyset$. Thus $\mathrm{h}_F(y) = \emptyset \supseteq \emptyset = \mathrm{h}_F(x \cdot y) \cap \mathrm{h}_F(x)$.

Hence, h_F is a hesitant fuzzy UP-filter of A.

Conversely, assume that h_F is a hesitant fuzzy UP-filter of A. Since $\mathrm{h}_F(0) \supseteq \mathrm{h}_F(x)$ for all $x \in A$, it follows from Lemma 2.1 that $0 \in F$. Next, let $x, y \in A$ be such that $x \cdot y \in F$ and $x \in F$. Then $\mathrm{h}_F(x \cdot y) = [0,1]$ and $\mathrm{h}_F(x) = [0,1]$. Thus $\mathrm{h}_F(y) \supseteq \mathrm{h}_F(x \cdot y) \cap \mathrm{h}_F(x) = [0,1]$, so $\mathrm{h}_F(y) = [0,1]$. Therefore, $y \in F$ and so F is a UP-filter of A. □

Theorem 2.4. *A nonempty subset* B *of* A *is a UP-ideal of* A *if and only if the characteristic hesitant fuzzy set* h_B *is a hesitant fuzzy UP-ideal of* A.

Proof. Assume that B is a UP-ideal of A. Since $0 \in B$, it follows from Lemma 2.1 that $\mathrm{h}_B(0) \supseteq \mathrm{h}_B(x)$ for all $x \in A$. Next, let $x, y, z \in A$.

Case 1: $x \cdot (y \cdot z) \in B$ and $y \in B$. Then $\mathrm{h}_B(x \cdot (y \cdot z)) = [0,1]$ and $\mathrm{h}_B(y) = [0,1]$. Thus $\mathrm{h}_B(x \cdot (y \cdot z)) \cap \mathrm{h}_B(y) = [0,1]$. Since $x \cdot (y \cdot z) \in B$ and $y \in B$, we have $x \cdot z \in B$ and so $\mathrm{h}_B(x \cdot z) = [0,1]$. Therefore, $\mathrm{h}_B(x \cdot z) = [0,1] \supseteq [0,1] = \mathrm{h}_B(x \cdot (y \cdot z)) \cap \mathrm{h}_B(y)$.

Case 2: $x \cdot (y \cdot z) \in B$ and $y \notin B$. Then $\mathrm{h}_B(x \cdot (y \cdot z)) = [0,1]$ and $\mathrm{h}_B(y) = \emptyset$. Thus $\mathrm{h}_B(x \cdot (y \cdot z)) \cap \mathrm{h}_B(y) = \emptyset$. Therefore, $\mathrm{h}_B(x \cdot z) \supseteq \emptyset = \mathrm{h}_B(x \cdot (y \cdot z)) \cap \mathrm{h}_B(y)$.

Case 3: $x \cdot (y \cdot z) \notin B$ and $y \in B$. Then $\mathrm{h}_B(x \cdot (y \cdot z)) = \emptyset$ and $\mathrm{h}_B(y) = [0,1]$. Thus $\mathrm{h}_B(x \cdot (y \cdot z)) \cap \mathrm{h}_B(y) = \emptyset$. Therefore, $\mathrm{h}_B(x \cdot z) \supseteq \emptyset = \mathrm{h}_B(x \cdot (y \cdot z)) \cap \mathrm{h}_B(y)$.

Case 4: $x \cdot (y \cdot z) \notin B$ and $y \notin B$. Then $\mathrm{h}_B(x \cdot (y \cdot z)) = \emptyset$ and $\mathrm{h}_B(y) = \emptyset$. Thus $\mathrm{h}_B(x \cdot (y \cdot z)) \cap \mathrm{h}_B(y) = \emptyset$. Therefore, $\mathrm{h}_B(x \cdot z) \supseteq \emptyset = \mathrm{h}_B(x \cdot (y \cdot z)) \cap \mathrm{h}_B(y)$.
Hence, h_B is a hesitant fuzzy UP-ideal of A.

Conversely, assume that h_B is a hesitant fuzzy UP-ideal of A. Since $\mathrm{h}_B(0) \supseteq \mathrm{h}_B(x)$ for all $x \in A$, it follows from Lemma 2.1 that $0 \in B$. Next, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in B$ and $y \in B$. Then $\mathrm{h}_B(x \cdot (y \cdot z)) = [0,1]$ and $\mathrm{h}_B(y) = [0,1]$. Thus $\mathrm{h}_B(x \cdot z) \supseteq \mathrm{h}_B(x \cdot (y \cdot z)) \cap \mathrm{h}_B(y) = [0,1]$, so $\mathrm{h}_B(x \cdot z) = [0,1]$. Therefore, $x \cdot z \in B$ and so B is a UP-ideal of A. □

Theorem 2.5. *A nonempty subset C of A is a strongly UP-ideal of A if and only if the characteristic hesitant fuzzy set h_C is a hesitant fuzzy strongly UP-ideal of A.*

Proof. Assume that C is a strongly UP-ideal of A. Since $0 \in C$, it follows form Lemma 2.1 that $\mathrm{h}_C(0) \supseteq \mathrm{h}_C(x)$ for all $x \in A$. Next, let $x, y, z \in A$.

Case 1: $(z \cdot y) \cdot (z \cdot x) \in C$ and $y \in C$. Then $\mathrm{h}_C((z \cdot y) \cdot (z \cdot x)) = [0,1]$ and $\mathrm{h}_C(y) = [0,1]$. Thus $\mathrm{h}_C((z \cdot y) \cdot (z \cdot x)) \cap \mathrm{h}_C(y) = [0,1]$. Since C is a strongly UP-ideal of A, we have $x \in C$ and so $\mathrm{h}_C(x) = [0,1]$. Therefore, $\mathrm{h}_C(x) = [0,1] \supseteq [0,1] = \mathrm{h}_C((z \cdot y) \cdot (z \cdot x)) \cap \mathrm{h}_C(y)$.

Case 2: $(z \cdot y) \cdot (z \cdot x) \in C$ and $y \notin C$. Then $\mathrm{h}_C((z \cdot y) \cdot (z \cdot x)) = [0,1]$ and $\mathrm{h}_C(y) = \emptyset$. Thus $\mathrm{h}_C((z \cdot y) \cdot (z \cdot x)) \cap \mathrm{h}_C(y) = \emptyset$. Therefore, $\mathrm{h}_C(x) \supseteq \emptyset = \mathrm{h}_C((z \cdot y) \cdot (z \cdot x)) \cap \mathrm{h}_C(y)$.

Case 3: $(z \cdot y) \cdot (z \cdot x) \notin C$ and $y \in C$. Then $\mathrm{h}_C((z \cdot y) \cdot (z \cdot x)) = \emptyset$ and $\mathrm{h}_C(y) = [0,1]$. Thus $\mathrm{h}_C((z \cdot y) \cdot (z \cdot x)) \cap \mathrm{h}_C(y) = \emptyset$. Therefore, $\mathrm{h}_C(x) \supseteq \emptyset = \mathrm{h}_C((z \cdot y) \cdot (z \cdot x)) \cap \mathrm{h}_C(y)$.

Case 4: $(z \cdot y) \cdot (z \cdot x) \notin C$ and $y \notin C$. Then $\mathrm{h}_C((z \cdot y) \cdot (z \cdot x)) = \emptyset$ and $\mathrm{h}_C(y) = \emptyset$. Thus $\mathrm{h}_C((z \cdot y) \cdot (z \cdot x)) \cap \mathrm{h}_C(y) = \emptyset$. Therefore, $\mathrm{h}_C(x) \supseteq \emptyset = \mathrm{h}_C((z \cdot y) \cdot (z \cdot x)) \cap \mathrm{h}_C(y)$.
Hence, h_C is a hesitant fuzzy strongly UP-ideal of A.

Conversely, assume that h_C is a hesitant fuzzy strongly UP-ideal of A. Since $\mathrm{h}_C(0) \supseteq \mathrm{h}_C(x)$ for all $x \in A$, it follows from Lemma 2.1 that $0 \in C$. Next, let $x, y, z \in A$ be such that $(z \cdot y) \cdot (z \cdot x) \in C$ and $y \in C$. Then $\mathrm{h}_C((z \cdot y) \cdot (z \cdot x)) = [0,1]$ and $\mathrm{h}_C(y) = [0,1]$. Thus $\mathrm{h}_C(x) \supseteq \mathrm{h}_C((z \cdot y) \cdot (z \cdot x)) \cap \mathrm{h}_C(y) = [0,1]$, so $\mathrm{h}_C(x) = [0,1]$. Therefore, $x \in C$ and so C is a strongly UP-ideal of A. □

Definition 2.1. A hesitant fuzzy set h on A is called a *prime hesitant fuzzy set* on A if it satisfies the following property: for any $x, y \in A$,

$$\mathrm{h}(x \cdot y) \subseteq \mathrm{h}(x) \cup \mathrm{h}(y).$$

Theorem 2.6. *A nonempty subset B of A is a prime subset of A if and only if the characteristic hesitant fuzzy set h_B is a prime hesitant fuzzy set on A.*

Proof. Assume that B is a prime subset of A and let $x, y \in A$.

Case 1: $x \cdot y \in B$. Then $\mathrm{h}_B(x \cdot y) = [0,1]$. Since B is a prime subset of A, we have $x \in B$ or $y \in B$. Then $\mathrm{h}_B(x) = [0,1]$ or $\mathrm{h}_B(y) = [0,1]$, so $\mathrm{h}_B(x) \cup \mathrm{h}_B(y) = [0,1]$. Therefore, $\mathrm{h}_B(x \cdot y) = [0,1] \subseteq [0,1] = \mathrm{h}_B(x) \cup \mathrm{h}_B(y)$.

Case 2: $x \cdot y \notin B$. Then $\mathrm{h}_B(x \cdot y) = \emptyset \subseteq \mathrm{h}_B(x) \cup \mathrm{h}_B(y)$.
Therefore, h_B is a prime hesitant fuzzy set on A.

Conversely, assume that h_B is a prime hesitant fuzzy set on A. Let $x, y \in A$ be such that $x \cdot y \in B$. Then $\mathrm{h}_B(x \cdot y) = [0,1]$, so $[0,1] = \mathrm{h}_B(x \cdot y) \subseteq \mathrm{h}_B(x) \cup \mathrm{h}_B(y)$.

Thus $\mathrm{h}_B(x) \cup \mathrm{h}_B(y) = [0,1]$, so $\mathrm{h}_B(x) = [0,1]$ or $\mathrm{h}_B(y) = [0,1]$. Hence, $x \in B$ or $y \in B$ and so B is a prime subset of A. $\square$

Definition 2.2. A hesitant fuzzy UP-subalgebra (resp. hesitant fuzzy UP-filter, hesitant fuzzy UP-ideal, hesitant fuzzy strongly UP-ideal) h of A is called a *prime hesitant fuzzy UP-subalgebra* (resp. *prime hesitant fuzzy UP-filter, prime hesitant fuzzy UP-ideal, prime hesitant fuzzy strongly UP-ideal*) if h is a prime hesitant fuzzy set on A.

Example 2.1. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	0	0	3
3	0	0	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a hesitant fuzzy set h on A as follows: for any $x \in A$,

$$\mathrm{h}(x) = \begin{cases} [0,1] & \text{if } x \neq 0, \\ \emptyset & \text{if } x = 0. \end{cases}$$

Then h is a prime hesitant fuzzy set on A. We define four constant hesitant fuzzy sets $\mathrm{h}_1, \mathrm{h}_2, \mathrm{h}_3$ and h_4 on A as follows: for any $x \in A$,

$$\mathrm{h}_1(x) = \emptyset, \mathrm{h}_2(x) = [0,1], \mathrm{h}_3(x) = \{0.5\}, \text{ and } \mathrm{h}_4(x) = \{0.2, 1\}.$$

Using this data, we can show that h_1 is a prime hesitant fuzzy UP-subalgebra of A, h_2 is a prime hesitant fuzzy UP-filter of A, h_3 is a prime hesitant fuzzy UP-ideal of A, and h_4 is a prime hesitant fuzzy strongly UP-ideal of A.

Theorem 2.7. *Let* h *be a hesitant fuzzy set on* A. *Then the following statements are equivalent:*

(1) h *is a prime hesitant fuzzy UP-subalgebra (resp. prime hesitant fuzzy UP-filter, prime hesitant fuzzy UP-ideal, prime hesitant fuzzy strongly UP-ideal) of* A*,*
(2) h *is a constant hesitant fuzzy set on* A*, and*
(3) h *is a hesitant fuzzy strongly UP-ideal of* A.

Proof. (1)⇒(2) Assume that h is a prime hesitant fuzzy UP-subalgebra (resp. prime hesitant fuzzy UP-filter, prime hesitant fuzzy UP-ideal, prime hesitant fuzzy strongly UP-ideal) of A. Then $\mathrm{h}(0) \supseteq \mathrm{h}(x)$ for all $x \in A$. By Proposition 1.1 (1), we have $\mathrm{h}(0) = \mathrm{h}(x \cdot x) \subseteq \mathrm{h}(x) \cup \mathrm{h}(x) = \mathrm{h}(x)$ for all $x \in A$ and so $\mathrm{h}(x) = \mathrm{h}(0)$ for all $x \in A$. Hence, h is a constant hesitant fuzzy set on A.

(2)⇒(1) Assume that h is a constant hesitant fuzzy set on A. Hence, we can easily show that h is a prime hesitant fuzzy UP-subalgebra (resp. prime hesitant fuzzy UP-filter, prime hesitant fuzzy UP-ideal, prime hesitant fuzzy strongly UP-ideal) of A.

(2)⇔(3) It is straightforward by Theorem 2.1. $\square$

Definition 2.3. [2] A nonempty subset B of A is called a *weakly prime subset* of A if it satisfies the following property: for any $x, y \in A$ and $x \neq y$,

$$x \cdot y \in B \text{ implies } x \in B \text{ or } y \in B.$$

Definition 2.4. [2] A UP-subalgebra (resp. UP-filter, UP-ideal, strongly UP-ideal) B of A is called a *weakly prime UP-subalgebra* (resp. *weakly prime UP-filter*, *weakly prime UP-ideal*, *weakly prime strongly UP-ideal*) of A if B is a weakly prime subset of A.

Definition 2.5. A hesitant fuzzy set h on A is called a *weakly prime hesitant fuzzy set* on A if it satisfies the following property: for any $x, y \in A$ and $x \neq y$,

$$\mathrm{h}(x \cdot y) \subseteq \mathrm{h}(x) \cup \mathrm{h}(y).$$

Definition 2.6. A hesitant fuzzy UP-subalgebra (resp. hesitant fuzzy UP-filter, hesitant fuzzy UP-ideal, hesitant fuzzy strongly UP-ideal) h of A is called a *weakly prime hesitant fuzzy UP-subalgebra* (resp. *weakly prime hesitant fuzzy UP-filter, weakly prime hesitant fuzzy UP-ideal, weakly prime hesitant fuzzy strongly UP-ideal*) if h is a weakly prime hesitant fuzzy set on A.

Theorem 2.8. *For UP-algebras, the notions of weakly prime hesitant fuzzy strongly UP-ideals and prime hesitant fuzzy strongly UP-ideals coincide.*

Proof. It is straightforward by Theorem 2.1. □

Example 2.2. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	3
3	0	1	2	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a hesitant fuzzy set h_1 on A as follows:

$$\mathrm{h}_1(0) = \{0.5, 0.6\}, \mathrm{h}_1(1) = \{0.5\}, \mathrm{h}_1(2) = \{0.6\}, \text{ and } \mathrm{h}_1(3) = [0, 1].$$

Using this data, we can show that h_1 is a weakly prime hesitant fuzzy UP-subalgebra of A. Since $\mathrm{h}_1(1 \cdot 1) = \mathrm{h}_1(0) = \{0.5, 0.6\} \nsubseteq \{0.5\} = \mathrm{h}_1(1) \cup \mathrm{h}_1(1)$, we have h_1 is not a prime hesitant fuzzy UP-subalgebra of A. We define a hesitant fuzzy set h_2 on A as follows:

$$\mathrm{h}_2(0) = [0, 1], \mathrm{h}_2(1) = \{0.2, 0.5\}, \mathrm{h}_2(2) = \emptyset, \text{ and } \mathrm{h}_2(3) = [0, 1].$$

Using this data, we can show that h_2 is a weakly prime hesitant fuzzy UP-filter of A. Since $\mathrm{h}_2(2 \cdot 2) = \mathrm{h}_2(0) = [0, 1] \nsubseteq \emptyset = \mathrm{h}_2(2) \cup \mathrm{h}_2(2)$, we have h_2 is not a prime hesitant fuzzy UP-filter of A. We define a hesitant fuzzy set h_3 on A as follows:

$$\mathrm{h}_3(0) = \{0.4, 0.5, 0.7\}, \mathrm{h}_3(1) = \{0.5, 0.7\}, \mathrm{h}_3(2) = \{0.5\}, \text{ and } \mathrm{h}_3(3) = \emptyset.$$

Using this data, we can show that h_3 is a weakly prime hesitant fuzzy UP-ideal of A. Since $\mathrm{h}_3(3 \cdot 3) = \mathrm{h}_3(0) = \{0.4, 0.5, 0.7\} \nsubseteq \emptyset = \mathrm{h}_3(3) \cup \mathrm{h}_3(3)$, we have h_3 is not a prime hesitant fuzzy UP-ideal of A.

By Definition 2.1 and 2.5, Example 2.2, and Theorem 2.8, we have that the notion of weakly prime hesitant fuzzy UP-subalgebras (resp. weakly prime hesitant fuzzy UP-filters, weakly hesitant fuzzy UP-ideals) is a generalization of prime hesitant fuzzy UP-subalgebras (resp. prime hesitant fuzzy UP-filters, prime hesitant fuzzy UP-ideals), and the notions of weakly prime hesitant fuzzy strongly UP-ideals and prime hesitant fuzzy strongly UP-ideals coincide.

Theorem 2.9. *A nonempty subset B of A is a weakly prime subset of A if and only if the characteristic hesitant fuzzy set h_B is a weakly prime hesitant fuzzy set on A.*

Proof. Assume that B is a weakly prime subset of A and let $x, y \in A$ be such that $x \neq y$.

Case 1: $x \cdot y \in B$. Then $\mathrm{h}_B(x \cdot y) = [0,1]$. Since B is a weakly prime subset of A, we have $x \in B$ or $y \in B$. Then $\mathrm{h}_B(x) = [0,1]$ or $\mathrm{h}_B(y) = [0,1]$, so $\mathrm{h}_B(x) \cup \mathrm{h}_B(y) = [0,1]$. Therefore, $\mathrm{h}_B(x \cdot y) = [0,1] \subseteq [0,1] = \mathrm{h}_B(x) \cup \mathrm{h}_B(y)$.

Case 2: $x \cdot y \notin B$. Then $\mathrm{h}_B(x \cdot y) = \emptyset \subseteq \mathrm{h}_B(x) \cup \mathrm{h}_B(y)$.

Therefore, h_B is a weakly prime hesitant fuzzy set on A.

Conversely, assume that h_B is a weakly prime hesitant fuzzy set on A. Let $x, y \in A$ be such that $x \cdot y \in B$ and $x \neq y$. Then $\mathrm{h}_B(x \cdot y) = [0,1]$, so $[0,1] = \mathrm{h}_B(x \cdot y) \subseteq \mathrm{h}_B(x) \cup \mathrm{h}_B(y)$. Thus $\mathrm{h}_B(x) \cup \mathrm{h}_B(y) = [0,1]$, so $\mathrm{h}_B(x) = [0,1]$ or $\mathrm{h}_B(y) = [0,1]$. Hence, $x \in B$ or $y \in B$ and so B is a weakly prime subset of A. □

Theorem 2.10. *A nonempty subset S of A is a weakly prime UP-subalgebra of A if and only if the characteristic hesitant fuzzy set h_S is a weakly prime hesitant fuzzy UP-subalgebra of A.*

Proof. It is straightforward by Theorem 2.2 and 2.9. □

Theorem 2.11. *A nonempty subset F of A is a weakly prime UP-filter of A if and only if the characteristic hesitant fuzzy set h_F is a weakly prime hesitant fuzzy UP-filter of A.*

Proof. It is straightforward by Theorem 2.3 and 2.9. □

Theorem 2.12. *A nonempty subset B of A is a weakly prime UP-ideal of A if and only if the characteristic hesitant fuzzy set h_B is a weakly prime hesitant fuzzy UP-ideal of A.*

Proof. It is straightforward by Theorem 2.4 and 2.9. □

Theorem 2.13. *A nonempty subset C of A is a weakly prime strongly UP-ideal of A if and only if the characteristic hesitant fuzzy set h_C is a weakly prime hesitant fuzzy strongly UP-ideal of A.*

Proof. It is straightforward by Theorem 2.5 and 2.9. □

3. Conclusions

In the present paper, we have introduced the notion of hesitant fuzzy sets which is a new extension of fuzzy sets on UP-algebras and the notions of hesitant fuzzy UP-subalgebras, hesitant fuzzy UP-filters, hesitant fuzzy UP-ideals and hesitant fuzzy strongly UP-ideals of UP-algebras and investigated some of its essential properties. We present the relation between characteristic hesitant fuzzy sets and UP-subalgebras (resp. UP-filters, UP-ideals and strongly UP-ideals). We think this work would enhance the scope for further study in this field of hesitant fuzzy set. It is our hope that this work would serve as a foundation for the further study in this field of hesitant fuzzy set in UP-algebras.

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