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Research Article

Sufficiency and duality for E-differentiable vector optimization problems under generalized convexity

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Abstract

In this paper, a new class of nonconvex vector optimization problems is considered. The concepts of E-B-pseudoinvexity and E-B-quasiinvexity are introduced for E-differentiable functions. Then, the sufficiency of the so-called E-Karush-Kuhn-Tucker optimality conditions is established for the considered E-differentiable vector optimization problems under (generalized) E-B-invexity. To illustrate the aforesaid results, an example of a nonsmooth vector programming problem with E-differentiable functions is given. For the E-differentiable vector optimization problem, the so-called vector Mond-Weir E-dual problem is defined, and several E-dual theorems are established under (generalized) E-B-invexity hypotheses.

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1. Introduction

The theory and applications of vector optimization problems have been closely tied with convex analysis and calculus analysis. During the last few years, generalizations of convexity related to optimality conditions and duality of nonlinear differentiable and nondifferentiable multiobjective optimization problems have received great interest from the authors in the areas of optimization theory and thus to explore the extent of optimality conditions and duality applicability in mathematical programming problems (see, for example, [1, 3, 7–9, 13, 16–20, 22–25], and others). One of such important generalizations of the convexity notion is the concept of invexity introduced by Hanson [15] in the

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case of differentiable scalar optimization problems. More precisely, Hanson established Karush-Kuhn-Tucker sufficient optimality conditions and duality theorems for a differentiable optimization problem involving an invex function. After that, Bector and Singh [12] introduced the class of B-vex functions as a generalization of convex functions. Later, Bector et al. [11] extended the concept of B-vexity of functions to B-invex functions.

Equally important in the optimization theory is the classes of nonconvex sets and nonconvex functions, called E-convex sets and E-convex functions, respectively, were introduced and studied by Youness [29]. These results inspired the authors to produce a great deal of subsequent research work in optimization theory (see, for example, [2, 4–6, 10, 14, 21, 26–28, 30], and others). Abdulaleem [1] introduced the concept of E-differentiable E-invexity in the case of (not necessarily) differentiable vector optimization problems with E-differentiable functions. Recently, Abdulaleem [3] introduced a new concept of generalized convexity as a generalization of several notions of generalized convexity previously introduced in the literature. In the meanwhile, Abdulaleem defined the concept of E-differentiable E-B-invexity in the case of (not necessarily) differentiable scalar optimization problems with E-differentiable functions and used this concept in proving sufficient optimality conditions for a new class of nonconvex E-differentiable scalar optimization problems.

However, in this paper, a nonconvex vector optimization problem is considered. The notions of E-B-pseudoinvexity and E-B-quasiinvexity are introduced for not necessarily differentiable functions. The sufficiency of the so-called E-Karush-Kuhn-Tucker optimality conditions is established for the considered E-differentiable vector optimization problems under (generalized) E-B-invexity. The result is illustrated by the example of a nonconvex E-differentiable optimization problem in which the involved functions are E-B-invex. Furthermore, for the considered E-differentiable vector optimization problem, its vector E-dual problem in the sense of vector Mond-Weir E-duality is defined. Then, various duality theorems between the considered E-differentiable vector optimization problem and its vector Mond-Weir E-dual problem are established under (generalized) E-B-invexity hypotheses.

2. Preliminaries

Throughout this paper, the following conventions vectors $x = (x_1, x_2, ..., x_n)^T$ and $y = (y_1, y_2, ..., y_n)^T$ in R^n will be followed:

(i) $x = y$ if and only if $x_i = y_i$ for all $i = 1, 2, ..., n$;
(ii) $x > y$ if and only if $x_i > y_i$ for all $i = 1, 2, ..., n$;
(iii) $x \geqq y$ if and only if $x_i \geqq y_i$ for all $i = 1, 2, ..., n$;
(iv) $x \geq y$ if and only if $x_i \geqq y_i$ for all $i = 1, 2, ..., n$ but $x \neq y$;
(v) $x \ngtr y$ is the negation of $x > y$.

Further, denote by $R^n_+ = \{y \in R^n : y \geqq 0\}$ and $R^n_{++} = \{y \in R^n : y > 0\}$ the nonnegative orthant and interior of nonnegative orthant of R^n, respectively.

Definition 2.1. [1] Let $E : R^n \to R^n$. A set $M \subseteq R^n$ is said to be an E-invex set with respect to $\eta : M \times M \to R^n$ if and only if the relation

$$E(u) + \lambda\eta(E(x), E(u)) \in M$$

holds for all $x, u \in M$ and any $\lambda \in [0, 1]$.

Let $M \subseteq R^n$ be a nonempty E-invex set with respect to η.

Definition 2.2. [27] Let $E : R^n \to R^n$. A function $f : M \to R$ is said to be E-B-preinvex on M with respect to $\eta : M \times M \to R^n$ and $b : M \times M \times [0, 1] \to R_+$ if and only if the following inequality

$$f(E(u) + \lambda\eta(E(x), E(u))) \leqq$$

$$\lambda b(E(x), E(u), \lambda) f(E(x)) + (1 - \lambda b(E(x), E(u), \lambda)) f(E(u))$$

holds for all $x, u \in M$ and any $\lambda \in [0, 1]$.

Definition 2.3. Let $E : R^n \to R^n$. A function $f : M \to R$ is said to be strictly E-B-preinvex on M with respect to $\eta : M \times M \to R^n$ and $b : M \times M \times [0, 1] \to R_+$ if and only if the following inequality

$$f(E(u) + \lambda\eta(E(x), E(u))) <$$
$$\lambda b(E(x), E(u), \lambda) f(E(x)) + (1 - \lambda b(E(x), E(u), \lambda)) f(E(u))$$

holds for all $x, u \in M$, $E(x) \neq E(u)$, and any $\lambda \in [0, 1]$.

Now, we introduce new class of functions called E-B-prequasiinvex, E-B-prepseudoinvex functions.

Definition 2.4. Let $E : R^n \to R^n$. A function $f : M \to R$ is said to be E-B-prequasiinvex on M with respect to $\eta : M \times M \to R^n$ and $b : M \times M \times [0, 1] \to R_+$ if and only if the following relation

$$f(E(x)) \leqq f(E(u)) \Rightarrow$$
$$b(E(x), E(u), \lambda) [f(E(u) + \lambda\eta(E(x), E(u))) - f(E(u))] \leqq 0$$

holds for all $x, u \in M$ and any $\lambda \in [0, 1]$.

Definition 2.5. Let $E : R^n \to R^n$. A function $f : M \to R$ is said to be E-B-prepseudoinvex on M with respect to $\eta : M \times M \to R^n$ and $b : M \times M \times [0, 1] \to R_+$ if and only if the following relation

$$f(E(x)) < f(E(u)) \Rightarrow$$
$$b(E(x), E(u), \lambda) [f(E(u) + \lambda\eta(E(x), E(u))) - f(E(u))] \leqq 0$$

holds for all $x, u \in M$, $E(x) \neq E(u)$, and any $\lambda \in [0, 1]$.

Definition 2.6. [21] Let $E : R^n \to R^n$ and $f : M \to R$ be a (not necessarily) differentiable function at a given point $u \in M$. It is said that f is an E-differentiable function at u if and only if $f \circ E$ is a differentiable function at u (in the usual sense), that is,

$$(f \circ E)(x) = (f \circ E)(u) + \nabla(f \circ E)(u)(x - u) + \theta(u, x - u) \|x - u\|$$

where $\theta(u, x - u) \to 0$ as $x \to u$.

Definition 2.7. [3] Let $E : R^n \to R^n$, $f : M \to R$ be an E-differentiable function at u on M. It is said that f is (strictly) E-B-invex at $u \in M$ with respect to η and b if, there exist $\eta : M \times M \to R^n$ and $b : M \times M \to R_+$ such that, for all $x \in M$, the inequality

$$b(E(x), E(u)) [f(E(x)) - f(E(u))] \geqq \nabla f(E(u))\eta(E(x), E(u)) \quad (>) \tag{2.1}$$

holds. If inequality (2.1) holds for any $u \in M$ $(E(x) \neq E(u))$, then f is (strictly) E-B-invex with respect to η and b on M.

Now, we introduce various classes of generalized E-B-invex functions with respect to η and b.

Definition 2.8. Let $E : R^n \to R^n$, $f : M \to R$ be an E-differentiable function at u on M. The function f is said to be E-B-pseudoinvex at $u \in M$ on M with respect to η and b if, there exist $\eta : M \times M \to R^n$ and $b : M \times M \to R_+$ such that, for all $x \in M$, the following relation

$$\nabla(f \circ E)(u)\eta(E(x), E(u)) \geqq 0 \Rightarrow b(E(x), E(u)) f(E(x)) \geqq b(E(x), E(u)) f(E(u)) \tag{2.2}$$

holds for all $x \in M$. If relation (2.2) holds for any $u \in M$, then f is E-B-pseudoinvex with respect to η and b on M.

Remark 2.9. a) Every E-B-invex function with respect to η and b is E-B-pseudoinvex with respect to η and b. However, the converse is not necessarily true.

b) Every B-pseudoinvex function with respect to η and b is E-B-pseudoinvex with respect to η and b. However, the converse is not necessarily true.
c) Every E-pseudoinvex function with respect to η is E-B-pseudoinvex with respect to η and b. However, the converse is not necessarily true.

Definition 2.10. Let $E : R^n \to R^n$, $f : M \to R$ be an E-differentiable function at u on M. The function f is said to be strictly E-B-pseudoinvex at $u \in M$ on M with respect to η and b if, there exist $\eta : M \times M \to R^n$ and $b : M \times M \to R_+$ such that, for all $x \in M$ $(x \neq u)$, the following relation

$$\nabla (f \circ E)(u)\, \eta(E(x), E(u)) \geqq 0 \Rightarrow b(E(x), E(u))f(E(x)) > b(E(x), E(u))f(E(u)) \quad (2.3)$$

holds for all $x \in M$ $(x \neq u)$. If relation (2.3) holds for any $u \in M$, then f is strictly E-B-pseudoinvex with respect to η and b on M.

Definition 2.11. Let $E : R^n \to R^n$, $f : M \to R$ be an E-differentiable function at u on M. The function f is said to be E-B-quasiinvex at $u \in M$ on M with respect to η and b if, there exist $\eta : M \times M \to R^n$ and $b : M \times M \to R_+$ such that, for all $x \in M$, the following relation

$$f(E(x)) \leqq f(E(u)) \Rightarrow \nabla (f \circ E)(u)\, b(E(x), E(u))\eta(E(x), E(u)) \leqq 0 \quad (2.4)$$

holds for all $x \in M$. If relation (2.4) holds for any $u \in M$, then f is E-B-quasiinvex with respect to η and b on M.

Remark 2.12. a) Every E-B-invex function with respect to η and b is E-B-quasiinvex with respect to η and b. However, the converse is not necessarily true.
b) Every B-quasiinvex function with respect to η and b is E-B-quasiinvex with respect to η and b. However, the converse is not necessarily true.
c) Every E-quasiinvex function with respect to η is E-B-quasiinvex with respect to η and b. However, the converse is not necessarily true.

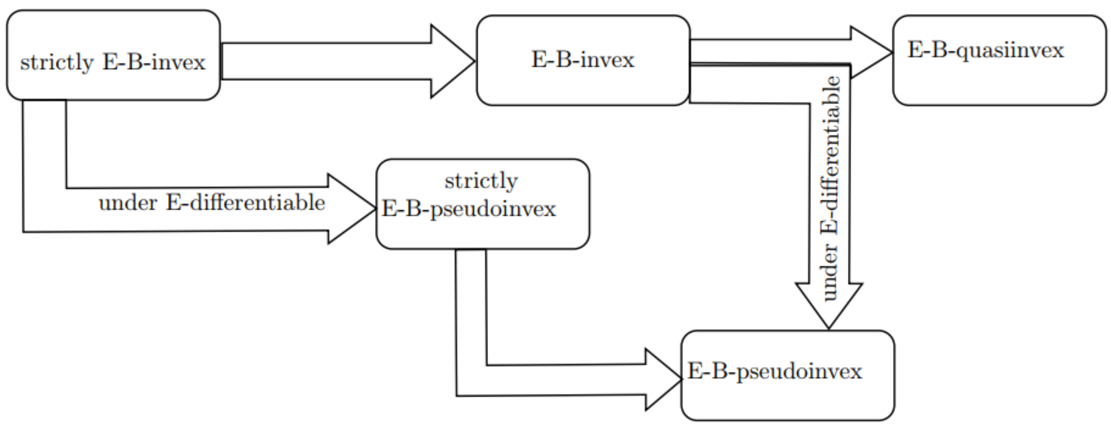

Figure 1. Relationships between E-differentiable E-B-invexity and different types of E-differentiable generalized E-B-invexity.

Now, we present an example of such an E-B-pseudoinvex function with respect to η and b which is not E-B-invex with respect to η and b, B-pseudoinvex with respect to η and b or E-pseudoinvex with respect to η.

Example 2.13. Let $E : R \to R$ and $f : (0, \frac{\pi}{2}) \to R$ be defined by

$$f(x) = \sin(\sqrt[3]{x}),$$
$$E(x) = x^3,$$
$$\eta(x,u) = \begin{cases} \sqrt[3]{x} - \sqrt[3]{u} & \text{if } x > u \\ 0 & \mathit{otherwise} \end{cases},$$
$$b(x,u) = \begin{cases} 1 & \text{if } x > u \\ 0 & \mathit{otherwise} \end{cases}.$$

Assume that $\nabla (f \circ E)(u)\, \eta(E(x), E(u)) \geqq 0$. Thus, we have $(x - u)\cos(x) \geqq 0$. This implies that $x \geqq u$ for all $x, u \in (0, \frac{\pi}{2})$. Moreover, we have

$$b(E(x), E(u))\, (f \circ E)(x) \geqq b(E(x), E(u))\, (f \circ E)(u)$$

for all $x, u \in (0, \frac{\pi}{2})$. Therefore, by Definition 2.8, f is E-B-pseudoinvex. Indeed, if we set $x = \frac{\pi}{3}$, $u = \frac{\pi}{6}$, then we have

$$b(E(x), E(u))\, [f(E(x)) - f(E(u))] < \nabla f(E(u))\eta(E(x), E(u)).$$

Hence, by Definition 2.7, it follows that f is not E-B-invex with respect to η and b given above. Further, f is not B-pseudoinvex with respect to η and b given above, if we set $x = \frac{\pi}{6}$, $u = \frac{\pi}{3}$, then we have $\nabla f(u)\eta(x,u) \geqq 0$, but $b(x,u)f(x) < b(x,u)f(u)$. Therefore, by definition of B-pseudoinvex (see, Bector et. al [11], Suneja et. al [25]), f is not B-pseudoinvex with respect to η and b given above. Also, f is not E-pseudoinvex with respect to η given above, if we set $x = \frac{\pi}{6}$, $u = \frac{\pi}{3}$, then we have $\nabla f(E(u))\eta(E(x), E(u)) \geqq 0$, but $(f \circ E)(x) < (f \circ E)(u)$. Thus, by the definition of E-pseudoinvex (see, Abdulaleem [2]), f is not E-pseudoinvex with respect to η given above.

Now, we present an example of such an E-B-quasiinvex function with respect to η and b which is not E-B-invex, B-quasiinvex or E-quasiinvex with respect to η.

Example 2.14. Let $E : R \to R$ and $f : [0, \frac{\pi}{2}] \to R$ be defined by

$$f(x) = \frac{3}{4}\sin(\sqrt[3]{x}) - \frac{1}{4}\sin(3\sqrt[3]{x}),$$
$$E(x) = x^3,$$
$$\eta(x,u) = \cos(\sqrt[3]{u})(\sin(\sqrt[3]{u}) - \sin(\sqrt[3]{x})),$$
$$b(x,u) = \begin{cases} \sqrt[3]{xu} & \text{if } x > u \\ 0 & \mathit{otherwise} \end{cases}.$$

Then, f is an E-B-quasiinvex function with respect to η and b given above. However, f is not E-B-invex with respect to η and b as can be seen by taking $x = \frac{\pi}{6}$, $u = \frac{\pi}{3}$, the following inequality

$$b(E(x), E(u))\, [f(E(x)) - f(E(u))] < \nabla f(E(u))\eta(E(x), E(u))$$

holds. Hence, by Definition 2.7, it follows that f is not E-B-invex with respect to η and b given above. Further, f is not an E-quasiinvex function with respect to η as can be seen by taking $x = \frac{\pi}{6}$, $u = \frac{\pi}{4}$, we have $f(E(x)) < f(E(u))$, but $\nabla f(E(u))\eta(E(x), E(u)) > 0$. Hence, by Definition of E-quasiinvex function (see, Abdulaleem [2]), f is not an E-quasiinvex function with respect to η given above. Moreover, f is only E-differentiable and it is not differentiable at $u = 0$. Therefore, for this reason, f is not B-quasiinvex or quasiinvex with with respect to η given above.

3. E-differentiable multiobjective programming

Consider the following (not necessarily differentiable) multiobjective programming problem (VP):

$$\begin{array}{c}\text{minimize } f(x) = (f_1(x), ..., f_q(x)) \\ \text{subject to } g_t(x) \leqq 0, \;\; t \in T := \{1, ..., m\}\end{array} \qquad \text{(VP)}$$

where $f_i : R^n \to R$, $i \in I := \{1, ..., q\}$ and $g_t : R^n \to R$, $t \in T$ are real-valued E-differentiable functions defined on R^n. We shall write $g := (g_1, ..., g_m) : R^n \to R^m$ for convenience. Let

$$\Omega := \{x \in R^n : g_t(x) \leqq 0, \;\; t \in T\}$$

be the set of all feasible solutions of (VP). Further, by $T(x)$, the set of inequality constraint indices that are active at a feasible solution x, that is, $T(x) = \{t \in T : g_t(x) = 0\}$.

Definition 3.1. A feasible point $\overline{x}$ is said to be a weak Pareto (weakly efficient) solution for (VP) if and only if there exists no feasible point x such that

$$f(x) < f(\overline{x}).$$

Definition 3.2. A feasible point $\overline{x}$ is said to be a Pareto (efficient) solution for (VP) if and only if there exists no feasible point x such that

$$f(x) \leq f(\overline{x}).$$

Now, for the considered multiobjective programming problem (VP), we define its associated differentiable vector optimization problem as follows:

$$\begin{array}{c}\text{minimize } (f \circ E)(x) = ((f_1 \circ E)(x), ..., (f_q \circ E)(x)) \\ \text{subject to } (g_t \circ E)(x) \leqq 0, \;\; t \in T = \{1, ..., m\}.\end{array} \qquad (\text{VP}_E)$$

Let

$$\Omega_E := \{x \in R^n : (g_t \circ E)(x) \leqq 0, \;\; t \in T\}$$

be the set of all feasible solutions of (VP_E). We call the problem (VP_E) an E-vector optimization problem.

Lemma 3.3. *[10] Let $E : R^n \to R^n$ be a one-to-one and onto. Then $E(\Omega_E) = \Omega$.*

Lemma 3.4. *[10] Let $\overline{x} \in \Omega$ be a Pareto solution (a weak Pareto solution) of (VP). Then, there exists $\overline{z} \in \Omega_E$ such that $\overline{x} = E(\overline{z})$ and $\overline{z}$ is a Pareto (a weak Pareto) solution of (VP_E).*

Theorem 3.5. *[1] (E-Karush-Kuhn-Tucker necessary optimality conditions). Let $\overline{x} \in \Omega_E$ be a weak Pareto solution of the E-vector optimization problem (VP_E) (and, thus, $E(\overline{x})$ be a weak E-Pareto solution of the considered problem (VP)). Further, f, g be E-differentiable at $\overline{x}$ and the E-Guignard constraint qualification [1] be satisfied at $\overline{x}$. Then there exist Lagrange multipliers $\overline{\lambda} \in R^q$, $\overline{\mu} \in R^m$ such that*

$$\sum_{i=1}^{q} \overline{\lambda}_i \nabla f_i(E(\overline{x})) + \sum_{t=1}^{m} \overline{\mu}_t \nabla g_t(E(\overline{x})) = 0, \tag{3.1}$$

$$\overline{\mu}_t g_t(E(\overline{x})) = 0, \;\; t = 1, ..., m, \tag{3.2}$$

$$\overline{\lambda} \geq 0, \, \overline{\mu} \geqq 0. \tag{3.3}$$

Definition 3.6. It is said that $(\overline{x}, \overline{\lambda}, \overline{\mu}) \in \Omega_E \times R^q \times R^m$ is a Karush-Kuhn-Tucker point for the E-vector optimization problem (VP_E) if the Karush-Kuhn-Tucker necessary optimality conditions (3.1)-(3.3) are satisfied at $\overline{x}$ with Lagrange multiplier $\overline{\lambda}$, $\overline{\mu}$.

Now, we prove the sufficient optimality conditions for the vector optimization problem (VP) under (generalized) E-B-invexity hypotheses that the functions constituting it are nondifferentiable (generalized) E-B-invex at a feasible point satisfying the E-Karush-Kuhn-Tucker necessary optimality conditions (3.1)-(3.3).

Theorem 3.7. *Let $(\overline{x}, \overline{\lambda}, \overline{\mu}) \in \Omega_E \times R^q \times R^m$ be a Karush-Kuhn-Tucker point for the E-vector optimization problem (VP$_E$). Further, assume the following hypotheses are fulfilled:*

a) *each objective function f_i, $i \in I$ is E-B_{f_i}-invex function with respect to η and b_{f_i} at $\overline{x}$ on Ω_E,*

b) *each inequality constraint g_t, $t \in T_E(\overline{x})$, is an E-B_{g_t}-invex function with respect to η and b_{g_t} at $\overline{x}$ on Ω_E.*

Then $\overline{x}$ is a weak Pareto solution of the problem (VP$_E$) and, thus, $E(\overline{x})$ is a weak E-Pareto solution of the problem (VP).

Proof. By assumption, $(\overline{x}, \overline{\lambda}, \overline{\mu}) \in \Omega_E \times R^q \times R^m$ is a Karush-Kuhn-Tucker point for the E-vector optimization problem (VP$_E$). Then, by Definition 3.6, the Karush-Kuhn-Tucker necessary optimality conditions (3.1)-(3.3) are satisfied at $\overline{x}$ with Lagrange multiplier $\overline{\lambda}$, $\overline{\mu}$. We proceed by contradiction. Suppose, contrary to the result, that $\overline{x}$ is not a weak Pareto solution of the problem (VP$_E$). Hence, by Definition 3.1, there exists another $\widehat{x} \in \Omega_E$ such that

$$f_i(E(\widehat{x})) < f_i(E(\overline{x})), \ i \in I. \tag{3.4}$$

Using hypotheses a)-b), by Definition 2.7, the following inequalities

$$b_{f_i}(E(\widehat{x}), E(\overline{x}))[f_i(E(\widehat{x})) - f_i(E(\overline{x}))] \geqq \nabla f_i(E(\overline{x}))\eta(E(\widehat{x}), E(\overline{x})) \tag{3.5}$$

for $i \in I$, and

$$b_{g_t}(E(\widehat{x}), E(\overline{x}))[g_t(E(\widehat{x})) - g_t(E(\overline{x}))] \geqq \nabla g_t(E(\overline{x}))\eta(E(\widehat{x}), E(\overline{x})) \tag{3.6}$$

for $t \in T(E(\overline{x}))$ hold, respectively. Multiplying inequalities (3.5) and (3.6) by the corresponding Lagrange multipliers, respectively, we obtain

$$\begin{aligned} \overline{\lambda}_i b_{f_i}(E(\widehat{x}), E(\overline{x}))[f_i(E(\widehat{x})) - f_i(E(\overline{x}))] &\geqq \\ \overline{\lambda}_i \nabla f_i(E(\overline{x}))\eta(E(\widehat{x}), E(\overline{x})), \ i \in I&, \end{aligned} \tag{3.7}$$

$$\begin{aligned} \overline{\mu}_t b_{g_t}(E(\widehat{x}), E(\overline{x}))[g_t(E(\widehat{x})) - g_t(E(\overline{x}))] &\geqq \\ \overline{\mu}_t \nabla g_t(E(\overline{x}))\eta(E(\widehat{x}), E(\overline{x})), \ t \in T(\overline{x})&. \end{aligned} \tag{3.8}$$

Since, $\overline{\lambda}_i \geq 0$, $b_{f_i}(E(\widehat{x}), E(\overline{x})) > 0$, $i \in I$ for all $\widehat{x} \in \Omega_E$. Combining (3.4) and (3.7), we get

$$\overline{\lambda}_i \nabla (f_i \circ E)(\overline{x})\eta(E(\widehat{x}), E(\overline{x})) < 0, \ i \in I.$$

Thus,

$$\sum_{i=1}^{q} \overline{\lambda}_i \nabla f_i(E(\overline{x}))\eta(E(\widehat{x}), E(\overline{x})) < 0. \tag{3.9}$$

Using the E-Karush-Kuhn-Tucker necessary optimality condition (3.2) together with $\widehat{x} \in \Omega_E$ and $\overline{x} \in \Omega_E$, $\overline{\mu}_t \geqq 0$, $b_{g_t}(E(\widehat{x}), E(\overline{x})) > 0$, $t \in T(E(\overline{x}))$, we have

$$\overline{\mu}_t \nabla g_t(E(\overline{x}))\eta(E(\widehat{x}), E(\overline{x})) \leqq 0, \ t \in T(E(\overline{x})).$$

Thus,

$$\sum_{t=1}^{m} \overline{\mu}_t \nabla g_t(E(\overline{x}))\eta(E(\widehat{x}), E(\overline{x})) \leqq 0. \tag{3.10}$$

Combining (3.9) and (3.10), we obtain that the following inequality

$$\left[\sum_{i=1}^{q} \overline{\lambda}_i \nabla f_i(E(\overline{x})) + \sum_{t=1}^{m} \overline{\mu}_t \nabla g_t(E(\overline{x}))\right] \eta(E(\widehat{x}), E(\overline{x})) < 0$$

holds, which is a contradiction to the the E-Karush-Kuhn-Tucker necessary optimality condition (3.1). By assumption, $E : R^n \to R^n$ is an one-to-one and onto operator. Since $\overline{x}$ is a weak Pareto solution of the problem (VP$_E$), by Lemma 3.4, $E(\overline{x})$ is a weak E-Pareto solution of the problem (VP). Thus, the proof of this theorem is completed. □

Theorem 3.8. *Let $(\overline{x}, \overline{\lambda}, \overline{\mu}) \in \Omega_E \times R^q \times R^m$ be a Karush-Kuhn-Tucker point for the E-vector optimization problem (VP$_E$). Further, assume the following hypotheses are fulfilled:*

a) *each objective function f_i, $i \in I$ is strictly E-B_{f_i}-invex function with respect to η and b_{f_i} at $\overline{x}$ on Ω_E,*
b) *each inequality constraint g_t, $t \in T_E(\overline{x})$, is an E-B_{g_t}-invex function with respect to η and b_{g_t} at $\overline{x}$ on Ω_E.*

Then $\overline{x}$ is a Pareto solution of the problem (VP$_E$) and, thus, $E(\overline{x})$ is a E-Pareto solution of the problem (VP).

Now, under the concepts of generalized E-B-invexity, we prove the sufficient optimality conditions for a feasible solution to be a weak E-Pareto solution of problem (VP).

Theorem 3.9. *Let $(\overline{x}, \overline{\lambda}, \overline{\mu}) \in \Omega_E \times R^q \times R^m$ be a Karush-Kuhn-Tucker point for the E-vector optimization problem (VP$_E$). Further, assume the following hypotheses are fulfilled:*

a) *each objective function f_i, $i \in I$ is E-B_{f_i}-pseudoinvex function with respect to η and b_{f_i} at $\overline{x}$ on Ω_E,*
b) *each inequality constraint g_t, $t \in T_E(\overline{x})$, is an E-B_{g_t}-quasiinvex function with respect to η and b_{g_t} at $\overline{x}$ on Ω_E.*

Then $\overline{x}$ is a weak Pareto solution of the problem (VP$_E$) and, thus, $E(\overline{x})$ is a weak E-Pareto solution of the problem (VP).

Proof. By assumption, $(\overline{x}, \overline{\lambda}, \overline{\mu}) \in \Omega_E \times R^q \times R^m$ is a Karush-Kuhn-Tucker point for the E-vector optimization problem (VP$_E$). Then, by Definition 3.6, the Karush-Kuhn-Tucker necessary optimality conditions (3.1)-(3.3) are satisfied at $\overline{x}$ with Lagrange multiplier $\overline{\lambda}$, $\overline{\mu}$. We proceed by contradiction. Suppose, contrary to the result, that $\overline{x}$ is not a weak Pareto solution of the problem (VP$_E$). Hence, by Definition 3.1, there exists another $\widehat{x} \in \Omega_E$ such that

$$f_i(E(\widehat{x})) < f_i(E(\overline{x})),\ i \in I.$$

Since, $b_{f_i}(E(\widehat{x}), E(\overline{x})) > 0,\ i \in I$, thus,

$$b_{f_i}(E(\widehat{x}), E(\overline{x}))\, f_i(E(\widehat{x})) < b_{f_i}(E(\widehat{x}), E(\overline{x}))\, f_i(E(\overline{x})),\ i \in I.$$

By hypothesis (a), each objective function $f_i, i \in I$ is E-B_{f_i}-pseudoinvex at $\overline{x}$ on Ω_E with respect to η and b_{f_i}. Then, Definition 2.8, gives

$$\nabla(f_i \circ E)(\overline{x})\, \eta(E(\widehat{x}), E(\overline{x})) < 0,\ i \in I. \tag{3.11}$$

By the E-Karush-Kuhn-Tucker necessary optimality condition (3.3), inequality (3.11) yields

$$\left[\sum_{i=1}^{q} \overline{\lambda}_i \nabla f_i(E(\overline{x}))\right] \eta(E(\widehat{x}), E(\overline{x})) < 0. \tag{3.12}$$

Since $\widehat{x} \in \Omega_E$, therefore, the E-Karush-Kuhn-Tucker necessary optimality conditions (3.2) and (3.3) imply

$$g_t(E(\widehat{x})) - g_t(E(\overline{x})) \leqq 0,\ \ t \in T(E(\overline{x})).$$

From the assumption, each inequality constraint g_j, $j \in J$, is an E-B_{g_t}-quasiinvex function with respect to η and b_{g_t} at $\overline{x}$ on Ω_E. Then, by Definition 2.11, we get

$$\nabla g_t(E(\overline{x}))\, b_{g_t}(E(\widehat{x}), E(\overline{x}))\, \eta(E(\widehat{x}), E(\overline{x})) \leqq 0,\ \ t \in T(E(\overline{x})).$$

Since, $b_{g_t}(E(\widehat{x}), E(\overline{x})) > 0,\ t \in T(E(\overline{x}))$, we get

$$\nabla g_t(E(\overline{x}))\, \eta(E(\widehat{x}), E(\overline{x})) \leqq 0,\ \ t \in T(E(\overline{x})). \tag{3.13}$$

Thus, by the E-Karush-Kuhn-Tucker necessary optimality condition (3.3), (3.13) gives

$$\sum_{t\in T(E(\overline{x}))} \overline{\mu}_t \nabla g_t \left(E\left(\overline{x}\right)\right) \eta \left(E\left(\widehat{x}\right), E\left(\overline{x}\right)\right) \leqq 0.$$

Hence, taking into account $\overline{\mu}_t = 0$, $t \notin T\left(E\left(\overline{x}\right)\right)$, we have

$$\sum_{t=1}^{m} \overline{\mu}_t \nabla g_t \left(E\left(\overline{x}\right)\right) \eta \left(E\left(\widehat{x}\right), E\left(\overline{x}\right)\right) \leqq 0. \tag{3.14}$$

Combining (3.12) and (3.14), we get that the following inequality

$$\left[\sum_{i=1}^{q} \overline{\lambda}_i \nabla f_i(E\left(\overline{x}\right)) + \sum_{t=1}^{m} \overline{\mu}_t \nabla g_t \left(E\left(\overline{x}\right)\right)\right] \eta \left(E\left(\widehat{x}\right), E\left(\overline{x}\right)\right) < 0,$$

which is a contradiction to the E-Karush-Kuhn-Tucker necessary optimality condition (3.1). The result that $E(\overline{x})$ is a weak E-Pareto solution follows directly from Lemma 3.4. Thus, the proof of this theorem is completed. □

In order to illustrate the sufficient optimality conditions established in the paper, we now present an example of an E-differentiable vector optimization problem in which the involved functions are (generalized) E-B-invex.

Example 3.10. Consider the following nondifferentiable vector optimization problem

$$\begin{gathered} f(x) = (f_1(x), f_2(x)) = \left(\sqrt[3]{x_1} + \sqrt[3]{x_2} + 1,\ \ \sqrt[3]{x_1^2} + \sqrt[3]{x_2} - 4\right) \to V\text{-}\min \\ g_1(x) = \sin \sqrt[3]{x_1} - 4 \sin \sqrt[3]{x_2} \leqq 0, \\ g_2(x) = 2\sqrt[3]{x_1^2} + 2\sqrt[3]{x_2^2} - 9 \leqq 0, \\ g_3(x) = -\sin \sqrt[3]{x_1} \leqq 0, \\ g_4(x) = -\sin \sqrt[3]{x_2} \leqq 0. \end{gathered} \tag{VP1}$$

Note that the set of all feasible solutions of (VP1) is

$$\begin{aligned} \Omega =& \{(x_1, x_2) \in R^2 : \sin \sqrt[3]{x_1} - 4 \sin \sqrt[3]{x_2} \leqq 0,\ 2\sqrt[3]{x_1^2} + 2\sqrt[3]{x_2^2} - 9 \leqq 0, \\ & -\sin \sqrt[3]{x_1} \leqq 0,\ \ -\sin \sqrt[3]{x_2} \leqq 0\}. \end{aligned}$$

Note that the functions constituting problem (VP1) are nondifferentiable at $(0, 0)$. Let

$$\begin{aligned} \eta(x, \overline{x}) &= (\sin \sqrt[3]{x_1} - \sin \overline{x}_1,\ \sin \sqrt[3]{x_2} - \sin \overline{x}_2), \\ b_{f_1}(x, \overline{x}) &= \frac{\sin \sqrt[3]{x_1} + \sin \sqrt[3]{x_2} - \sin \overline{x}_1 - \sin \overline{x}_2}{\sqrt[3]{x_1} + \sqrt[3]{x_2}}, \\ b_{f_2}(x, \overline{x}) &= \frac{\sin \sqrt[3]{x_1} + 2 \sin \sqrt[3]{x_2} - \sin \overline{x}_1 - 2 \sin \overline{x}_2}{\sqrt[3]{x_1} + \sqrt[3]{x_2}}, \\ b_{g_1}(x, \overline{x}) &= b_{g_3}(x, \overline{x}) = b_{g_4}(x, \overline{x}) = 1, \\ b_{g_2}(x, \overline{x}) &= \sqrt[3]{x_1^2} + \sqrt[3]{x_2^2} - \overline{x}_1^2, \end{aligned}$$

and $E : R^2 \to R^2$ be defined as follows

$$E\left(x_1, x_2\right) = \left(x_1^3, x_2^3\right).$$

Now, for the considered E-differentiable multiobjective programming problem (VP1), we define its associated E-vector optimization problem (VP1$_E$) as follows

$$\begin{aligned} f(E(x)) = (f_1(E(x)), f_2(E(x))) = (x_1 + x_2 + 1,\ \ x_1 + x_2 - 4) \to V\text{-}\min \\ g_1(E(x)) = \sin x_1 - 4\sin x_2 \leqq 0, \\ g_2(E(x)) = 2x_1^2 + 2x_2^2 - 9 \leqq 0, \\ g_3(E(x)) = -\sin x_1 \leqq 0, \\ g_4(E(x)) = -\sin x_2 \leqq 0. \end{aligned} \qquad (\text{VP1}_E)$$

Note that the set of all feasible solutions of the considered E-vector optimization problem (VP1$_E$) is $\Omega_E = \{(x_1, x_2) \in R^2 : \sin x_1 - 4\sin x_2 \leqq 0,\ \ 2x_1^2 + 2x_2^2 - 9 \leqq 0,\ -\sin x_1 \leqq 0,\ -\sin x_2 \leqq 0\}$, (see, Figure 2). Further, it is not hard to see that all functions constituting the problem (VP1$_E$) are differentiable at $(0, 0)$. Then, it can also be shown that the E-Karush-Kuhn-Tucker necessary optimality conditions (3.1)-(3.3) are fulfilled at $(0, 0)$. Moreover, all hypotheses of Theorem 3.9 are fulfilled, it can be proved that each objective function f_i, $i = 1, 2$, is an E-B_{f_i}-pseudoinvex function with respect to η and b_{f_i} given above at $\overline{x}$ on Ω_E, each inequality constraint g_t, $t = 1, 2, 3, 4$, is an E-B_{g_t}-quasiinvex function with respect to η and b_{g_t} given above at $\overline{x}$ on Ω_E. Hence, $\overline{x} = (0, 0)$ is a Pareto solution of the E-vector optimization problem (VP1$_E$) (see Fig. 2). Furthermore, that the sufficient optimality conditions under E-differentiable E-invexity are not applicable since not all functions constituting problem (VP1) are E-invex with respect to η given above.

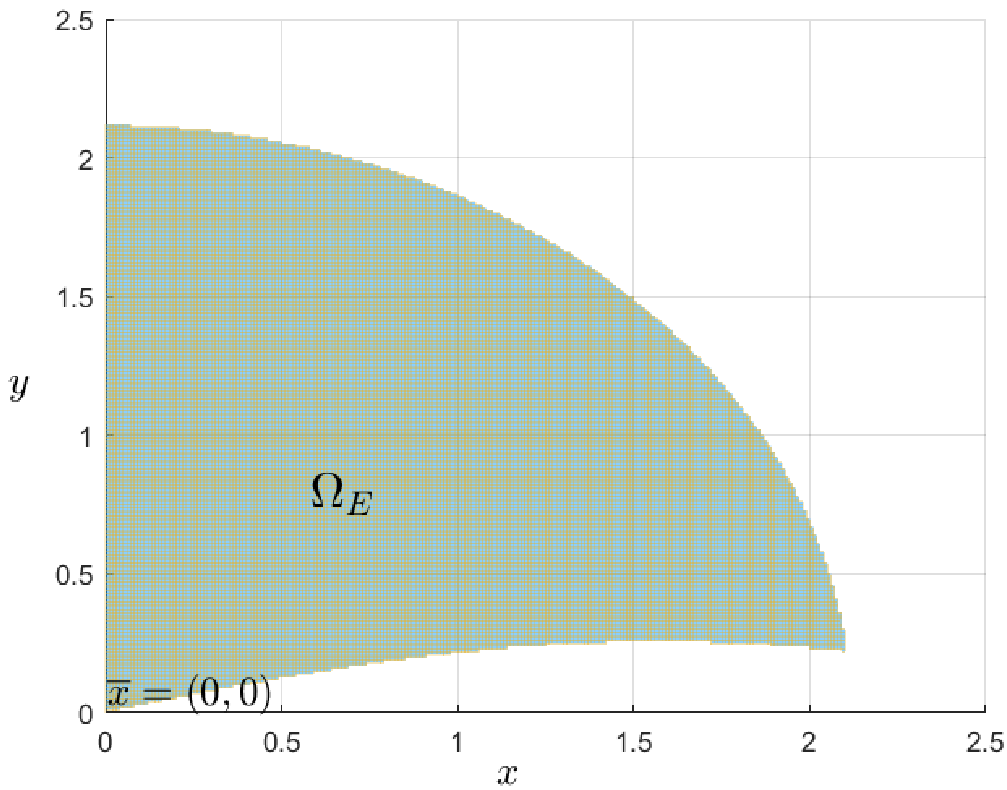

Figure 2. The set of all feasible solutions of (VP1$_E$) in example 3.10.

4. Mond-Weir E-duality

Let $E : R^n \to R^n$ be a given one-to-one and onto operator, for the differentiable vector E-optimization problem (VP$_E$), we define its dual problem in the sense of vector Mond-Weir duality as follows:

$$\begin{aligned} &(f \circ E)(y) = (f_1(E(y)), ..., f_q(E(y))) \to V - \max \\ \text{s.t. } & \textstyle\sum_{i=1}^{q} \lambda_i \nabla (f_i \circ E)(y) + \sum_{t=1}^{m} \mu_t \nabla (g_t \circ E)(y) = 0, \\ & \textstyle\sum_{t=1}^{m} \mu_t (g_t \circ E)(y) \geqq 0, \\ & \lambda \in R^q, \lambda \geq 0 \ , \ \mu \in R^m, \mu \geqq 0. \end{aligned} \qquad (\text{MWD}_E)$$

Let

$$\Gamma_E = \Big\{ (y, \lambda, \mu) \in R^n \times R^q \times R^m : \textstyle\sum_{i=1}^{q} \lambda_i \nabla (f_i \circ E)(y) + \sum_{t=1}^{m} \mu_t \nabla (g_t \circ E)(y) = 0,$$
$$\textstyle\sum_{t=1}^{m} \mu_t (g_t \circ E)(y) \geqq 0, \ \lambda \geq 0, \ \mu \geqq 0 \Big\}$$

be the set of all feasible solutions of the problem (MWD$_E$). Let us denote, $Y_E = \{y \in R^n : (y, \lambda, \mu) \in \Gamma_E\}$.

Now, under E-B-invexity hypotheses, we prove duality results between (VP$_E$) and (MWD$_E$) and, thus, E-duality results between (VP) and (MWD$_E$).

Theorem 4.1. *(Mond-Weir weak duality between (VP$_E$) and (MWD$_E$)). Let x and (y, λ, μ) be any feasible solutions of the problems (VP$_E$) and (MWD$_E$), respectively. Further, assume that at least one of the following hypotheses is fulfilled:*

A) *each objective function f_i, $i \in I$ is E-B_{f_i}-invex with respect to η and b_{f_i} at y on $\Omega_E \cup Y_E$, each constraint function g_t, $t \in T$, is E-B_{g_t}-invex with respect to η and b_{g_t} at y on $\Omega_E \cup Y_E$.*

B) *$\lambda_i f_i$, $i \in I$ is E-B_{f_i}-pseudoinvex with respect to η and b_{f_i} at y on $\Omega_E \cup Y_E$, $\mu_t g_t$, $t \in T$ is E-B_{g_t}-quasiinvex with respect to η and b_{g_t} at y on $\Omega_E \cup Y_E$.*

Then

$$(f \circ E)(x) \nless (f \circ E)(y).$$

Proof. Let x and (y, λ, μ) be any feasible solutions of problems (VP$_E$) and (VMD$_E$), respectively. We, first, consider the case that under hypothesis A) is satisfied. If $x = y$, then the weak duality trivially holds. Now, we prove the weak duality theorem when $x \neq y$. We proceed by contradiction. Suppose, contrary to the result, that the inequality

$$(f \circ E)(x) < (f \circ E)(y) \qquad (4.1)$$

holds. By the feasibility of (y, λ, μ) in problem (MWD$_E$), the above inequality yields

$$\sum_{i=1}^{q} \lambda_i f_i(E(x)) < \sum_{i=1}^{q} \lambda_i f_i(E((y)). \qquad (4.2)$$

By assumption, x and (y, λ, μ) are feasible solutions for problems (VP$_E$) and (MWD$_E$), respectively. Since each function f_i, $i \in I$ is E-B_{f_i}-invex with respect to η and b_{f_i} at y on $\Omega_E \cup Y_E$, each constraint function g_t, $t \in T$, is E-B_{g_t}-invex with respect to η and b_{g_t} at y on $\Omega_E \cup Y_E$, by Definition 2.7, the following inequalities hold

$$b_{f_i}(E(x), E(y)) [f_i(E(x)) - f_i(E(y))] \geqq \nabla f_i(E(y)) \eta(E(x), E(y)) \qquad (4.3)$$

for all $i \in I$, and

$$b_{g_t}(E(x), E(y)) [g_t(E(x)) - g_t(E(y))] \geqq \nabla g_t(E(y)) \eta(E(x), E(y)) \qquad (4.4)$$

for all $t \in T(E(y))$, respectively. Multiplying inequalities (4.3)-(4.4) by the corresponding Lagrange multiplier, respectively, we obtain the inequalities

$$\lambda_i b_{f_i}(E(x), E(y))[f_i(E(x)) - f_i(E(y))] \geqq \lambda_i \nabla f_i(E(y))\eta(E(x), E(y)) \tag{4.5}$$

for all $i \in I$, and

$$\mu_t b_{g_t}(E(x), E(y))[g_t(E(x)) - g_t(E(y))] \geqq \mu_t \nabla g_t(E(y))\eta(E(x), E(y)) \tag{4.6}$$

for all $t \in T(E(y))$. Since, $\lambda_i \geq 0$, $b_{f_i}(E(x), E(y)) > 0$, $i \in I$ for all $x \in \Omega_E$. Combining (4.2) and (4.5), we get

$$\lambda_i \nabla (f_i \circ E)(y)\eta(E(x), E(y)) < 0, \ i \in I.$$

Thus,

$$\sum_{i=1}^{q} \lambda_i \nabla f_i(E(y))\eta(E(x), E(y)) < 0. \tag{4.7}$$

Using the E-Karush-Kuhn-Tucker necessary optimality condition (3.2) together with $x \in \Omega_E$, (4.6) and $y \in \Omega_E$, $\mu_t \geqq 0$, $b_{g_t}(E(x), E(y)) > 0$, $t \in T(E(y))$, we have

$$\mu_t \nabla g_t(E(y))\eta(E(x), E(y)) \leqq 0, \ t \in T(E(y)).$$

Thus,

$$\sum_{t=1}^{m} \mu_t \nabla g_t(E(y))\eta(E(x), E(y)) \leqq 0. \tag{4.8}$$

Combining (4.7) and (4.8), we obtain that the following inequality

$$\left[\sum_{i=1}^{q} \lambda_i \nabla f_i(E(y)) + \sum_{t=1}^{m} \mu_t \nabla g_t(E(y))\right] \eta(E(x), E(y)) < 0$$

holds. This is a contradiction to the first constraint of the vector Mond-Weir E-dual problem (MWD$_E$) which means that the proof of the Mond-Weir weak duality theorem between the E-vector optimization problems (VP$_E$) and (MWD$_E$) is completed under hypothesis A).

The proof of this theorem under hypothesis B). We proceed by contradiction. Suppose, contrary to the result, that (4.1) holds. Since $\lambda_i f_i$, $i \in I$ is E-B_{f_i}-pseudoinvex with respect to η and b_{f_i} at y on $\Omega_E \cup Y_E$, by Definition 2.8, the inequality

$$\sum_{i=1}^{q} \lambda_i \nabla (f_i \circ E)(y)\eta(E(x), E(y)) < 0 \tag{4.9}$$

holds. Since $\mu_t g_t$ is E-B_{g_t}-quasiinvex with respect to η and b_{g_t} at y on $\Omega_E \cup Y_E$, $b(E(x), E(y)) > 0$, Definition 2.11 implies that the inequality

$$\sum_{t=1}^{m} \mu_t \nabla (g_t \circ E)(y)\eta(E(x), E(y)) \leqq 0 \tag{4.10}$$

holds. Combining (4.9) and (4.10), it follows that the inequality

$$\left[\sum_{i=1}^{q} \lambda_i \nabla f_i(E(y)) + \sum_{t=1}^{m} \mu_t \nabla g_t(E(y))\right] \eta(E(x), E(y)) < 0$$

holds. This contradicts with the first constraint of the vector Mond-Weir E-dual problem (MWD$_E$). This means that the proof of the Mond-Weir weak duality theorem between the E-vector optimization problems (VP$_E$) and (MWD$_E$) is completed under hypothesis B). □

Theorem 4.2. *(Mond-Weir weak E-duality between (VP) and (MWD_E)). Let $E(x)$ and (y, λ, μ) be a feasible solutions of the problems (VP) and (MWD_E), respectively. Further, assume that all hypotheses of Theorem 4.1 are fulfilled. Then, Mond-Weir weak E-duality between (VP) and (MWD_E) holds, that is,*

$$(f \circ E)(x) \nless (f \circ E)(y).$$

Proof. Let $E(x)$ and (y, λ, μ) be any feasible solutions of the problems (VP) and (MWD_E), respectively. Then, by Lemma 3.3, it follows that x is any feasible solution of (VP_E). Since all hypotheses of Theorem 4.1 are fulfilled, the Mond-Weir weak E-duality theorem between the problems (VP) and (MWD_E) follows directly from Theorem 4.1. □

If some stronger E-B-invexity hypotheses are imposed on the functions constituting the considered E-differentiable multiobjective programming problem, then the following stronger result is true.

Theorem 4.3. *(Mond-Weir weak duality between (VP_E) and (MWD_E)). Let x and (y, λ, μ) be any feasible solutions of the problems (VP_E) and (MWD_E), respectively. Further, assume that at least one of the following hypotheses is fulfilled:*

A) *each objective function f_i, $i \in I$ is strictly E-B_{f_i}-invex with respect to η and b_{f_i} at y on $\Omega_E \cup Y_E$, each constraint function g_t, $t \in T$, is E-B_{g_t}-invex with respect to η and b_{g_t} at y on $\Omega_E \cup Y_E$.*
B) *$\lambda_i f_i$, $i \in I$ is strictly E-B_{f_i}-pseudoinvex with respect to η and b_{f_i} at y on $\Omega_E \cup Y_E$, $\mu_t g_t$, $t \in T$ is E-B_{g_t}-quasiinvex with respect to η and b_{g_t} at y on $\Omega_E \cup Y_E$.*

Then

$$(f \circ E)(x) \nleq (f \circ E)(y).$$

Theorem 4.4. *(Mond-Weir weak E-duality between (VP) and (MWD_E)). Let $E(x)$ and (y, λ, μ) be any feasible solutions of the problems (VP) and (MWD_E), respectively. Further, assume that all hypotheses of Theorem 4.3 are fulfilled. Then, weak E-duality between (VP) and (VMD_E) holds, that is,*

$$(f \circ E)(x) \nleq (f \circ E)(y).$$

Theorem 4.5. *(Mond-Weir strong duality between (VP_E) and (MWD_E) and also Mond-Weir strong E-duality between (VP) and (MWD_E)). Let $\overline{x} \in \Omega_E$ be a weak Pareto solution (a Pareto solution) of the E-vector optimization problem (VP_E) (and, thus, $E(\overline{x})$ be a weak E-Pareto solution (an E-Pareto solution) of the E-vector optimization problem (MOP)). Further, assume that the E-Guignard constraint qualification proposed in [1] be satisfied at $\overline{x}$. Then, there exist $\overline{\lambda} \in R^q$, $\overline{\mu} \in R^m$, $\overline{\mu} \geqq 0$ such that $\left(\overline{x}, \overline{\lambda}, \overline{\mu}\right)$ is feasible for the problem (MWD_E) and the objective functions of (VP_E) and (MWD_E) are equal at these points. If also all hypotheses of the Mond-Weir weak duality (Theorem 4.1 (Theorem 4.3)) are satisfied, then $\left(\overline{x}, \overline{\lambda}, \overline{\mu}\right)$ is a weak efficient (an efficient) solution of a maximum type in the problem (MWD_E).*
In other words, if $E(\overline{x}) \in \Omega$ is a (weak) E-Pareto solution of the multiobjective programming problem (VP), then $\left(\overline{x}, \overline{\lambda}, \overline{\mu}\right)$ is a (weak) efficient solution of a maximum type in the dual problem (MWD_E) in the sense of Mond-Weir. This means that the Mond-Weir strong E-duality holds between the problems (VP) and (MWD_E).

Proof. Since $\overline{x} \in \Omega_E$ is a (weak) Pareto solution of the problem (VP_E) and the E-Guignard constraint qualification be satisfied at $\overline{x}$, by Theorem 3.5, there exist $\overline{\lambda} \in R^q$, $\overline{\mu} \in R^m$, $\overline{\mu} \geqq 0$ such that $\left(\overline{x}, \overline{\lambda}, \overline{\mu}\right)$ is a feasible solution of problem (MWD_E). This means that the objective functions of (VP_E) and (MWD_E) are equal. If we assume that all hypotheses of the Mond-Weir weak duality (Theorem 4.1 (Theorem 4.3)) are fulfilled, then

$\left(\overline{x}, \overline{\lambda}, \overline{\mu}\right)$ is a (weak) efficient solution of a maximum type in the dual problem (MWD$_E$) in the sense of Mond-Weir.

Moreover, we have, by Lemma 3.3, that $E\left(\overline{x}\right) \in \Omega$. Since $\overline{x} \in \Omega_E$ is a weak Pareto solution of the problem (VP$_E$), by Lemma 3.4, it follows that $E\left(\overline{x}\right)$ is a weak E-Pareto solution in the problem (VP). Then, by the Mond-Weir strong duality between (VP$_E$) and (MWD$_E$), we conclude that also the Mond-Weir strong E-duality holds between the problems (VP) and (MWD$_E$). This means that if $E\left(\overline{x}\right) \in \Omega$ is a weak E-Pareto solution of the problem (VP), there exist $\overline{\lambda} \in R^q$, $\overline{\mu} \in R^m$, $\overline{\mu} \geqq 0$ such that $\left(\overline{x}, \overline{\lambda}, \overline{\mu}\right)$ is a weakly efficient solution of a maximum type in the Mond-Weir dual problem (MWD$_E$). □

Theorem 4.6. *(Mond-Weir converse duality between (VP$_E$) and (MWD$_E$)). Let $\left(\overline{x}, \overline{\lambda}, \overline{\mu}\right)$ be a (weakly) efficient solution of a maximum type in Mond-Weir dual problem (MWD$_E$) such that $\overline{x} \in \Omega_E$. Moreover, assume that at least one of the following hypotheses is fulfilled:*

A) *each objective function f_i, $i \in I$ is (strictly) E-B_{f_i}-invex with respect to η and b_{f_i} at $\overline{x}$ on $\Omega_E \cup Y_E$, each constraint function g_t, $t \in T$, is E-B_{g_t}-invex with respect to η and b_{g_t} at $\overline{x}$ on $\Omega_E \cup Y_E$.*

B) *$\overline{\lambda}_i f_i$, $i \in I$ is E-B_{f_i}-pseudoinvex with respect to η and b_{f_i} at $\overline{x}$ on $\Omega_E \cup Y_E$, $\overline{\mu}_t g_t$ $t \in T$ is E-B_{g_t}-quasiinvex with respect to η and b_{g_t} at $\overline{x}$ on $\Omega_E \cup Y_E$.*

Then $\overline{x}$ is a (weak) Pareto solution of the problem (VP$_E$).

Proof. Let $\left(\overline{x}, \overline{\lambda}, \overline{\mu}\right)$ be a (weakly) efficient solution of a maximum type in Mond-Weir dual problem (MWD$_E$) such that $\overline{x} \in \Omega_E$. By means of contradiction, we suppose that there exists $\widehat{x} \in \Omega_E$ such that the inequalty

$$(f \circ E)\left(\widehat{x}\right) < (f \circ E)\left(\overline{x}\right) \tag{4.11}$$

holds. By assumption $\left(\overline{x}, \overline{\lambda}, \overline{\mu}\right)$ is a (weakly) efficient solution of a maximum type in the problem (MWD$_E$), the above inequality yields

$$\sum_{i=1}^{q} \overline{\lambda}_i f_i(E(\widehat{x})) < \sum_{i=1}^{q} \overline{\lambda}_i f_i(E((\overline{x})). \tag{4.12}$$

Since functions f_i, $i \in I$ are E-B_{f_i}-invex with respect to η and b_{f_i} at $\overline{x}$ on $\Omega_E \cup Y_E$, each constraint function g_t, $t \in T$, is E-B_{g_t}-invex with respect to η and b_{g_t} at $\overline{x}$ on $\Omega_E \cup Y_E$, by Definition 2.7, the following inequalities hold

$$b_{f_i}\left(E\left(\widehat{x}\right), E\left(\overline{x}\right)\right)\left[f_i\left(E\left(\widehat{x}\right)\right) - f_i\left(E\left(\overline{x}\right)\right)\right] \geqq \nabla f_i\left(E\left(\overline{x}\right)\right)\eta\left(E\left(\widehat{x}\right), E\left(\overline{x}\right)\right) \tag{4.13}$$

for all $i \in I$, and

$$b_{g_t}\left(E\left(\widehat{x}\right), E\left(\overline{x}\right)\right)\left[g_t\left(E\left(\widehat{x}\right)\right) - g_t\left(E\left(\overline{x}\right)\right)\right] \geqq \nabla g_t\left(E\left(\overline{x}\right)\right)\eta\left(E\left(\widehat{x}\right), E\left(\overline{x}\right)\right) \tag{4.14}$$

for all $t \in T\left(E(\overline{x})\right)$, respectively. Multiplying inequalities (4.13)-(4.14) by the corresponding Lagrange multiplier, respectively, we obtain that the inequality

$$\overline{\lambda}_i b_{f_i}\left(E\left(\widehat{x}\right), E\left(\overline{x}\right)\right)\left[f_i\left(E\left(\widehat{x}\right)\right) - f_i\left(E\left(\overline{x}\right)\right)\right] \geqq \overline{\lambda}_i \nabla f_i\left(E\left(\overline{x}\right)\right)\eta\left(E\left(\widehat{x}\right), E\left(\overline{x}\right)\right) \tag{4.15}$$

for all $i \in I$, and

$$\overline{\mu}_t b_{g_t}\left(E\left(\widehat{x}\right), E\left(\overline{x}\right)\right)\left[g_t\left(E\left(\widehat{x}\right)\right) - g_t\left(E\left(\overline{x}\right)\right)\right] \geqq \overline{\mu}_t \nabla g_t\left(E\left(\overline{x}\right)\right)\eta\left(E\left(\widehat{x}\right), E\left(\overline{x}\right)\right)$$

for all $t \in T\left(E(\overline{x})\right)$. Since, $\overline{\lambda}_i > 0$, $b_{f_i}\left(E\left(\widehat{x}\right), E\left(\overline{x}\right)\right) > 0$, $i \in I$ for all $\widehat{x} \in \Omega_E$. Combining (4.12) and (4.15), we get

$$\sum_{i=1}^{q} \overline{\lambda}_i \nabla f_i\left(E(\overline{x})\right)\eta\left(E\left(\widehat{x}\right), E\left(\overline{x}\right)\right) < 0. \tag{4.16}$$

Using the E-Karush-Kuhn-Tucker necessary optimality condition (3.2) together with $\widehat{x} \in \Omega_E$, (4.6) and $\overline{x} \in \Omega_E$, $\overline{\mu}_t > 0$, $b_{g_t}(E(\widehat{x}), E(\overline{x})) > 0$, $t \in T(E(\overline{x}))$, we have

$$\overline{\mu}_t \nabla g_t(E(\overline{x}))\, \eta(E(\widehat{x}), E(\overline{x})) \leqq 0,\ t \in T(E(\overline{x})).$$

Thus,

$$\sum_{t=1}^{m} \overline{\mu}_t \nabla g_t(E(\overline{x}))\, \eta(E(\widehat{x}), E(\overline{x})) \leqq 0. \tag{4.17}$$

Combining (4.16) and (4.17), we obtain the following inequality

$$\left[\sum_{i=1}^{q} \overline{\lambda}_i \nabla f_i(E(\overline{x})) + \sum_{t=1}^{m} \overline{\mu}_t \nabla g_t(E(\overline{x}))\right] \eta(E(\widehat{x}), E(\overline{x})) < 0.$$

This contradicts with the (weakly) efficient solution of a maximum type of $\left(\overline{x}, \overline{\lambda}, \overline{\mu}\right)$ in (MWD$_E$). This means that the proof of the converse duality theorem between the E-vector optimization problems (VP$_E$) and (MWD$_E$) is completed under hypothesis A).

The proof of this theorem under hypothesis B). We proceed by contradiction. Suppose, contrary to the result, that (4.11) holds. Since $\overline{\lambda}_i f_i$, $i \in I$ is E-B_{f_i}-pseudoinvex with respect to η and b_{f_i} at $\overline{x}$ on $\Omega_E \cup Y_E$, by Definition 2.8, the inequality

$$\sum_{i=1}^{q} \overline{\lambda}_i \nabla (f_i \circ E)(\overline{x}) \eta(E(\widehat{x}), E(\overline{x})) < 0 \tag{4.18}$$

holds. Since $\overline{\mu}_t g_t$ is E-B_{g_t}-quasiinvex with respect to η and b_{g_t} at $\overline{x}$ on $\Omega_E \cup Y_E$, Definition 2.11 implies that the inequality

$$\sum_{t=1}^{m} \overline{\mu}_t \nabla (g_t \circ E)(\overline{x}) b_{g_t}(E(\widehat{x}), E(\overline{x}))\, \eta(E(\widehat{x}), E(\overline{x})) \leqq 0$$

holds. Since $b_{g_t}(E(\widehat{x}), E(\overline{x})) > 0$, we obtain

$$\sum_{t=1}^{m} \overline{\mu}_t \nabla (g_t \circ E)(\overline{x}) \eta(E(\widehat{x}), E(\overline{x})) \leqq 0. \tag{4.19}$$

Combining (4.18) and (4.19), it follows that the inequality

$$\left[\sum_{i=1}^{q} \overline{\lambda}_i \nabla f_i(E(\overline{x})) + \sum_{t=1}^{m} \overline{\mu}_t \nabla g_t(E(\overline{x}))\right] \eta(E(\widehat{x}), E(\overline{x})) < 0$$

holds. This contradicts with the first constraint of the vector Mond-Weir E-dual problem (MWD$_E$). This means that the proof of the converse duality theorem between the E-vector optimization problems (VP$_E$) and (MWD$_E$) is completed under hypothesis B). □

Theorem 4.7. *(Mond-Weir converse E-duality between (VP) and (MWD$_E$)). Let $\left(\overline{x}, \overline{\lambda}, \overline{\mu}\right)$ be a (weakly) efficient solution of a maximum type in Mond-Weir dual problem (MWD$_E$). Further, assume that all hypotheses of Theorem 4.6 are fulfilled. Then $E(\overline{x}) \in \Omega$ is a (weak) E-Pareto solution of the problem (VP).*

Proof. The proof of this theorem follows directly from Lemma 3.4 and Theorem 4.6. □

5. Conclusion

In this paper, new classes of generalized E-B-invex functions called E-B-pseudoinvexity and E-B-quasiinvexity with respect to η and b are introduced. These classes of functions are defined by relaxing the definitions of the classes of E-pseudo-invex and E-quasi-invex functions (with respect to η) defined by Abdulaleem [1], and the classes of B-pseudo-invex and B-quasi-invex functions (with respect to η and b) defined by Bector et al. [11]. Several sufficient optimality conditions have been derived for E-differentiable vector optimization problems under (generalized) E-B-invexity hypotheses. Further, the so-called vector Mond-Weir E-duality problem has been investigated for the considered E-differentiable vector optimization problem. Various duality theorems between the E-differentiable vector optimization problem and its vector Mond-Weir E-dual problem have been proved under (generalized) E-B-invexity hypotheses.

However, some interesting topics for further research remain. It would be of interest to investigate whether it is possible to prove similar results for other classes of E-differentiable vector optimization problems. We shall investigate these questions in subsequent papers.

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