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Research Article

Ricci Solitons on Pseudosymmetric (κ,μ)-Paracontact Metric Manifolds

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Abstract
The object of the present paper is to study some types of Ricci pseudosymmetric (κ,μ)-paracontact metric manifolds whose metric admits Ricci soliton. We researched the conditions when Ricci soliton on Ricci pseudosymmetric, concircular Ricci pseudosymmetric, W_3-Ricci pseudosymmetric, Weyly projective Ricci pseudosymmetric and conharmonic Ricci pseudosmettric conditions on a (κ,μ)-paracontact metric manifold. According to these conditions, we have evaluated the manifold to be shrinking, steady and expanding. Finally, we have also constructed a non-trivial example of (κ,μ)-paracontact metric manifolds whose metric admits Ricci soliton and found the functions for the Ricci pseudosymmetric conditions.

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1. Introduction

The study of paracontact geometry was initiated by Kaneyuki and Williams [1]. Then, Zamkovey started working on paracontact metric manifolds and their subclasses [2]. Since many geometers interested in paracontact metric manifolds have investigated various important properties of these manifolds, they have obtained some interesting results.

The geometry of paracontact metric manifolds can be related to the theory of Legendre foliations. One of the class of paracontact manifolds for which the characteristic vector field ξ-belongs to the (κ,μ)-nullity condition for some real constants κ and μ. Such manifolds are known as (κ,μ)-paracontact metric manifolds [3].

The notion of Ricci soliton wa appeared after Hamilton introduced the Ricci flow in 1982 and the self similar solutions of such a flow is Ricci soliton [4]. According to the definition of Hamilton, a Riemannian manifold (M,g) is called a Ricci soliton if it admits a smooth vector field V on M such that

$$\ell_V g+2Ric+2\lambda g=0, \tag{1}$$

where ℓ_V is a Lie-derivative of the vector field V, Ric denotes the Ricci tensor of (M,g) and λ is also an arbitrary constant. Hence the Ricci soliton is denoted by (M,g,V,λ).

A smooth vector field V is called the potential field of the Ricci soliton. A Ricci soliton (M,g,V,λ) is said to be shrinking, steady or expanding according to $\lambda<0$, $\lambda=0$ or $\lambda>0$, respectively.

During the last two decades, the geometry of Ricci solitons has been the focus of attention of many mathematicians. In particular, it has become more important after Perelman applied Ricci solitons to solve the long standing Poincare conjecture posed. Sharma studied the Ricci solitons in contact geometry in [5]. After then, Ricci solitons in contact metric manifolds have been studied by various many mathematicians [6–17].

Motivated by the above studies, this paper presents Ricci pseudosymmetric paracontact (κ,μ)-manifolds whose metric is a Ricci soliton.

2. Preliminaries

A $(2n+1)$-dimensional smooth manifold $\widetilde{M}$ is said to be a paracontact metric manifold if it admits a $(1,1)$-type tensor field ϕ, a unit spacelike vector field ξ, 1-form η and a semi-Riemannian metric tensor g which satisfy

$$\phi^2X = X-\eta(X)\xi, \;\; \eta(X)=g(X,\xi), \tag{2}$$

$$g(\phi X,\phi Y) = -g(X,Y)+\eta(X)\eta(Y), \;\; \eta\circ\phi = 0, \tag{3}$$

and

$$d\eta(X,Y)=g(X,\phi Y), \tag{4}$$

for all $X,Y\in\Gamma(T\widetilde{M})$, where $\Gamma(T\widetilde{M})$ denote the set of the differentiable vector fields on $\widetilde{M}$.

In a paracontact metric manifold $(\widetilde{M},\phi,\eta,\xi,g)$, we define a $(1,1)$-type tensor field by $h=\frac{1}{2}\ell_\xi\phi$, where ℓ denotes the Lie-derivative. One can easily see that h is symmetric and satisfies

$$h\xi = 0, \;\; h\phi=-\phi h \;\text{ and }\; Trh=0, \tag{5}$$

and

$$2hX = (\ell_\xi\phi)X = \ell_\xi\phi X - \phi\ell_\xi X = [\xi,\phi X]-\phi[\xi,X]. \tag{6}$$

By $\widetilde{\nabla}$, we denote the Levi-Civita connection of g, then we have

$$\widetilde{\nabla}_X\xi = -\phi X+\phi hX, \tag{7}$$

for all $X\in\Gamma(TM)$.

A paracontact metric manifold $\widetilde{M}$ is said to be a (κ,μ)-space form if its the Riemannian curvature tensor $\widetilde{R}$ satisfies

$$R(X,Y)\xi = \kappa\{\eta(Y)X-\eta(X)Y\}+\mu\{\eta(Y)hX-\eta(X)hY\}, \tag{8}$$

for all $X,Y\in\Gamma(TM)$, where κ,μ are real constant. In a (κ,μ)-paracontact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$, we have

$$(\nabla_X\phi)Y = -g(X-hX,Y)\xi+\eta(Y)(X-hX), \tag{9}$$

$$S(X,Y) = [2(1-n)+n\mu]g(X,Y)+[2(n-1)+\mu]g(hX,Y)+[2(n-1)+n(2\kappa-\mu)]\eta(X)\eta(Y), \tag{10}$$

$$h^2 = (1+\kappa)\phi^2, \tag{11}$$

and

$$QX = [2(1-n)+n\mu]X+[2(1-n)+\mu]hX+[2(n-1)+n(2\kappa-\mu)]\eta(X)\xi, \tag{12}$$

where S and Q denote the Ricci tensor and Ricci operatory defined $S(X,Y)=g(QX,Y)$.

On a semi-Riemannian manifold (M,g), for a $(0,k)$-type tensor field T and $(0,2)$-type tensor field A, $(0,k+2)$-type Tachibana tensor field $Q(A,T)$ is defined as

$$Q(A,T)(X_1,X_2,...,X_k;X,Y) = -T((X\wedge_A Y)X_1,X_2,...,X_k)-...-T(X_1,X_2,...,(X\wedge_A Y)X_k), \tag{13}$$

for all $X_1, X_2, ..., X_k, X, Y \in \Gamma(TM)$, where

$$(X \wedge_A Y)X_1 = A(Y, X_1)X - A(X, X_1)Y. \tag{14}$$

For a Riemannian manifold (M^n, g), the invariant of a concircular transformation is the concircular curvature tensor C, the W_3-curvature, the conharmonic curvature, the conharmonic curvature tensor H, and the Weyly projective curvature tensors are, respectively, given by

$$C(X,Y)Z = R(X,Y)Z - \frac{\tau}{n(n-1)}\{g(Y,Z)X - g(X,Z)Y\}, \tag{15}$$

$$W_3(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}\{g(Y,Z)QX - S(X,Z)Y\}, \tag{16}$$

$$H(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}\{g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y\}, \tag{17}$$

and

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}\{S(Y,Z)X - S(X,Z)Y\}, \tag{18}$$

for all $X, Y, Z \in \Gamma(TM)$, where R, Q and τ denote the Riemannian curvature tensor, Ricci operatory, and scalar curvature of M^n, respectively, We note that a Riemannian manifold with vanishing concircular curvature tensor is of constant curvature, that is, the concircular curvature tensor is a measure of the failure of a Riemannian manifold to be a constant curvature. Likewise, other tensors measure different properties of manifolds.

3. Ricci solitons on Ricci-pseudosymmetric (κ, μ)-paracontact metric manifolds

Let (g, ξ, λ) be a Ricci soliton on (κ, μ)-paracontact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$. Then, we have

$$(\ell_\xi g)(X,Y) = g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) = g(-\phi X + \phi hX, Y) + g(-\phi Y + \phi hY, X),$$

and so

$$(\ell_\xi g)(X,Y) = 2g(\phi hX, Y). \tag{19}$$

On the other hand, from equation (1), we reach at

$$(\ell_\xi g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0,$$

that is,

$$g(\phi hX, Y) + S(X,Y) + \lambda g(X,Y) = 0. \tag{20}$$

This yields to $Y = \xi$,

$$S(X,\xi) + \lambda \eta(X) = 0 \;\text{ or }\; \lambda = -2n\kappa. \tag{21}$$

The notion of Ricci pseudosymmetric manifold was introduced by Deszcz in a Riemannian manifold [18]. Then, a geometrical interpretation of Ricci pseudosymmetric manifolds in the Riemannian case is given in [19].

In the same way, a (κ, μ)-paracontact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is called Ricci pseudosymmetric if the tensor $R \cdot S$ and the Tachibana tensor $Q(g, S)$ are linearly dependent, where

$$(R(X,Y) \cdot S)(U,V) = -S(R(X,Y)U, V) - S(U, R(X,Y)V), \tag{22}$$

for all $X, Y, U, V \in \Gamma(TM)$.

Now, we suppose that (M,g,ξ,λ) is a Ricci soliton on Ricci pseudosymmetric (κ,μ)-paracontact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$. Then, there is a function L_1 on M such that

$$(R(X,Y)\cdot S)(Z,U)=L_1Q(g,S)(Z,U;X,Y),$$

which implies that

$$S(R(X,Y)Z,U)+S(Z,R(X,Y)U)=L_1\{S((X\wedge_g Y)Z,U)+S(Z,(X\wedge_g Y)U)\}. \tag{23}$$

Setting $Z=\xi$ in equation (23), we have

$$S(R(X,Y)\xi,U)+S(R(X,Y)U,\xi)=L_1\{S(\eta(Y)X-\eta(X)Y,U)+S(\xi,X)g(Y,U)-g(X,U)S(Y,\xi)\}.$$

By view of equation (8) and equation (21), we have

$$\begin{aligned} &S(\kappa[\eta(Y)X-\eta(X)Y],U)+S(\mu[\eta(Y)hX-\eta(X)hY],U)\\ &-2n\kappa\{\kappa[\eta(Y)g(X,U)-\eta(X)g(Y,U)]+\mu[\eta(Y)g(hX,U)-\eta(X)g(hY,U)]\} \quad =L_1\{S(\eta(Y)X-\eta(X)Y,U)\\ &\qquad +2n\kappa[\eta(X)g(Y,U)-\eta(Y)g(X,U)]\},\end{aligned}$$

which from,

$$\begin{aligned} &\kappa S(\eta(Y)X-\eta(X)Y,U)+\mu S(\eta(Y)hX-\eta(X)hY,U)\\ &-2n\kappa^2[\eta(Y)g(X,U)-\eta(X)g(Y,U)]-2n\kappa\mu[\eta(Y)g(hX,U)-\eta(X)g(hY,U)] \quad =L_1\{S(\eta(Y)X-\eta(X)Y,U)\\ &\qquad +2n\kappa[\eta(X)g(Y,U)-\eta(Y)g(X,U)]\},\end{aligned}$$

which implies for $Y=\xi$

$$L_1[S(X,U)-2n\kappa g(X,U)]=\kappa S(X,U)-2n\kappa^2 g(X,U)+\mu S(hX,U)-2n\kappa\mu g(hX,U).$$

By virtue of equation (20), we have

$$L_1 g(\phi hX,U)=\kappa g(\phi hX,U)+\mu(\kappa+1)g(\phi X,U),$$

which is equivalent to

$$(L_1-\kappa)\phi hX=\mu(\kappa+1)\phi X. \tag{24}$$

Substituting hX for X in equation (24) and taking into account equation (11), we have

$$(L_1-\kappa)\phi h^2X=\mu(\kappa+1)\phi hX,$$

Provided $\kappa+1\neq 0$, we obtain

$$(L_1-\kappa)\phi X=\mu\phi hX. \tag{25}$$

From equation (24) and equation (25), we conclude that

$$[L_1-\kappa]^2-\mu^2(\kappa+1)=0. \tag{26}$$

Thus, we have the following theorem.

Theorem 1. *If (g,ξ,λ) is a Ricci soliton on a Ricci pseudosymmetric (κ,μ)-paracontact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$, then the function L_1 satisfies*

$$L_1=\kappa\mp\mu\sqrt{\kappa+1},\ \ \lambda=-2n\kappa.$$

Corollary 2. *If (g,ξ,λ) is a Ricci soliton on a Ricci semi-symmetric (κ,μ)-paracontact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$, then we have*

$$\kappa=\mp\mu\sqrt{\kappa+1},\ \ \lambda=-2n\kappa.$$

Now, from equations (8) and (15) we have for the concircular curvature tensor C

$$C(X,Y)\xi = \left(\kappa - \frac{\tau}{2n(2n+1)}\right)(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \tag{27}$$

and

$$\eta(C(X,Y)Z) = \left(\kappa - \frac{\tau}{2n(2n+1)}\right)(\eta(X)g(Y,Z) - \eta(Y)g(X,Z)) - \mu(\eta(Y)g(hX,Z) - \eta(X)g(Y,Z)). \tag{28}$$

Now, we suppose that (M,g,ξ,λ) is a Ricci soliton on concircular Ricci pseudosymmetric (κ,μ)-paracontact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$. Then, there exists a function L_2 on M such that

$$(C(X,Y)\cdot S)(U,V) = L_2 Q(g,S)(U,V;X,Y),$$

for all $X,Y,U,V \in \Gamma(TM)$. That means

$$S(C(X,Y)U,V) + S(U,C(X,Y)V) = L_2\{S((X\wedge_g X)U,V) + S(U,(X\wedge_g Y)V)\}.$$

Putting $V=\xi$, by using equation (21), equation (27) and equation (28), we have

$$\begin{aligned} 2n\kappa\eta(C(X,Y)U) &= S(U,(\kappa - \frac{\tau}{2n(2n+1)})(\eta(Y)X - \eta(Y)X) + \mu(\eta(Y)hX - \eta(Y)hY)) \\ &= L_2\{g(Y,U)S(X,\xi) - g(X,U)S(Y,\xi)S(U,\eta(X)Y - \eta(X)Y)\}, \end{aligned}$$

or

$$\left(L_2 - (\kappa - \frac{\tau}{2n(2n+1)})\right)g(\phi U,\eta(X)hY - \eta(Y)h) + 2n\kappa\mu g(U,\eta(Y)hX - \eta(X)hY) = 0,$$

which is equivalent to

$$\left(L_2 - (\kappa - \frac{\tau}{2n(2n+1)})\right)\phi U - 2n\kappa\mu U = 0.$$

This implies that

$$L_2 = \kappa - \frac{\tau}{2n(2n+1)} \text{ and } \mu = 0. \tag{29}$$

Thus, we have the following theorem.

Theorem 3. *If (g,ξ,λ) is a Ricci soliton on concircular Ricci pseudosymmetric (κ,μ)-paracontact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$, then the function L_2 satisfies*

$$L_2 = \kappa - \frac{\tau}{2n(2n+1)} \text{ and } \mu = 0.$$

From Theorem 3, we have the following corollary.

Corollary 4. *If (g,ξ,λ) is a Ricci soliton on concircular Ricci semi-symmetric (κ,μ)-paracontact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$, then we have*

$$\tau = 2n(2n+1)\kappa \text{ and } \mu = 0.$$

Now we study Ricci solitons on W_3-Ricci pseudosymmetric paracontact metric (κ,μ)-spaces. In [20], Pokhariyal introduced the notion of a new curvature tensor, denoted by W_3 and studied its relativistic significance.

Now, making use of equations (8) and (16), for the Weyly projective curvature tensor W_3, we have

$$W_3(X,Y)\xi = \kappa\{\eta(Y)X - 2\eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\} + \frac{1}{2n}\eta(Y)QX, \tag{30}$$

and

$$\eta(W_3(X,Y)Z) = \kappa\{2\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\} - \mu\{\eta(Y)g(hX,Z) - \eta(X)g(hY,Z)\} - \frac{1}{2n}\eta(Y)S(X,Z). \tag{31}$$

Next, we are taking into account that the Weyly Ricci pseudosymmetric (κ,μ) paracontact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$ whose metric admits Ricci soliton (g,ξ,λ). For this, we have

$$(W_3(X,Y)\cdot S)(U,Z) = L_3 Q(g,S)(U,Z;X,Y), \tag{32}$$

for all $X,Y,U,Z \in \Gamma(TM)$, where L_3 is a function on M. That's means

$$L_3\{S(g(Y,U)X - g(X,U)Y,Z) + S(U,g(Y,Z)X - g(X,Z)Y\} = S(W_3(X,Y)U,Z) + S(U,W_3(X,Y)Z).$$

This implies for $Z=\xi$

$$\begin{aligned} L_3\{g(Y,U)S(X,\xi) - g(X,U)S(Y,\xi) + \eta(Y)S(X,U) - \eta(X)S(Y,U)\} = \; & 2n\kappa\eta(W_3(X,Y)U) \\ & + S(U,\kappa[\eta(Y)X - 2\eta(X)Y] \\ & + \mu[\eta(Y)hX - \eta(X)hY] + \frac{1}{2n}\eta(Y)QX), \end{aligned}$$

or

$$\begin{aligned} L_3\{2n\kappa\eta(X)g(Y,U) - 2n\kappa\eta(Y)g(X,U) + \eta(Y)S(X,U) - \eta(X)S(U,Y)\} = \; & 2n\kappa\{-\kappa[\eta(Y)g(X,U) - 2\eta(X)g(Y,U)] \\ & - \mu[\eta(Y)g(hX,U) - \eta(X)g(hY,U)] \\ & - \frac{1}{2n}\eta(Y)S(X,U)\} \\ & + S(U,\kappa[\eta(Y)X - 2\eta(X)Y]) \\ & + \mu S(U,\eta(Y)hX - \eta(X)hY) \\ & + \frac{1}{2n}\eta(Y)S(U,QX), \end{aligned}$$

which implies that for $X=\xi$

$$L_3\{2n\kappa g(Y,U) - S(U,Y)\} = 2n\kappa\mu g(hY,U) + \kappa\eta(Y)S(U,\xi) - 2\kappa S(U,Y) - \mu S(U,hY) + \kappa\eta(Y)S(U,\xi).$$

By using of equation (20), we have

$$L_3 g(\phi hU,Y) = 2n\kappa\mu g(hY,U) - 4n\kappa^2[g(Y,U) - \eta(U)\eta(Y)] + 2\kappa g(\phi hU,Y) - 2n\kappa\mu g(U,hY) + \mu g(\phi hU,hY).$$

Also taking into account equation (3), we obtain

$$(L_3 - 2\kappa)g(\phi hU,Y) = n\kappa\mu g(hY,U) + 4n\kappa^2 g(\phi U,\phi Y) - \mu g(\phi h^2 U,Y),$$

$$(L_3 - 2\kappa)g(\phi hU,Y) = n\kappa\mu g(hY,U) + 4n\kappa^2 g(\phi U,\phi Y) - \mu(\kappa+1)g(\phi^3 U,Y),$$

$$(L_3 - 2\kappa)g(\phi U,hY) = n\kappa\mu g(hY,U) + 4n\kappa^2 g(\phi U,\phi Y) - \mu(\kappa+1)g(\phi U,Y).$$

Consequently, we have

$$(L_3 - 2\kappa)hY - n\kappa\mu\phi hY = 4n\kappa^2\phi Y - \mu(\kappa+1)Y = 0,$$

which implies that

$$\mu(\kappa+1) = 0. \tag{33}$$

Hence, we have

$$(2\kappa - L_3)hY + n\kappa\mu\phi hY = 4n\kappa^2\phi Y. \tag{34}$$

Substituting hY for Y in equation (34) and by using equation (11), we have

$$(2\kappa - L_3)h^2 Y + n\kappa\mu\phi h^2 Y = 4n\kappa^2\phi hY,$$

or

$$(2\kappa - L_3)(\kappa+1)\phi^2 Y + n\kappa\mu(\kappa+1)\phi Y = 4n\kappa^2\phi hY. \tag{35}$$

From equation (34) and equation (35), we observe

$$[(L_3 - 2\kappa)^2(\kappa+1) - 16n^2\kappa^4]\phi Y = 0. \tag{36}$$

This leads to the following.

Theorem 5. *If (g,ξ,λ) is a Ricci soliton on W_3-Ricci pseudosymmetric (κ,μ)-paracontact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$, then the function L_3 satisfies*

$$L_3 = \kappa\left(2 \pm \frac{4n\kappa}{\sqrt{\kappa+1}}\right).$$

From Theorem 5, we have the following corollary.

Corollary 6. *If (g,ξ,λ) is a Ricci soliton on W_3-Ricci semi-symmetric (κ,μ)-paracontact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$, then we have*

$$\kappa(2n \pm \sqrt{\kappa+1}) = 0.$$

Now, we suppose that the Weyly projective Ricci pseudosymmetric (κ,μ)-paracontact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$ admits Ricci-soliton (g,ξ,λ), then there is a function on M such that

$$S(P(X,Y)U,V) + S(U,P(X,Y)V) = L_4\{S((X\wedge_g Y)U,V) + S(U,(X\wedge_g Y)V)\},$$

for all $X,Y,U,V \in \Gamma(TM)$. For $V=\xi$, this means

$$S(P(X,Y)U,\xi) + S(U,P(X,Y)\xi) = L_4\{S(g(Y,U)X - g(X,U)Y,\xi) + S(U,\eta(Y)X - \eta(X)Y)\},$$

that is,

$$-\lambda\eta(P(X,Y)U) + S(U,P(X,Y)\xi) = L_4\{g(Y,U)S(X,\xi) - g(X,U)S(Y,\xi) + \eta(Y)S(X,U) - \eta(X)S(U,Y)\}. \tag{37}$$

On the other hand, by direct calculations, using equation (8) and equation (18), we obtain

$$P(X,Y)\xi = \mu\{\eta(Y)hX - \eta(X)hY\}, \tag{38}$$

and

$$\eta(P(X,Y)U) = \mu\{\eta(X)g(hY,U) - \eta(Y)g(hX,U)\}. \tag{39}$$

By virtue of equation (37), equation (38) and equation (39), we have

$$\begin{aligned} \mu\{\eta(Y)[S(hX,U)+\lambda g(hX,U)] - \eta(X)[S(hY,U)+\lambda g(hY,U)]\} &= L_4\{\eta(Y)[S(X,U)+\lambda g(X,U)] \\ &\quad -\eta(X)[S(U,Y)+\lambda g(Y,U)]\}. \end{aligned}$$

From equation (11) and equation (20), we obtain

$$L_4\{\eta(X)g(\phi hY,U) - \eta(Y)g(\phi hX,U)\} = \mu\{\eta(X)g(\phi h^2 Y,U) - \eta(Y)g(\phi h^2 X,U)\},$$

or

$$L_4 g(\eta(X)\phi hY - \eta(Y)\phi hX,U) = \mu(\kappa+1)g(\eta(X)\phi hY - \eta(Y)\phi hX,U)$$

which implies that

$$L_4 = \mu(\kappa+1). \tag{40}$$

Thus, we have the following theorem.

Theorem 7. *If (g,ξ,λ) is a Ricci soliton on the Weyly projective Ricci pseudosymmetric (κ,μ)-paracontact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$, then the critical value for L_4 is $\mu(\kappa+1)$.*

Corollary 8. *If (g,ξ,λ) is a Ricci soliton on the Weyly projective Ricci semi-symmetric (κ,μ)-paracontact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$, then we have*

$$\mu=0 \ \textit{provided} \ \kappa+1\neq 0.$$

Now, we will calculate the conharmonic curvature tensor H for

$$H(X,Y)\xi=-\frac{\kappa}{2n-1}[\eta(Y)X-\eta(X)Y]-\frac{1}{2n-1}[\eta(Y)QX-\eta(X)QY]+\mu[\eta(Y)hX-\eta(X)hY], \tag{41}$$

and

$$\begin{aligned}\eta(H(X,Y)Z) &= \frac{\kappa}{2n-1}[\eta(Y)g(X,Z)-\eta(X)g(Y,Z)]+\frac{1}{2n-1}[\eta(Y)S(X,Z)-\eta(X)S(Y,Z)]\\ &\quad+\mu[\eta(X)g(hY,Z)-\eta(Y)g(hX,Z)].\end{aligned} \tag{42}$$

Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a conharmonic Ricci pseudosymmetric manifold whose metric tensor admits Ricci soliton (g,ξ,λ). Then, there is a function L_5 on M such that

$$(H(X,Y)\cdot S)(U,V)=L_5Q(g,S)(U,V;X,Y), \tag{43}$$

for all $X,Y,U,V\in\Gamma(TM)$. That means

$$S(H(X,Y)U,V)+S(U,H(X,Y)V)=L_5\{S((X\wedge_g Y)U,V)+S(U,(X\wedge_g Y)V)\}.$$

Here, taking $V=\xi$ and by means of equation (13) and equation (21), we have

$$2n\kappa\eta(H(X,Y)U)+S(H(X,Y)\xi,U)=L_5\{g(Y,U)S(X,\xi)-g(X,U)S(Y,\xi)+\eta(Y)S(X,U)-\eta(X)S(U,Y)\}.$$

By virtue of equation (41) and equation (42), we have

$$\begin{aligned}&2n\kappa\eta(H(X,Y)U)-\tfrac{\kappa}{2n-1}S(\eta(Y)X-\eta(X)Y,U)+\mu S(\eta(Y)hX-\eta(X)hY,U)\\ &-\frac{1}{2n-1}S(\eta(Y)QX-\eta(X)QY,U)=L_5\{g(Y,U)S(X,\xi)-g(X,U)S(Y,\xi)+\eta(Y)S(X,U)-\eta(X)S(U,Y)\},\end{aligned}$$

or

$$\begin{aligned}L_5\{S(\eta(Y)X-\eta(X)Y,U)+\lambda g(\eta(Y)X-\eta(X)Y,U) &= -\lambda(\frac{\kappa}{2n-1}[g(\eta(Y)X-\eta(X)Y,U)]\\ &\quad+\frac{1}{2n-1}[S(\eta(Y)X-\eta(X)Y,U]\\ &\quad+\mu g(\eta(X)hY-\eta(Y)hX,U))\\ &\quad+S(-\tfrac{\kappa}{2n-1}(\eta(X)Y-\eta(Y)X)\\ &\quad+\mu[\eta(Y)hX-\eta(X)hY]\\ &\quad-\frac{1}{2n-1}[\eta(Y)QX-\eta(X)QY],U),\end{aligned}$$

that is,

$$\begin{aligned}L_5\{S(\eta(Y)X-\eta(X)Y,U)+\lambda g(\eta(Y)X-\eta(X)Y,U)\} &= -\frac{\kappa}{2n-1}\{S(\eta(Y)X-\eta(X)Y,U)\\ &\quad+\lambda g(\eta(Y)X-\eta(X)Y,U)\}+\mu\{S(\eta(Y)hX-\eta(X)hY,U)\\ &\quad+\lambda g(\eta(Y)hX-\eta(X)hY,U)\}-\frac{1}{2n-1}\{S(\eta(Y)QX-\eta(X)QY,U)\\ &\quad+\lambda S(\eta(Y)X-\eta(X)Y,U)\}.\end{aligned}$$

Making use of equation (20), we reach at

$$L_5 g(\eta(X)Y-\eta(Y)X,\phi hU) = \frac{\kappa}{2n-1}g(\eta(Y)X-\eta(X)Y,\phi hU)+\mu g(\eta(Y)X-\eta(X)Y,\phi h^2U) + \tfrac{1}{2n-1}g(\eta(Y)X-\eta(X)Y,\phi hQU)$$

Again, by using equation (11) and equation (20), we obtain

$$\left(L_5+\frac{\kappa}{2n-1}\right)g(\eta(X)Y-\eta(Y)X,\phi hU)+\mu(\kappa+1)g(\eta(X)Y-\eta(X)Y,\phi U)+\frac{1}{2n-1}g(\eta(X)Y-\eta(Y)X,\phi hQU)=0.$$

This implies that

$$\left(L_5+\frac{\kappa}{2n-1}\right)\phi hU+\mu(\kappa+1)\phi U+\frac{1}{2n-1}\phi hQU=0. \tag{44}$$

From equation (12) and equation (44) we have

$$\left(L_5+\frac{\kappa+n\mu+2(1-n)}{2n-1}\right)\phi hU+(\kappa+1)\left(\mu+\frac{2(n-1)+\mu}{2n-1}\right)\phi U=0. \tag{45}$$

Replacing hU for U in equation (45), using equation (2) and equation (11) and taking into account that $\kappa+1\neq 0$, we have

$$\left(L_5+\frac{\kappa+n\mu+2(1-n)}{2n-1}\right)\phi U+\left(\mu+\frac{2(n-1)+\mu}{2n-1}\right)\phi hU=0. \tag{46}$$

From equations (45) and (46) we conclude

$$-\left[(\kappa+1)\left(\mu+\frac{2(n-1)+\mu}{2n-1}\right)\right]^2+\left[L_5+\frac{\kappa+n\mu+2(1-n)}{2n-1}\right]^2=0. \tag{47}$$

Thus, we have the following theorem.

Theorem 9. *If $M^{2n+1}(\phi,\xi,\eta,g)$ be a conharmonic Ricci pseudosymmetric manifold whose metric tensor admits Ricci soliton (g,ξ,λ), then the function L_5 satisfies equation* (47).

Thus, we have the following corollary.

Corollary 10. *If $M^{2n+1}(\phi,\xi,\eta,g)$ be a conharmonic Ricci semi-symmetric manifold whose metric tensor admits Ricci soliton (g,ξ,λ), then we have*

$$\kappa+n\mu-2(n-1)\mp 2(\kappa+1)[n(\mu+1)-1]=0.$$

Example 11. *We consider a 3-dimensional manifold $M=\{(x,y,z)\in\mathbb{R}^3 : z>0\}$, where (x,y,z) denote the standard coordinates in $\mathbb{R}^3$. We defined the linearly independent vector fields*

$$e_1=3x^3\frac{\partial}{\partial x}+6\sqrt{z}\frac{\partial}{\partial y}+\frac{\partial}{\partial z},\quad e_2=\frac{\partial}{\partial y},\quad e_3=\frac{\partial}{\partial z}.$$

Let g be a metric tensor defined by

$$g(e_1,e_1)=g(e_2,e_2)=-g(e_3,e_3)=1,$$

and

$$g(e_i,e_j)=0,$$

for $i\neq j$. Furthermore, let η be the 1-form defined by $\eta(X)=g(X,e_2)$ for all $X\in\Gamma(TM)$. On the other hand, we define the paracontact structure ϕ by

$$\phi e_1=-e_3,\quad \phi e_3=-e_1 \ \text{and}\ \phi e_2=0.$$

Thus, we have

$$\phi^2 X = X - \eta(X), \eta(e_2) = 1,$$

and

$$g(\phi X, \phi Y) = -g(X,Y) + \eta(X)\eta(Y),$$

for all $X,Y \in \Gamma(TM)$. Let ∇ be the Levi-Civita connection for the metric tensor g and R be the curvature tensor of g. Then, by direct calculations, we have

$$[e_1,e_2] = 0, \ \ [e_1,e_3] = -\frac{3}{\sqrt{z}}e_2, \ \ [e_2,e_3] = 0.$$

By using Koszul formula for the metric tensor g, we can easily see that,

$$\begin{array}{lcllcllcl}
\nabla_{e_1}e_1 & = & 0, & \nabla_{e_1}e_2 & = & -\frac{3}{2\sqrt{z}}e_3, & \nabla_{e_1}e_3 & = & -\frac{3}{2\sqrt{z}}e_2 \\
\nabla_{e_2}e_1 & = & -\frac{3}{2\sqrt{z}}e_3, & \nabla_{e_2}e_2 & = & 0, & \nabla_{e_2}e_3 & = & -\frac{3}{2\sqrt{z}}e_1 \\
\nabla_{e_3}e_1 & = & \frac{3}{2\sqrt{z}}e_2, & \nabla_{e_3}e_2 & = & -\frac{3}{2\sqrt{z}}e_1, & \nabla_{e_3}e_3 & = & 0.
\end{array}$$

From $\nabla_X e_2 = -\phi X + \phi hX$, one can easily see that

$$he_1 = -\left(1+\frac{3}{2\sqrt{z}}\right)e_1 \ \text{ and } \ he_3 = -\left(1+\frac{3}{2\sqrt{z}}\right)e_3.$$

On the other hand, direct calculations, we observe

$$\begin{array}{rl}
R(e_1,e_2)e_2 = & \kappa\{\eta(e_2)e_1 - \eta(e_1)e_2\} + \mu\{\eta(e_2)he_1 - \eta(e_1)he_2\} \\
= & \kappa e_1 - \mu(1+\frac{3}{2\sqrt{z}})e_1 \\
= & -\frac{9}{4z}e_1
\end{array}$$

and

$$\begin{array}{rl}
R(e_2,e_3)e_2 = & \kappa\{\eta(e_3)e_2 - \eta(e_2)e_3\} + \mu\{\eta(e_3)he_2 - \eta(e_2)he_3\} \\
= & \kappa e_3 - \mu(1+\frac{3}{2\sqrt{z}})e_3 \\
= & -\frac{9}{4z}e_3.
\end{array}$$

Thus, we have

$$\kappa = \left(1+\frac{3}{2\sqrt{z}}\right)^2 - 1 \ \text{ and } \ \mu = \frac{3}{\sqrt{z}}.$$

By direct calculations, we obtain the non-zero components of the Riemannian curvature tensor such as

$$\begin{cases}
R(e_2,e_1)e_1 & = -\frac{9}{4z}e_2, \\
R(e_3,e_1)e_1 & = -\frac{9}{4z}e_3, \\
R(e_1,e_2)e_2 & = -\frac{9}{4z}e_1, \\
R(e_3,e_2)e_2 & = -\frac{9}{4z}e_3, \\
R(e_1,e_3)e_3 & = \frac{3}{4z^{\frac{3}{2}}}e_2 - \frac{27}{4z}e_1, \\
R(e_2,e_3)e_3 & = \frac{3}{4z^{\frac{3}{2}}}e_1 + \frac{9}{4z}e_2.
\end{cases}$$

Thus, we have

$$\lambda = 6\left[1 - \left(1+\frac{3}{2\sqrt{z}}\right)^2\right].$$

Thus, the manifold in the example is always shrinking. Furthermore, for Theorem 1, when it is Ricci pseudosymmetric,

$$L_1 = \left(1+\frac{3}{2\sqrt{z}}\right)^2 - 1 \pm \frac{3}{\sqrt{z}}\left(1+\frac{3}{2\sqrt{z}}\right),$$

for the W_3-Ricci pseudosymmetric,

$$L_3 = \left[\left(1+\frac{3}{2\sqrt{z}}\right)^2 - 1\right]\left[2 \pm \frac{2\sqrt{z}}{3+2\sqrt{z}}12\left(\left(1+\frac{3}{2\sqrt{z}}\right)^2 - 1\right)\right]$$

and for the Weyly projective Ricci pseudosymmetric,

$$L_4 = \frac{3}{\sqrt{z}}\left(1+\frac{3}{2\sqrt{z}}\right)^2.$$

The manifold given in this example is not concircular Ricci pseudosymmetric because $z > 0$.

4. Conclusions

The object of the present paper is to study some types of Ricci pseudosymmetric (κ,μ)-paracontact metric manifolds whose metric admits Ricci soliton. We researched the conditions when Ricci soliton on Ricci pseudosymmetric, concircular Ricci pseudosymmetric, W_3-Ricci pseudosymmetric, Weyly projective Ricci pseudosymmetric and conharmonic Ricci pseudosmettric conditions on a (κ,μ)-paracontact metric manifold. According to these conditions, we have evaluated the manifold to be shrinking, steady and expanding. Finally, we have also constructed a non-trivial example of (κ,μ)-paracontact metric manifolds whose metric admits Ricci soliton and found the functions for the Ricci pseudosymmetric conditions.

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