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Ruled Surfaces with $\{\overline{u}_1\,\overline{u}_3\}$-Smarandache Base Curve Obtained From the Successor Frame

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Abstract

In this study, ruled surfaces formed by the movement of the Frenet vectors of the Successor curve along the Smarandache curve obtained from the tangent and binormal vectors of the Successor curve of a curve are defined. Then, the Gaussian and mean curvatures of each ruled surface are calculated. It is shown that the ruled surface formed by the movement of the tangent vector of the Successor curve along the $\{\overline{u}_1\,\overline{u}_3\}$ curve is a developable minimal surface and the ruled surface formed by the movement of the binormal vector is only a developable surface. It is also stated that if the principal curve is a planar curve, the ruled surface formed by the principal normal vector of the Successor curve along the $\{\overline{u}_1\,\overline{u}_3\}$ curve is also a developable minimal surface. Conditions for other surfaces to be developable or minimal surfaces are given.

Keywords: *Ruled surfaces, Gaussian curvature, Successor curve.*
2010 Mathematics Subject Classification: *53A04, 53A05.*

1. Introduction

The image of a function with two real variables in three-dimensional space is a surface. Surfaces are used in many fields, such as architecture and engineering (see [1]). The curvature of surfaces was defined by Gauss in the 19th century, and therefore it was named Gaussian curvature (see [2]). Gaussian curvatures are related to the dimensions of the surface [3]. Since the average curvature of the surface is a ratio, it is independent of the size of the surface. Thus far, many studies [4]-[9] on the Gaussian curvatures of surfaces have been conducted. In 1795, Monge defined the striped surface as the surface formed by the movement of the line along the curve. For more details, see [10]-[15].
There are many special curves in differential geometry. One of them is the successor curve. This curve is defined as, there is a new curve, such that the tangent of one curve the principal normal of the other curve, by Menninger in 2014. Later, Masal investigated the relationships between the position vectors of this curve and defined Successor planes. You can see [16]-[21]. And other special curve is Smarandache curve. This curve were first defined in Minkowski space. Related studies with Smarandache curves are available in [22]-[25].
In this paper, we present some special ruled surface with $\{\overline{u}_1\,\overline{u}_3\}$-Smarandache base curve obtained from the successor frame. Then we examine the properties of these ruled surfaces by means of Gaussian and mean curvatures.

2. Preliminaries

In this section, we recall some basic notions of which we refer through out the paper.
Let $\alpha(s)$ be a differentiable curve in E^3. Then, its Frenet frame and curvatures are $\{u_1,u_2,u_3,k_1,k_2\}$. Here,

$$u_1=\alpha',\ u_2=\frac{\alpha''}{\|\alpha''\|},\ u_3=u_1\wedge u_2,\ k_1=\|\alpha''\|,\ k_2=\langle u_2',u_3\rangle,$$

$$u_1'=k_1u_2,\ u_2'=-k_1u_1+k_2u_3,\ u_3'=-k_2u_2.$$

The surface formed by a line moving depending on the parameter of a curve is called a ruled surface, and its parametric expression is as follows:

$$X(s,v)=\alpha(s)+vr(s).$$

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The normal vector field, the Gaussian curvatures, and the mean curvatures of $X(s,v)$ are as follows:

$$N_X = \frac{X_s \wedge X_v}{\|X_s \wedge X_v\|}, \tag{2.1}$$

$$K = \frac{eg - f^2}{EG - F^2}, \quad H = \frac{Eg - 2fF + eG}{2(EG - F^2)} \tag{2.2}$$

respectively. Here, the coefficients of the first and the second fundamental forms are defined as follows:

$$E = \langle X_s, X_s \rangle, \; F = \langle X_s, X_v \rangle, \; G = \langle X_v, X_v \rangle, \tag{2.3}$$

$$e = \langle X_{ss}, N_X \rangle, \; f = \langle X_{sv}, N_X \rangle, \; g = \langle X_{vv}, N_X \rangle. \tag{2.4}$$

Definition 2.1. *[16, 17] Let α and β be curves with unit speed in E^3. If the unit tangent vector of the α curve is the principal normal vector at the same point on the β curve, the β curve is called the Successor curve of the α curve.*

Theorem 2.2. *[16] Let the Successor curve of the β curve be α. Frenet apparatus of an $\alpha = \alpha(s)$ curve with unit speed be $\{u_1, u_2, u_3, k_1, k_2\}$ and Frenet apparatus of a $\beta = \beta(s)$ curve be $\{\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{k}_1, \bar{k}_2\}$. Frenet apparatus of β curve is as follows:*

$$\bar{u}_1 = -\cos\theta u_2 + \sin\theta u_3, \; \bar{u}_2 = u_1, \; \bar{u}_3 = \sin\theta u_2 + \cos\theta u_3,$$

$$\bar{k}_1 = k_1 \cos\theta, \; \bar{k}_2 = k_1 \sin\theta, \; \theta(s) = \theta_0 + \int k_2(s) ds.$$

Here, θ is the angle between binormal vector u_3 and binormal vector $\bar{u}_3$.

3. Ruled Surfaces with $\{\bar{u}_1 \bar{u}_3\}$-Smarandache Base Curve Obtained From the Successor Frame

In this section, firstly we define some special ruled surfaces with $\{\bar{u}_1 \bar{u}_3\}$-Smarandache base curve obtained from the successor frame. Then we calculate the properties of these ruled surfaces by means of Gaussian and mean curvatures and we examine whether these surfaces are develepoble or minimal surface. Finally, we illustrate the shapes of the ruled surfaces with four examples.

Definition 3.1. *Let the Successor curve of the β curve be α. The ruled surface formed by tangent vector $\bar{u}_1$ the vector along the $\{\bar{u}_1 \bar{u}_3\}$ Smarandache curve obtained from the $\bar{u}_1$ tangent vector and $\bar{u}_3$ binormal vector s of the β curve is as follows:*

$$\begin{aligned} \Phi(s,v) &= \frac{1}{\sqrt{2}}(\bar{u}_1 + \bar{u}_3) + v\bar{u}_1 \\ &= \frac{1}{\sqrt{2}}\big((\sin\theta - \cos\theta)u_2 + (\sin\theta + \cos\theta)u_3\big) + v(-\cos\theta u_2 + \sin\theta u_3). \end{aligned} \tag{3.1}$$

Theorem 3.2. *Let the Successor curve of the β curve be α. Then, the Gaussian and mean curvature of the $\Phi(s,v)$ ruled surface are as follows:*

$$K_\Phi = H_\Phi = 0.$$

Proof. Partial derivatives of Equality 3.1 are,

$$\Phi_s = \frac{k_1\big((1+\sqrt{2})\cos\theta \sin\theta\big)u_1}{\sqrt{2}}, \; \Phi_v = -\cos\theta u_2 + \sin\theta u_3, \; \Phi_{sv} = k_1 \cos\theta u_1,$$

$$\Phi_{ss} = -\frac{\big(k_1'\big(\sin\theta + (v\sqrt{2}-1)\cos\theta\big) + k_1 k_2(\cos\theta + (1 - v\sqrt{2})\sin\theta)\big)u_1 + \big({k_1}^2\big(\sin\theta + (v\sqrt{2}-1)\cos\theta\big)\big)u_2}{\sqrt{2}}, \; \Phi_{vv} = 0,$$

Thus, from Equality 2.1 the normal of the surface N_Φ is given as

$$N_\Phi = -\sin\theta u_2 - \cos\theta u_3.$$

Moreover, in Equalities 2.3 and 2.4 the coefficients of fundamental forms are

$$E_\Phi = \frac{k_1\big((1+v\sqrt{2})\cos\theta - \sin\theta\big)}{2}, \quad F_\Phi = G_\Phi = 0,$$
$$e_\Phi = \frac{{k_1}^2 \sin\theta\big(\sin\theta + (v\sqrt{2}-1)\cos\theta\big)}{\sqrt{2}}, \quad f_\Phi = g_\Phi = 0,$$

respectively. Thus, by using Equality 2.2, the Gaussian and mean curvatures are found. □

Corollary 3.3. *The ruled surface $\Phi(s,v)$ is a developable minimal surface.*

Definition 3.4. *Let the Successor curve of the β curve be α. The ruled surface formed by principal normal vector $\overline{u}_2$ along the $\{\overline{u}_1\,\overline{u}_3\}$ Smarandache curve obtained from the $\overline{u}_1$ tangent vector and $\overline{u}_3$ binormal vectors of the β curve is as follows:*

$$\begin{aligned} Q(s,v) &= \tfrac{1}{\sqrt{2}}(\overline{u}_1+\overline{u}_3)+v\overline{u}_2 \\ &= \tfrac{1}{\sqrt{2}}\big((\sin\theta-\cos\theta)u_2+(\sin\theta-\cos\theta)u_3\big)+vu_1. \end{aligned} \tag{3.2}$$

Theorem 3.5. *Let the Successor curve of the β curve be α. Then, the Gaussian and mean curvature of the $Q(s,v)$ ruled surface are as follows:*

$$K_Q=0,\ H_Q=-\frac{vk_2}{2k_1}.$$

Proof. Partial derivatives of Equality 3.2 are,

$$Q_s=\frac{k_1\big((\cos\theta-\sin\theta)u_1+v\sqrt{2}\big)}{\sqrt{2}},\quad Q_v=u_1,\ Q_{sv}=k_1u_2,\ Q_{vv}=0,$$

$$Q_{ss}=-\frac{\big(k_1{}'(\sin\theta-\cos\theta)-k_1k_2(\sin\theta+\cos\theta)-v\sqrt{2}k_1{}^2\big)u_1-v\sqrt{2}k_1{}'u_2-v\sqrt{2}k_1k_2u_3}{\sqrt{2}}.$$

Thus, from Equality 2.1 the normal of the surface N_Q is given as $N_Q=-u_3$. Moreover, in Equalities 2.3 and 2.4 the coefficients of fundamental forms are

$$E_Q=\frac{k_1{}^2(3-\sin 2\theta)}{2},\ F_Q=\frac{k_1(\cos\theta-\sin\theta)}{\sqrt{2}},\ G_Q=1,$$

$$e_Q=-vk_1k_2,\ f_Q=g_Q=0$$

respectively. Thus, by using Equality 2.2, the Gaussian and mean curvatures are found. □

Corollary 3.6. *Let the Successor curve of the β curve be α. If α curve is planar, the ruled surface $Q(s,v)$ is the minimal developable surface.*

Definition 3.7. *Let the Successor curve of the β curve be α. The ruled surface formed by binormal vector $\overline{u}_3$ the vector along the $\{\overline{u}_1\,\overline{u}_3\}$ Smarandache curve obtained from the $\overline{u}_1$ tangent vector and $\overline{u}_3$ binormal vectors of the β curve is as follows:*

$$\begin{aligned} M(s,v) &= \frac{1}{\sqrt{2}}(\overline{u}_1+\overline{u}_3)+v\overline{u}_3 \\ &= \frac{1}{\sqrt{2}}\big((\sin\theta-\cos\theta)u_2+(\sin\theta-\cos\theta)u_3\big)+v(\sin\theta u_2+\cos\theta u_3). \end{aligned} \tag{3.3}$$

Theorem 3.8. *Let the Successor curve of the β curve be α. The Gaussian and mean curvature of the $M(s,v)$ ruled surface are as follows:*

$$K_M=0,\ H_M=\cos\theta-(1-v\sqrt{2})\sin\theta.$$

Proof. Partial derivatives of Equality 3.3 are,

$$M_s=\frac{k_1\big(\cos\theta-(1-v\sqrt{2})\sin\theta\big)}{\sqrt{2}},\quad M_v=\sin\theta u_2+\cos\theta u_3,\ M_{sv}=-k_1\sin\theta u_1,$$

$$M_{ss}=\frac{\big(k_1{}'(\cos\theta-(1-v\sqrt{2})\sin\theta)-k_1k_2(\sin\theta+(1+v\sqrt{2})\cos\theta)\big)u_1+\big(k_1{}^2(\cos\theta-(1-v\sqrt{2})\sin\theta)\big)u_2}{\sqrt{2}},\ M_{vv}=0.$$

Thus, from Equality 2.1 the normal of the surface N_M is given as

$$N_M=-\cos\theta u_2+\sin\theta u_3.$$

Moreover, in Equalities 2.3 and 2.4 the coefficients of fundamental forms are

$$E_M=\frac{k_1{}^2\big(\cos\theta-(1-v\sqrt{2})\sin\theta\big)^2}{2},\ F_M=0,\ G_M=1,$$

$$e_M=k_1{}^2\cos\theta\big((1-v\sqrt{2})\sin\theta-\cos\theta\big),\ f_M=g_M=0$$

respectively. Thus, by using Equality 2.2, the Gaussian and mean curvatures are found. □

Corollary 3.9. *The ruled surface $M(s,v)$ is a developable surface.*

Definition 3.10. *Let the Successor curve of the β curve be α. The ruled surface formed by $\{\bar{u}_1\ \bar{u}_2\}$ the vector along the $\{\bar{u}_1\,\bar{u}_3\}$ Smarandache curve obtained from the $\bar{u}_1$ tangent vector and and $\bar{u}_3$ binormal vectors of the β curve is as follows:*

$$
\begin{aligned}
\Sigma(s,v) &= \frac{1}{\sqrt{2}}(\bar{u}_1+\bar{u}_3)+\frac{v}{\sqrt{2}}(\bar{u}_1+\bar{u}_2) \\
&= \frac{1}{\sqrt{2}}\big((\sin\theta-\cos\theta)u_2+(\sin\theta-\cos\theta)u_3\big)+\frac{v}{\sqrt{2}}(u_1-\cos\theta u_2+\sin\theta u_3).
\end{aligned}
\tag{3.4}
$$

Theorem 3.11. *Let the Successor curve of the β curve be α. The Gaussian and mean curvature of the $\Sigma(s,v)$ ruled surface are as follows:*

$$
K_\Sigma = \frac{\sin^2\theta(v\cos\theta+\sin\theta)^2}{\left(\sin^2\theta+\sin^2\theta((1-v)\cos\theta-\sin\theta)^2+\cos^2\theta\big((1-v)\cos\theta-\sin\theta+1)^2\right)\left(((1-v)\cos\theta-\sin\theta)^2-(\sin\theta+v\cos\theta)^2+1\right)},
$$

$$
H_\Sigma = \frac{(v\cos\theta+\sin\theta)(v\sqrt{2}\sin\theta-1)+k_2\big((1-v)+\cos\theta\big)+k_1\sin\theta\big((1-v)^2\cos^2\theta-(1-v)\sin 2\theta+\sin^2\theta+1\big)}{\sqrt{2\left(\sin^2\theta+\sin^2\theta((1-v)\cos\theta-\sin\theta)^2+\cos^2\theta\big((1-v)\cos\theta-\sin\theta+1)^2\right)}\left(((1-v)\cos\theta-\sin\theta)^2-(\sin\theta+v\cos\theta)^2+1\right)}.
$$

Proof. Partial derivatives of Equality 3.4 are,

$$
\Sigma_s=\frac{k_1\big((1-v)\cos\theta-\sin\theta)u_1+u_2\big)}{\sqrt{2}},\ \Sigma_v=\frac{u_1-\cos\theta u_2+\sin\theta u_3}{\sqrt{2}},\ \Sigma_{sv}=\frac{k_1(\cos\theta u_1+u_2)}{\sqrt{2}},
$$

$$
\Sigma_{ss}=\frac{\big(k_1{}'(1-v)\cos\theta-\sin\theta)-k_1k_2((1-v)\sin\theta+\cos\theta)-k_1{}^2\big)u_1+\big(k_1{}'+k_1{}^2((1-v)\cos\theta-\sin\theta)\big)u_2+k_1k_2(1+v)u_3}{\sqrt{2}},\ \Sigma_{vv}=0.
$$

Thus, from Equality 2.1 the normal of the surface N_Σ is given as

$$
N_\Sigma=\frac{\sin\theta u_1-\sin\theta((1-v)\cos\theta-\sin\theta)u_2+\cos\theta\big((1-v)\cos\theta-\sin\theta+1\big)u_3}{\sqrt{\sin^2\theta+\sin^2\theta\big((1-v)\cos\theta-\sin\theta\big)^2+\cos^2\theta\big((1-v)\cos\theta-\sin\theta+1\big)^2}}.
$$

Moreover, in Equalities 2.3 and 2.4 the coefficients of fundamental forms are

$$
E_\Sigma=\frac{k_1{}^2\big((1-v)\cos\theta-\sin\theta)^2+1\big)}{2},\ F_\Sigma=-\frac{k_1(\sin\theta+\cos\theta)}{2},\ G_\Sigma=1,
$$

$$
e_\Sigma=\frac{k_1{}^2\sin\theta\big((1-v)^2\cos^2\theta-(1-v)\sin 2\theta+\sin^2\theta+1\big)+k_1k_2\big((1-v)+\cos\theta\big)}{\sqrt{2}\left(\sin^2\theta+\sin^2\theta\big((1-v)\cos\theta-\sin\theta\big)^2+\cos^2\theta\big((1-v)\cos\theta-\sin\theta+1\big)^2\right.},
$$

$$
f_\Sigma=\frac{k_1\sin\theta(v\cos\theta+\sin\theta)}{\sqrt{2}\left(\sin^2\theta+\sin^2\theta\big((1-v)\cos\theta-\sin\theta\big)^2+\cos^2\theta\big((1-v)\cos\theta-\sin\theta+1\big)^2\right.},
$$

$$
g_\Sigma=0
$$

respectively. Thus, by using Equality 2.2 the Gaussian and mean curvatures are found. □

Corollary 3.12. $\theta=k\pi$ $(k\in\mathbb{N})$ *the ruled surface $\Sigma(s,v)$ is a developable surface.*

Definition 3.13. *Let the Successor curve of the β curve be α. The ruled surface formed by $\{\bar{u}_1\,\bar{u}_3\}$ the vector along the $\{\bar{u}_1\,\bar{u}_3\}$ Smarandache curve obtained from the $\bar{u}_1$ tangent vector and $\bar{u}_3$ binormal vectors of the β curve is as follows:*

$$
\begin{aligned}
\lambda(s,v) &= \tfrac{1}{\sqrt{2}}(\bar{u}_1+\bar{u}_3)+v\bar{u}_3 \\
&= \tfrac{1}{\sqrt{2}}\big((\sin\theta-\cos\theta)u_2+(\sin\theta+\cos\theta)u_3\big)+v(\sin\theta u_2+\cos\theta u_3).
\end{aligned}
\tag{3.5}
$$

Theorem 3.14. *Let the Successor curve of the β curve be α. The Gaussian and mean curvature of the $\lambda(s,v)$ ruled surface are as follows:*

$$
K_\lambda=0,\ H_\lambda=-\frac{\cos\theta+\sin\theta}{\sqrt{2}(1+v)^2\sqrt{(\sin^2\theta-\cos^2\theta)^2+(\cos\theta\sin\theta-1)^2}}.
$$

Proof. Partial derivatives of Equality 3.5 are,

$$\lambda_s = \frac{k_1(1+v)(\cos\theta - \sin\theta)u_1}{\sqrt{2}}, \ \lambda_v = \frac{(\sin\theta - \cos\theta)u_2 + (\sin\theta + \cos\theta)u_3}{\sqrt{2}},$$

$$\lambda_{sv} = \frac{k_1(\cos\theta - \sin\theta)u_1}{\sqrt{2}}, \ \lambda_{vv} = 0,$$

$$\lambda_{ss} = \frac{1}{\sqrt{2}}\big(k_1{}'(\cos\theta - \sin\theta) - k_1k_2(\sin\theta - \cos\theta)\big)u_1 + \big(k_1{}^2(\cos\theta - \sin\theta)\big)u_2.$$

Thus, from Equality 2.1 the normal of the surface N_λ is given as

$$N_\lambda = \frac{(\sin^2\theta - \cos^2\theta)u_2 + (\sin\theta\cos\theta - 1)u_3}{\sqrt{(\sin^2\theta - \cos^2\theta)^2 + (\sin\theta\cos\theta - 1)^2}}.$$

Moreover, in Equalities 2.3 and 2.4 the coefficients of fundamental forms are

$$E_\lambda = \frac{k_1{}^2(1+v)^2(\cos\theta - \sin\theta)^2}{2}, \ F_\lambda = 0, \ G_\lambda = 1,$$

$$e_\lambda = \frac{k_1{}^2(\cos\theta - \sin\theta)\sin^2\theta - \cos^2\theta}{\sqrt{2}\sqrt{k_1{}^2(\cos\theta - \sin\theta)\sin^2\theta - \cos^2\theta}}, \ f_\lambda = g_\lambda = 0$$

respectively. Thus, by using Equality 2.2 the Gaussian and mean curvatures are found. □

Corollary 3.15. $\theta = \dfrac{\pi}{4} + k\pi$ $(k \in \mathbb{N})$ *the ruled surface $\lambda(s,v)$ is a developable minimal surface.*

Definition 3.16. *Let the Successor curve of the β curve be α. The ruled surface formed by $\{\overline{u}_2\ \overline{u}_3\}$ the vector along the $\{\overline{u}_1\ \overline{u}_3\}$ Smarandache curve obtained from the $\overline{u}_1$ tangent vector and $\overline{u}_3$ binormal vectors of the β curve is as follows:*

$$\begin{aligned} \eta(s,v) &= \frac{1}{\sqrt{2}}(\overline{u}_1 + \overline{u}_3) + \frac{v}{\sqrt{2}}(\overline{u}_2 + \overline{u}_3) \\ &= \frac{1}{\sqrt{2}}\big((\sin\theta - \cos\theta)u_2 + (\sin\theta + \cos\theta)u_3\big) + \frac{v}{\sqrt{2}}(u_1 + \sin\theta u_2 + \cos\theta u_3). \end{aligned} \tag{3.6}$$

Theorem 3.17. *Let the Successor curve of the β curve be α. The Gaussian and mean curvature of the $\eta(s,v)$ ruled surface are as follows:*

$$K_\eta = \frac{\sin^2\theta + \sin 2\theta}{2\kappa^2\Big(v^2\cos^2\theta + \big((1+v)\cos\theta\sin\theta - cos^2\theta\big)^2 + \big((\sin\theta\cos\theta - (1+v)\sin^2\theta) - v\big)^2\Big)\Big(\big(\cos\theta - (1+v)\sin\theta\big)^2 + v^2 - 1 + \sin 2\theta\Big)},$$

$$H_\eta = \frac{\cos\theta(\sin\theta - \cos\theta)^2 + vk_2(1+2v) + k_1\cos\theta\big(v^2 + (1 + v\sin\theta(\cos\theta - (1+v)\sin\theta)^2)\big)}{\sqrt{2}k_1{}^2\sqrt{\Big(v^2\cos^2\theta + \big((1+v)\cos\theta\sin\theta - cos^2\theta\big)^2 + \big((\sin\theta\cos\theta - (1+v)\sin^2\theta) - v\big)^2\Big)}\Big(\big(\cos\theta - (1+v)\sin\theta\big)^2 + v^2 - 1 + \sin 2\theta\Big)}.$$

Proof. Partial derivatives of Equality 3.6 are,

$$\eta_s = \frac{k_1\big((\cos\theta - (1+v)\sin\theta)u_1 + vu_2\big)}{\sqrt{2}}, \ \eta_v = \frac{u_1 + \sin\theta u_2 + \cos\theta u_3}{\sqrt{2}}, \ \eta_{sv} = \frac{-k_1(\sin\theta u_1 - u_2)}{\sqrt{2}},$$

$$\eta_{ss} = \frac{\big(k_1{}'(\cos\theta - (1+v)\sin\theta)\big) - k_1k_2\big(\sin\theta + (1+v)\cos\theta\big) - vk_1{}^2\big)u_1 + \big(k_1{}^2(\cos\theta - (1+v)\sin\theta)\big) - vk_1{}'\big)u_2 + vk_1k_2u_3}{\sqrt{2}}, \ \eta_{vv} = 0.$$

Thus, from Equality 2.1 the normal of the surface N_λ is given as

$$N_\eta = \frac{v\cos\theta u_1 + \big((1+v)\cos\theta\sin\theta - \cos^2\theta\big)u_2 + \big((\sin\theta\cos\theta - (1+v)\sin^2\theta) - v\big)u_3}{\sqrt{v^2\cos^2\theta + \big((1+v)\cos\theta\sin\theta - \cos^2\theta\big)^2 + \big((\sin\theta\cos\theta - (1+v)\sin^2\theta) - v\big)^2}}.$$

Moreover, in Equalities 2.3 and 2.4 the coefficients of fundamental forms are

$$E_\eta = \frac{k_1{}^2\big((\cos\theta - (1+v)\sin\theta)^2 + v^2\big)}{2}, \ F_\eta = \frac{k_1(\cos\theta - \sin\theta)}{2}, \ G_\eta = 1, \ g_\eta = 0,$$

$$e_\eta = \frac{-k_1\Big(k_2v(1+2v) + k_1\cos\theta\big(v^2 + (1+v)\sin\theta(\cos\theta - (1+v)\sin\theta)^2\big)\Big)}{\sqrt{2}\sqrt{v^2\cos^2\theta + \big((1+v)\cos\theta\sin\theta - \cos^2\theta\big)^2 + \big((\sin\theta\cos\theta - (1+v)\sin^2\theta) - v\big)^2}},$$

$$f_\eta = \frac{\cos\theta(\sin\theta - \cos\theta)}{\sqrt{2}\sqrt{v^2\cos^2\theta + \big((1+v)\cos\theta\sin\theta - \cos^2\theta\big)^2 + \big((\sin\theta\cos\theta - (1+v)\sin^2\theta) - v\big)^2}}$$

respectively. Thus, by using Equality 2.2 the Gaussian and mean curvatures are found. □

Corollary 3.18. $\theta = \dfrac{\pi}{4} + k\pi$ $(k \in \mathbb{N})$ *the ruled surface $\eta(s,v)$ is a developable surface.*

Definition 3.19. *Let the Successor curve of the β curve be α. The ruled surface formed by $\{\overline{u}_1\ \overline{u}_2\ \overline{u}_3\}$ the vector along the $\{\overline{u}_1\ \overline{u}_3\}$ Smarandache curve obtained from the $\overline{u}_1$ tangent vector and $\overline{u}_3$ binormal vectors of the β curve is as follows:*

$$\begin{aligned}\Gamma(s,v) &= \frac{1}{\sqrt{2}}(\overline{u}_1+\overline{u}_3)+\frac{v}{\sqrt{3}}(\overline{u}_1+\overline{u}_2+\overline{u}_3)\\ &= \frac{1}{\sqrt{2}}\big((\sin\theta-\cos\theta)u_2+(\sin\theta+\cos\theta)u_3\big)+\frac{v}{\sqrt{3}}\big(u_1+(\sin\theta-\cos\theta)u_2+(\sin\theta+\cos\theta)u_3\big).\end{aligned} \tag{3.7}$$

Theorem 3.20. *Let the Successor curve of the β curve be α. The Gaussian and mean curvature of the $\Gamma(s,v)$ ruled surface are as follows:*

$$K_\Gamma = -\frac{3{k_1}^2(\sin^2\theta-\cos^2\theta)^2}{4\Big(2v^2(1+sin2\theta)+(\sqrt{3}+v\sqrt{2})^2(\sin^2\theta-\cos^2\theta)^2+\big((\sqrt{3}+v\sqrt{2})(\sin 2\theta-1)v\sqrt{2}\big)^2\Big)\Big((1+v\sqrt{6}+2v^2)(1-\sin 2\theta)^2\Big)},$$

$$H_\Gamma = -\frac{\sqrt{6}\Big(-\sqrt{2}(\sin^2\theta-\cos^2\theta)(\cos\theta-\sin\theta)\Big)-\sqrt{6}\Big(k_2(v\sqrt{6}+2v^2)(2\sin 2\theta-v\sqrt{2})\Big)+\sqrt{6}\Big(k_1(\cos\theta+\sin\theta)(2v^2+(\sqrt{3}+v\sqrt{2})^2 sin2\theta)\Big)}{2k_1\big(((\sqrt{3}+v\sqrt{2})^2-1)(1-\sin 2\theta)+2v^2\big)2k_1\Big(2v^2(1+\sin 2\theta)^2+(\sqrt{3}+v\sqrt{2})^2(\sin^2\theta-\cos^2\theta)+\big((\sqrt{3}+v\sqrt{2})(\sin 2\theta-1)\big)-v\sqrt{2}\Big)^{\frac{1}{2}}}.$$

Proof. Partial derivatives of Equality 3.7 are,

$$\Gamma_s = \frac{k_1\Big(\big((\cos\theta-\sin\theta)(\sqrt{3}+v\sqrt{2})\big)u_1+v\sqrt{2}u_2\Big)}{\sqrt{6}},\ \Gamma_{sv} = \frac{k_1\big((\cos\theta-\sin\theta)u_1+u_2\big)}{\sqrt{3}},$$

$$\Gamma_v = \frac{u_1+(\sin\theta-\cos\theta)u_2+(\sin\theta+\cos\theta)u_3}{\sqrt{3}},\ \Gamma_{vv}=0,$$

$$\Gamma_{ss} = \frac{\big({k_1}'(\sqrt{3}+v\sqrt{2})(\cos\theta-\sin\theta)-k_1k_2(\sqrt{3}+v\sqrt{2})(\cos\theta+\sin\theta)-v\sqrt{2}{k_1}^2\big)u_1+\big({k_1}^2(\sqrt{3}+v\sqrt{2})(\cos\theta-\sin\theta)-v\sqrt{2}{k_1}'\big)u_2+v\sqrt{2}k_1k_2u_3}{\sqrt{6}}.$$

Thus, from Equality 2.1 the normal of the surface N_λ is given as

$$N_\Gamma = \frac{\big(v\sqrt{2}(\cos\theta+\sin\theta)\big)u_1+(\sqrt{3}+v\sqrt{2})(\sin^2\theta-\cos^2\theta)u_2+\big((\sqrt{3}+v\sqrt{2})(\sin 2\theta-1)-v\sqrt{2}\big)u_3}{\Big(2v^2(1+\sin 2\theta)+(\sqrt{3}+v\sqrt{2})^2(\sin^2\theta-\cos^2\theta)^2+\big((\sqrt{3}+v\sqrt{2})(\sin 2\theta-1)-v\sqrt{2}\big)^2\Big)^{\frac{1}{2}}}$$

Moreover, in Equalities 2.3 and 2.4 the coefficients of fundamental forms are

$$E_\Gamma = \frac{{k_1}^2\big((\sqrt{3}+v\sqrt{2})(1-\sin 2\theta)-2v^2\big)}{6},\ F_\Gamma = \frac{k_1(\cos\theta-\sin\theta)}{6},\ G_\Gamma = 1,\ g_\Gamma = 0,$$

$$e_\Gamma = \frac{-k_1(\cos\theta+\sin\theta)\big(2v^2+(\sqrt{3}+v\sqrt{2})^2(\sin 2\theta)\big)-k_2v\sqrt{2}(\sqrt{3}+v\sqrt{2})(2\sin 2\theta-1)}{\Big(12v^2(1+\sin 2\theta)+6(\sqrt{3}+v\sqrt{2})^2+(\sin^2\theta-\cos^2\theta)^2+6\big((\sqrt{3}+v\sqrt{2})(\sin 2\theta-1)-v\sqrt{2}\big)\Big)^{\frac{1}{2}}}$$

$$f_\Gamma = \frac{\sin^2\theta-\cos^2\theta}{\Big(12v^2(1+\sin 2\theta)+6(\sqrt{3}+v\sqrt{2})^2+(\sin^2\theta-\cos^2\theta)^2+6\big((\sqrt{3}+v\sqrt{2})(\sin 2\theta-1)-v\sqrt{2}\big)\Big)^{\frac{1}{2}}}$$

respectively. Thus, by using Equality 2.2 the Gaussian and mean curvatures are found. □

Example 3.21. *Let the Successor curve of α curve be β Salkowski curve (see [26]). The equation of this curve for $m = \dfrac{1}{3}$ is as follows:*

$$\beta(s) = \frac{3}{2\sqrt{10}}\left(\begin{array}{c} -\dfrac{\sqrt{10}-1}{2\sqrt{10}+4}\sin\left(s+\frac{2s}{\sqrt{10}}\right)-\dfrac{\sqrt{10}-1}{2\sqrt{10}+4}\sin\left(s-\frac{2s}{\sqrt{10}}\right)-\sin s,\\ -\dfrac{\sqrt{10}-1}{2\sqrt{10}+4}\cos\left(s+\frac{2s}{\sqrt{10}}\right)+\dfrac{\sqrt{10}-1}{2\sqrt{10}+4}\cos\left(s-\frac{2s}{\sqrt{10}}\right)+\cos s,\ \dfrac{3}{2}\cos\left(\dfrac{2s}{\sqrt{10}}\right)\end{array}\right)$$

The Successor frames of β curve $\{\overline{u}_1,\overline{u}_2,\overline{u}_3\}$ are as follows:

$$\begin{aligned}\overline{u}_1(s) &= \left(\begin{array}{c} -\cos s\cos\dfrac{s}{\sqrt{10}}-\dfrac{1}{\sqrt{10}}\sin s\sin\dfrac{s}{\sqrt{10}},\\ -\sin s\cos\dfrac{s}{\sqrt{10}}+\dfrac{1}{\sqrt{10}}\cos s\sin\dfrac{s}{\sqrt{10}},\ \dfrac{3}{\sqrt{10}}\sin\dfrac{s}{\sqrt{10}}\end{array}\right),\\ \overline{u}_2(s) &= \left(\frac{3}{\sqrt{10}}\sin s,\ -\frac{3}{\sqrt{10}}\cos s,\ -\frac{1}{\sqrt{10}}\right),\\ \overline{u}_3(s) &= \left(\begin{array}{c} -\cos s\sin\dfrac{s}{\sqrt{10}}-\dfrac{1}{\sqrt{10}}\sin s\cos\dfrac{s}{\sqrt{10}},\\ -\sin s\cos\dfrac{s}{\sqrt{10}}+\dfrac{1}{\sqrt{10}}\cos s\cos\dfrac{s}{\sqrt{10}},\ \dfrac{3}{\sqrt{10}}\cos\dfrac{s}{\sqrt{10}}\end{array}\right).\end{aligned}$$

The graphs of the ruled surfaces obtained from these frames for $s \in [-\pi, \pi]$ and $v \in [-1, 1]$ are shown in Figures 3.1-3.7:

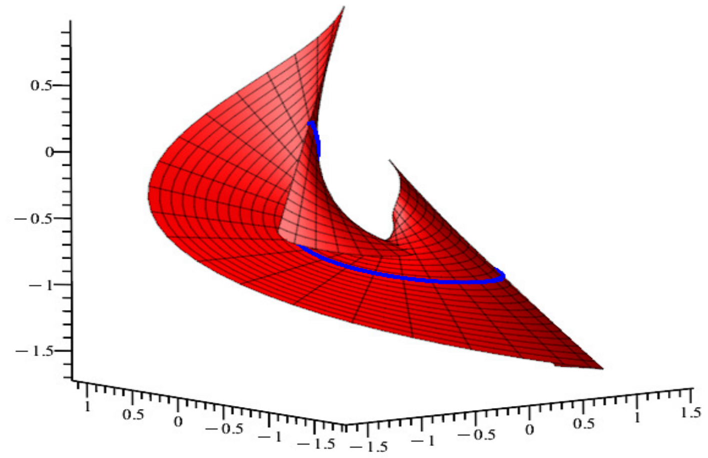

Figure 3.1: $\Phi(s,v) = \frac{1}{\sqrt{2}}(\overline{u}_1 + \overline{u}_3) + v\overline{u}_1$

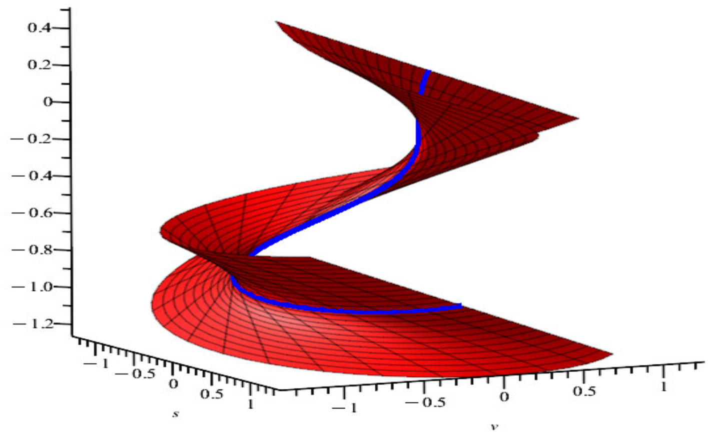

Figure 3.2: $Q(s,v) = \frac{1}{\sqrt{2}}(\overline{u}_1 + \overline{u}_3) + v\overline{u}_2$

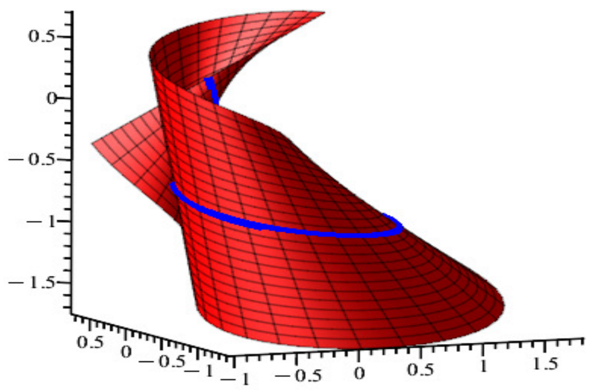

Figure 3.3: $M(s,v) = \frac{1}{\sqrt{2}}(\overline{u}_1 + \overline{u}_3) + v\overline{u}_3$

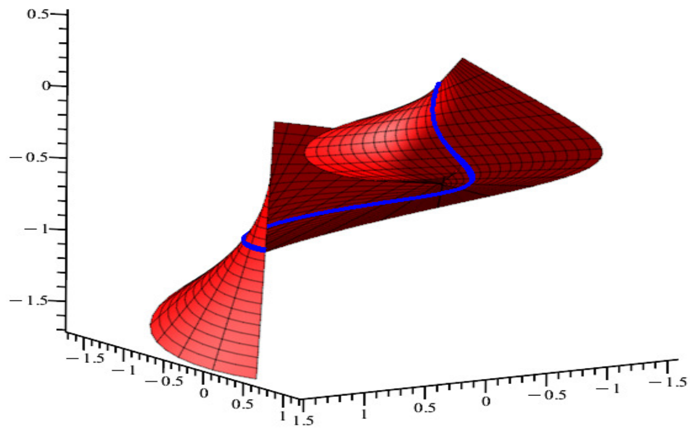

Figure 3.4: $\Sigma(s,v)=\frac{1}{\sqrt{2}}(\bar{u}_1+\bar{u}_3)+\frac{v}{\sqrt{2}}(\bar{u}_1+\bar{u}_2)$

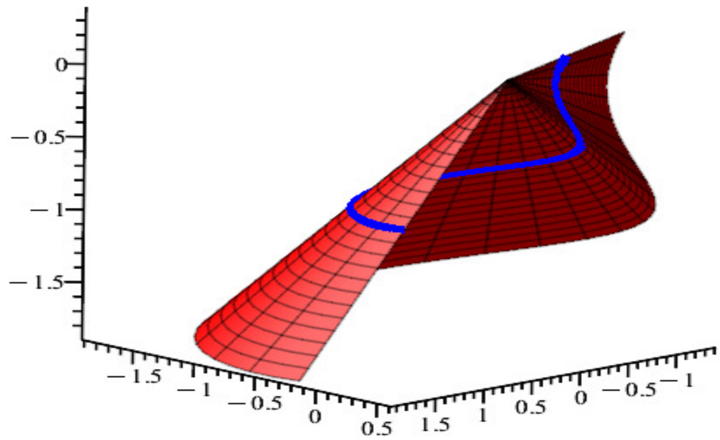

Figure 3.5: $\lambda(s,v)=\frac{1}{\sqrt{2}}(\bar{u}_1+\bar{u}_3)+\frac{v}{\sqrt{2}}(\bar{u}_1+\bar{u}_3)$

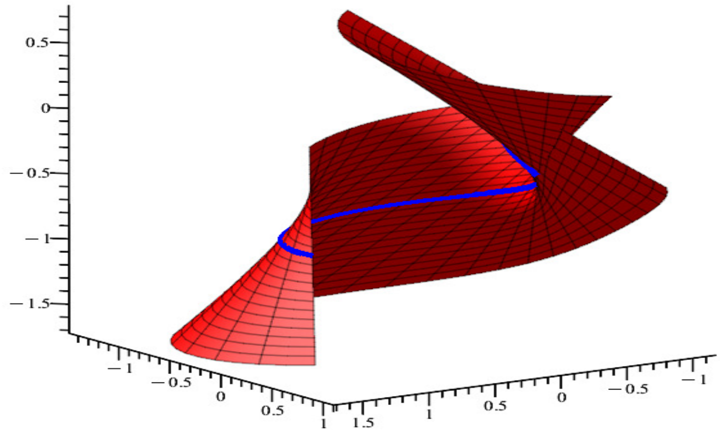

Figure 3.6: $\eta(s,v)=\frac{1}{\sqrt{2}}(\bar{u}_1+\bar{u}_3)+\frac{v}{\sqrt{2}}(\bar{u}_2+\bar{u}_3)$

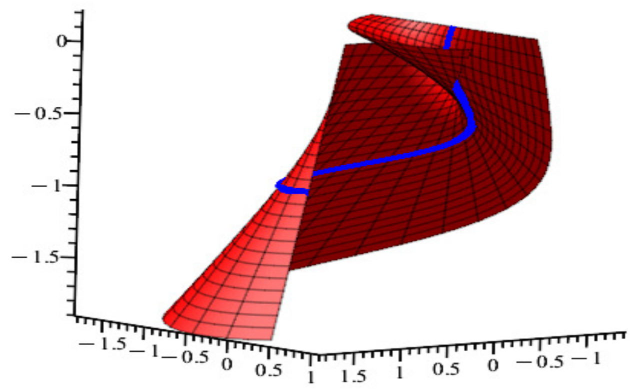

Figure 3.7: $\Gamma(s,v)=\frac{1}{\sqrt{2}}(\bar{u}_1+\bar{u}_3)+\frac{v}{\sqrt{3}}(\bar{u}_1+\bar{u}_2+\bar{u}_3)$

Example 3.22. *Let the Salkowski curve in Example 3.21 be the main curve. From [26] and Theorem 2.2, the Successor frames are as follows:*

$$\bar{u}_1(s)=\begin{pmatrix} -\cos\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(\frac{3}{\sqrt{10}}\sin s\right) \\ +\sin\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(-\cos s\sin\frac{s}{\sqrt{10}}-\frac{1}{\sqrt{10}}\sin s\cos\frac{s}{\sqrt{10}}\right), \\ \cos\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(\frac{3}{\sqrt{10}}\cos s\right) \\ -\sin\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(-\sin s\sin\frac{s}{\sqrt{10}}+\frac{1}{\sqrt{10}}\cos s\cos\frac{s}{\sqrt{10}}\right), \\ \cos\left(\int tan\frac{s}{\sqrt{10}}ds\right)\frac{1}{\sqrt{10}}+\sin\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(\frac{3}{\sqrt{10}}\cos\frac{s}{\sqrt{10}}\right) \end{pmatrix},$$

$$\bar{u}_2(s)=\begin{pmatrix} -\cos s\cos\frac{s}{\sqrt{10}}-\frac{1}{\sqrt{10}}\sin s\sin\frac{s}{\sqrt{10}}, \\ -\sin s\cos\frac{s}{\sqrt{10}}+\frac{1}{\sqrt{10}}\sin s\sin\frac{s}{\sqrt{10}},\ \frac{3}{\sqrt{10}}\sin\frac{s}{\sqrt{10}} \end{pmatrix},$$

$$\bar{u}_3(s)=\begin{pmatrix} \sin\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(\frac{3}{\sqrt{10}}\sin s\right) \\ -\cos\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(\cos s\sin\frac{s}{\sqrt{10}}+\frac{1}{\sqrt{10}}\sin s\cos\frac{s}{\sqrt{10}}\right), \\ -\sin\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(\frac{3}{\sqrt{10}}\cos s\right) \\ -\cos\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(\sin s\sin\frac{s}{\sqrt{10}}-\frac{1}{\sqrt{10}}\cos s\cos\frac{s}{\sqrt{10}}\right), \\ -\sin\left(\int tan\frac{s}{\sqrt{10}}ds\right)\frac{1}{\sqrt{10}}+\cos\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(\frac{3}{\sqrt{10}}\cos\frac{s}{\sqrt{10}}\right) \end{pmatrix}.$$

The graphs of the ruled surfaces obtained from these frames for $s\in[-\pi,\pi]$ and $v\in[-1,1]$ are shown in Figures 3.8-3.14:

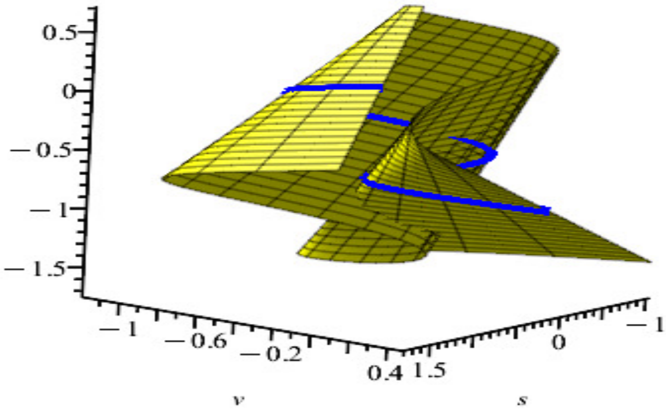

Figure 3.8: $\Phi(s,v) = \frac{1}{\sqrt{2}}(\overline{u}_1 + \overline{u}_3) + v\overline{u}_1$

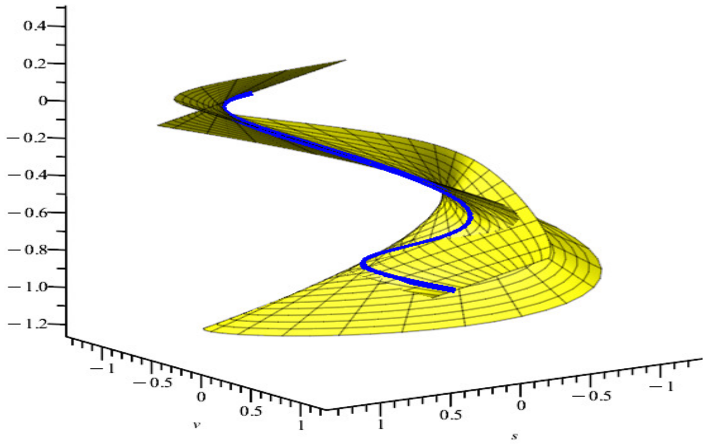

Figure 3.9: $Q(s,v) = \frac{1}{\sqrt{2}}(\overline{u}_1 + \overline{u}_3) + v\overline{u}_2$

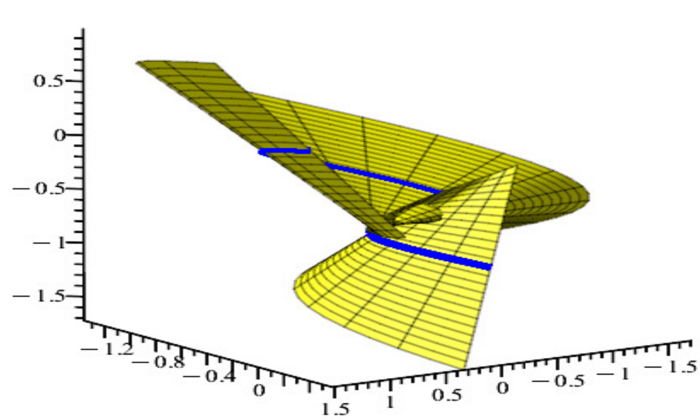

Figure 3.10: $M(s,v) = \frac{1}{\sqrt{2}}(\overline{u}_1 + \overline{u}_3) + v\overline{u}_3$

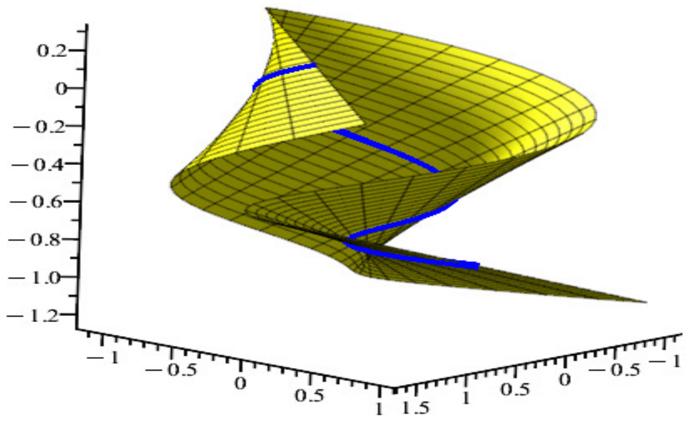

Figure 3.11: $\Sigma(s,v)=\frac{1}{\sqrt{2}}(\bar{u}_1+\bar{u}_3)+\frac{v}{\sqrt{2}}(\bar{u}_1+\bar{u}_2)$

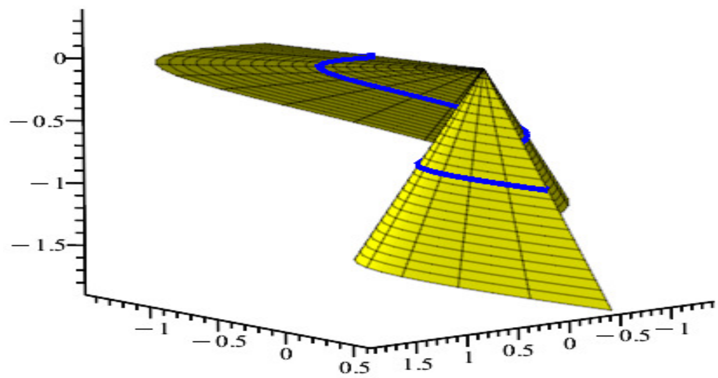

Figure 3.12: $\lambda(s,v)=\frac{1}{\sqrt{2}}(\bar{u}_1+\bar{u}_3)+\frac{v}{\sqrt{2}}(\bar{u}_1+\bar{u}_3)$

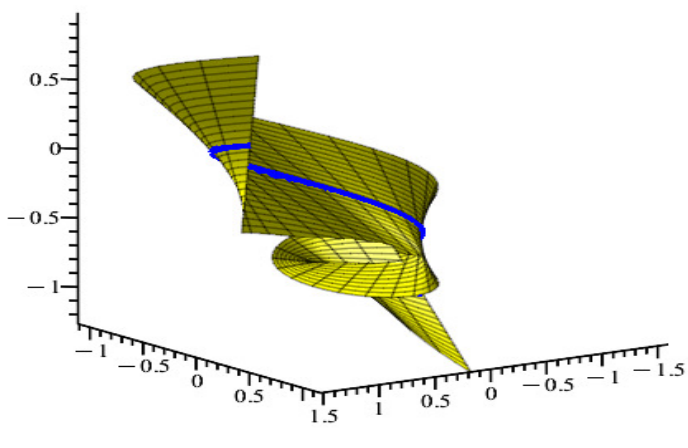

Figure 3.13: $\eta(s,v)=\frac{1}{\sqrt{2}}(\bar{u}_1+\bar{u}_3)+\frac{v}{\sqrt{2}}(\bar{u}_2+\bar{u}_3)$

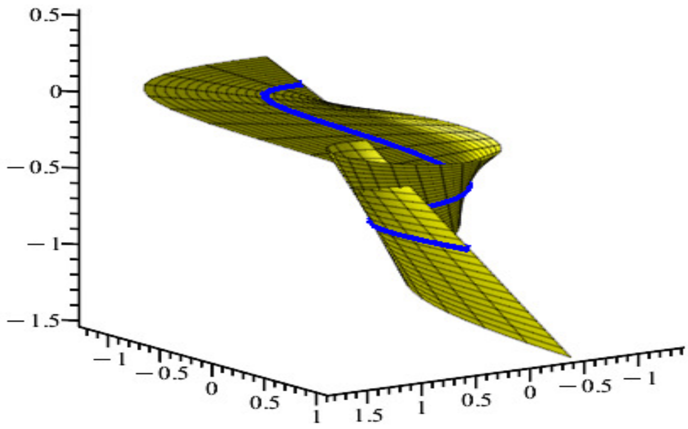

Figure 3.14: $\Gamma(s,v) = \frac{1}{\sqrt{2}}(\overline{u}_1+\overline{u}_3)+\frac{v}{\sqrt{3}}(\overline{u}_1+\overline{u}_2+\overline{u}_3)$

Example 3.23. *Let the Successor curve of α curve be $\beta^*(s)$ anti Salkowski curve [26]. The equation of this curve for $m=\dfrac{1}{3}$ is as follows:*

$$\beta^*(s)=\frac{\sqrt{10}}{40}\begin{pmatrix} -\dfrac{5}{2\sqrt{10}}\Big(\dfrac{3}{\sqrt{10}}\cos(\dfrac{1}{5}+\cos(\dfrac{2}{\sqrt{10}})s)\Big)+\dfrac{6}{5}\sin s\sin\dfrac{2}{\sqrt{10}}s, \\ -\dfrac{5}{2\sqrt{10}}\Big(\dfrac{3}{\sqrt{10}}\sin(\dfrac{1}{5}+\cos(\dfrac{2}{\sqrt{10}})s)\Big)+\dfrac{6}{5}\cos s\sin\dfrac{2}{\sqrt{10}}s, \\ -\dfrac{9\sqrt{10}}{40}\Big(\dfrac{2}{\sqrt{10}}s+\sin(\dfrac{2}{\sqrt{10}})s\Big) \end{pmatrix}.$$

The Successor frames of β^* curve $\{\overline{u}_1^*,\overline{u}_2^*,\overline{u}_3^*\}$ are as follows:

$$\overline{u}_1^*(s)=\begin{pmatrix} -\cos s\sin\dfrac{s}{\sqrt{10}}+\dfrac{1}{\sqrt{10}}\sin s\cos\dfrac{s}{\sqrt{10}}, \\ -\sin s\sin\dfrac{s}{\sqrt{10}}-\dfrac{1}{\sqrt{10}}\cos s\cos\dfrac{s}{\sqrt{10}}, -\dfrac{3}{\sqrt{10}}\cos\dfrac{s}{\sqrt{10}} \end{pmatrix},$$

$$\overline{u}_2^*(s)=\Big(\frac{3}{\sqrt{10}}\sin s,\ -\frac{3}{\sqrt{10}}\cos s,\ \frac{1}{\sqrt{10}}\Big),$$

$$\overline{u}_3^*(s)=\begin{pmatrix} -\cos s\cos\dfrac{s}{\sqrt{10}}-\dfrac{1}{\sqrt{10}}\sin s\sin\dfrac{s}{\sqrt{10}}, \\ -\sin s\cos\dfrac{s}{\sqrt{10}}+\dfrac{1}{\sqrt{10}}\cos s\sin\dfrac{s}{\sqrt{10}}, \dfrac{3}{\sqrt{10}}\sin\dfrac{s}{\sqrt{10}} \end{pmatrix}.$$

The graphs of the ruled surfaces obtained from these frames for $s\in[-\pi,\pi]$ and $v\in[-1,1]$ are shown in Figures 3.15-3.21:

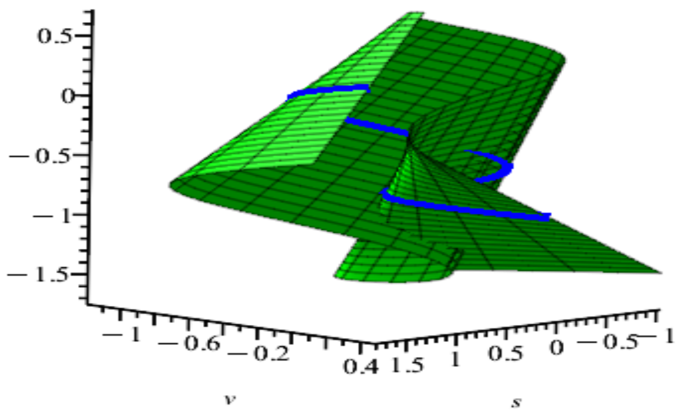

Figure 3.15: $\Phi^*(s,v)=\frac{1}{\sqrt{2}}(\overline{u}_1^*+\overline{u}_3^*)+v\overline{u}_1^*$

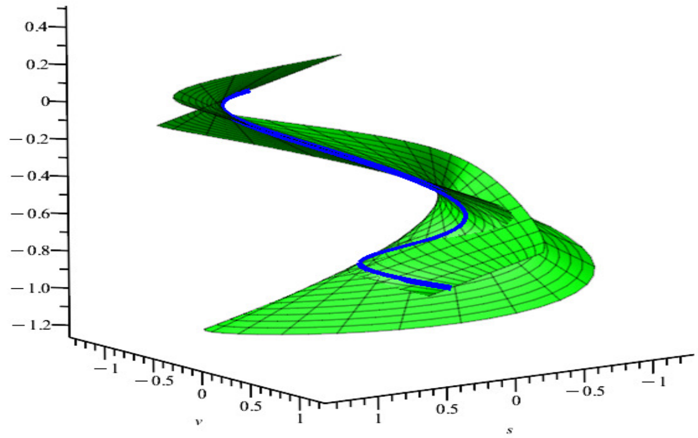

Figure 3.16: $Q^*(s,v) = \frac{1}{\sqrt{2}}(\overline{u}_1^* + \overline{u}_3^*) + v\overline{u}_2^*$

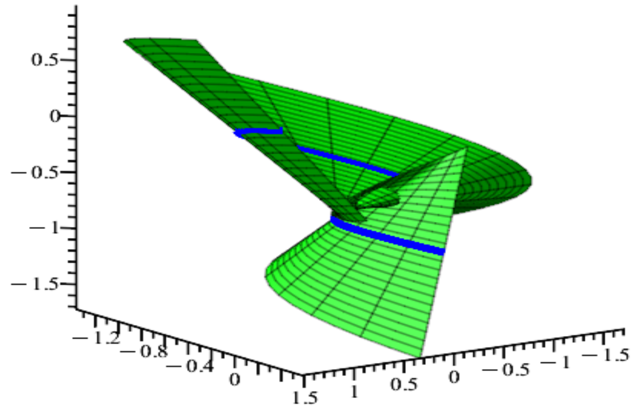

Figure 3.17: $M^*(s,v) = \frac{1}{\sqrt{2}}(\overline{u}_1^* + \overline{u}_3^*) + v\overline{u}_3^*$

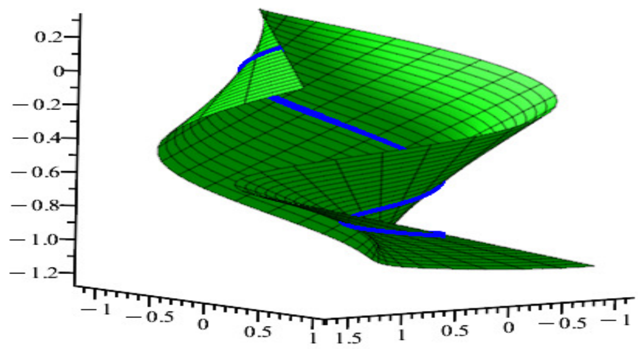

Figure 3.18: $\Sigma^*(s,v) = \frac{1}{\sqrt{2}}(\overline{u}_1^* + \overline{u}_3^*) + \frac{v}{\sqrt{2}}(\overline{u}_1^* + \overline{u}_2^*)$

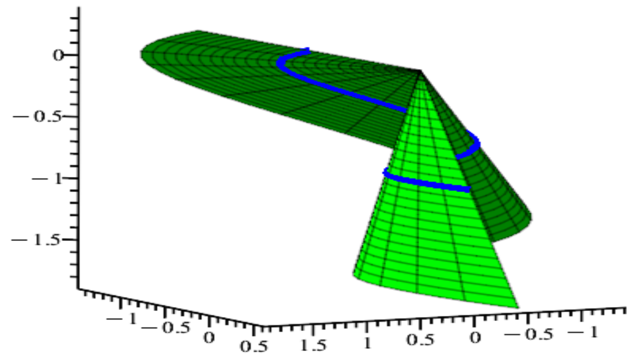

Figure 3.19: $\lambda^*(s,v) = \frac{1}{\sqrt{2}}(\overline{u}_1^* + \overline{u}_3^*) + \frac{v}{\sqrt{2}}(\overline{u}_1^* + \overline{u}_3^*)$

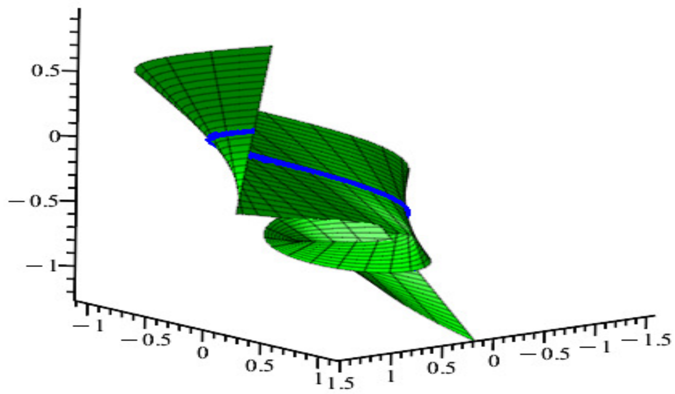

Figure 3.20: $\eta^*(s,v) = \frac{1}{\sqrt{2}}(\overline{u}_1^* + \overline{u}_3^*) + \frac{v}{\sqrt{2}}(\overline{u}_2^* + \overline{u}_3^*)$

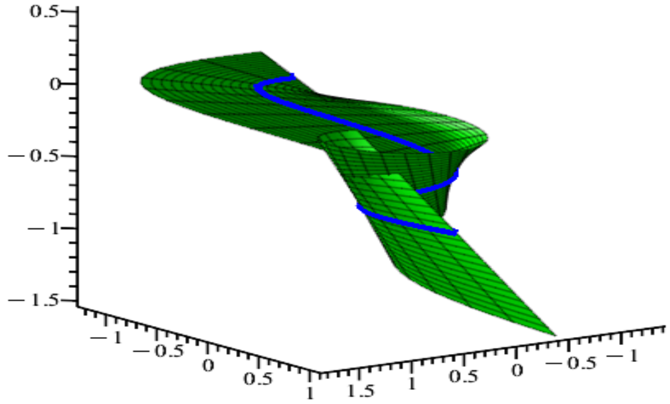

Figure 3.21: $\Gamma^*(s,v) = \frac{1}{\sqrt{2}}(\overline{u}_1^* + \overline{u}_3^*) + \frac{v}{\sqrt{3}}(\overline{u}_1^* + \overline{u}_2^* + \overline{u}_3^*)$

Example 3.24. *Let the Salkowski curve in Example 3.23 be the main curve. From [26] and Theorem 2.2 the Successor frames are as follows:*

$$\overline{u}_1^*(s) = \begin{pmatrix} -\cos(s+c)\big(\frac{3}{\sqrt{10}}\sin s\big) + \sin(s+c)\big(-\cos s \cos\frac{s}{\sqrt{10}} - \frac{1}{\sqrt{10}}\sin s \sin\frac{s}{\sqrt{10}}\big), \\ \cos(s+c)\frac{3}{\sqrt{10}}\cos s + \sin(s+c)\big(-\sin s \cos\frac{s}{\sqrt{10}} + \frac{1}{\sqrt{10}}\cos s \sin\frac{s}{\sqrt{10}}\big), \\ -\cos(s+c)\frac{1}{\sqrt{10}} + \sin(s+c)\big(\frac{3}{\sqrt{10}}\sin\frac{s}{\sqrt{10}}\big) \end{pmatrix}$$

$$\overline{u}_2^*(s)=\begin{pmatrix}-\cos s\sin\dfrac{s}{\sqrt{10}}+\dfrac{1}{\sqrt{10}}\sin s\cos\dfrac{s}{\sqrt{10}},-\sin s\sin\dfrac{s}{\sqrt{10}}-\dfrac{1}{\sqrt{10}}\cos s\cos\dfrac{3}{\sqrt{10}},\\ -\dfrac{3}{\sqrt{10}}\cos\dfrac{s}{\sqrt{10}}\end{pmatrix}$$

$$\overline{u}_3^*(s)=\begin{pmatrix}\sin(s+c)\left(\dfrac{3}{\sqrt{10}}\sin s\right)+\cos(s+c)\left(-\cos s\cos\dfrac{s}{\sqrt{10}}-\dfrac{1}{\sqrt{10}}\sin s\sin\dfrac{s}{\sqrt{10}}\right),\\ \sin(s+c)\dfrac{3}{\sqrt{10}}\cos s+\cos(s+c)\left(-\cos s\cos\dfrac{s}{\sqrt{10}}-\dfrac{1}{\sqrt{10}}\sin s\sin\dfrac{s}{\sqrt{10}}\right),\\ \sin(s+c)\dfrac{1}{\sqrt{10}}+\cos(s+c)\left(\dfrac{3}{\sqrt{10}}\sin\dfrac{s}{\sqrt{10}}\right)\end{pmatrix}$$

The graphs of the ruled surfaces obtained from these frames for $s\in[-\pi,\pi]$ and $v\in[-1,1]$ are shown in Figures 3.22-3.28:

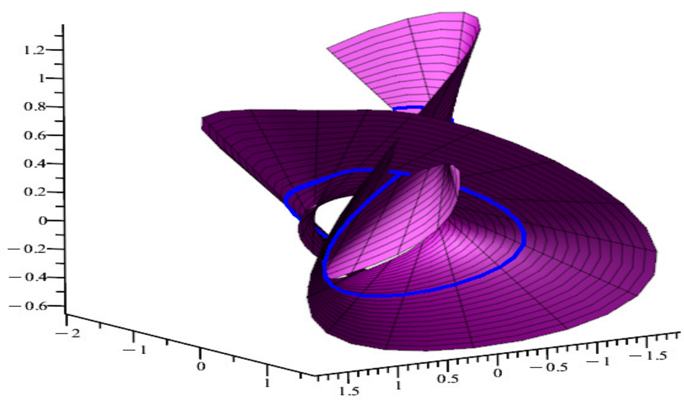

Figure 3.22: $\Phi^*(s,v)=\frac{1}{\sqrt{2}}(\overline{u}_1^*+\overline{u}_3^*)+v\overline{u}_1^*$

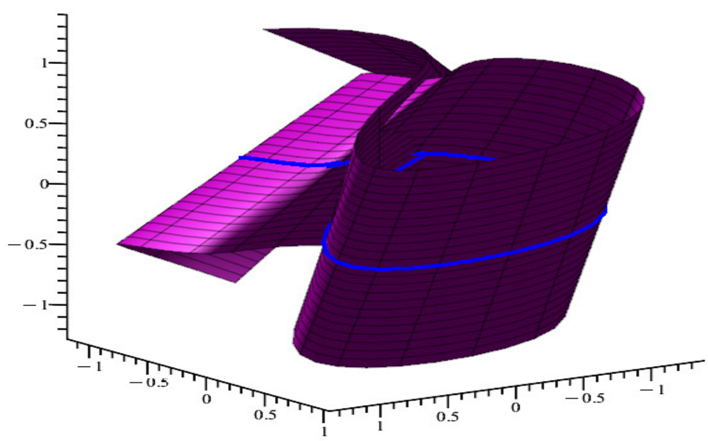

Figure 3.23: $Q^*(s,v)=\frac{1}{\sqrt{2}}(\overline{u}_1+\overline{u}_3)+v\overline{u}_2$

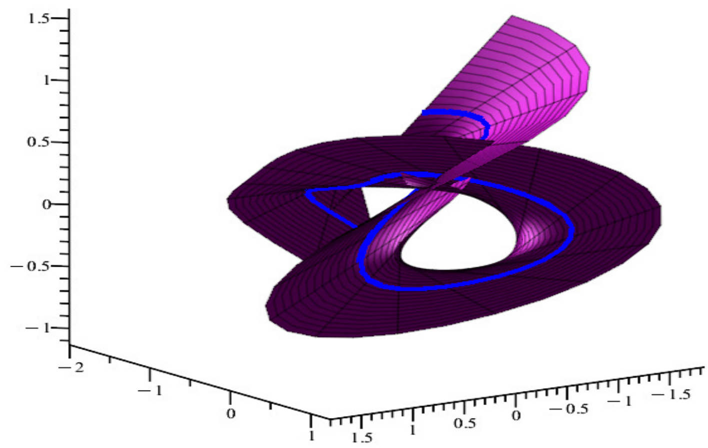

Figure 3.24: $M^*(s,v) = \frac{1}{\sqrt{2}}(\overline{u}_1^* + \overline{u}_3^*) + v\overline{u}_3^*$

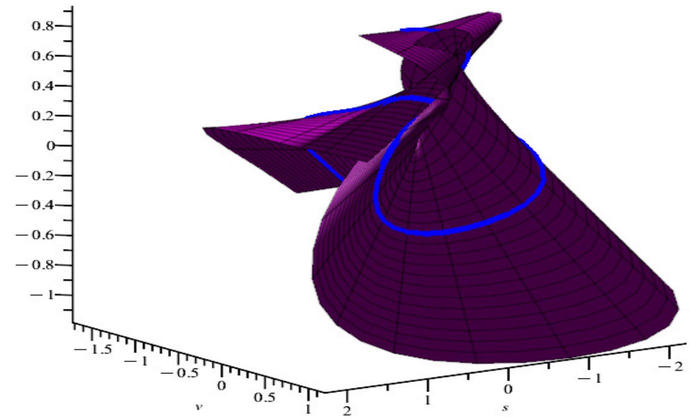

Figure 3.25: $\Sigma^*(s,v) = \frac{1}{\sqrt{2}}(\overline{u}_1^* + \overline{u}_3^*) + \frac{v}{\sqrt{2}}(\overline{u}_1^* + \overline{u}_2^*)$

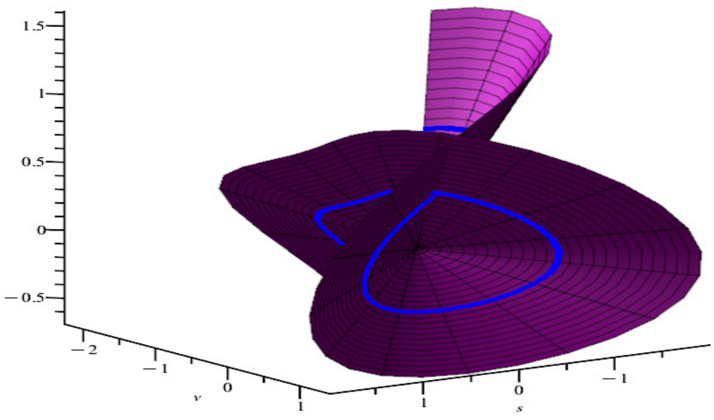

Figure 3.26: $\lambda^*(s,v) = \frac{1}{\sqrt{2}}(\overline{u}_1^* + \overline{u}_3^*) + \frac{v}{\sqrt{2}}(\overline{u}_1^* + \overline{u}_3^*)$

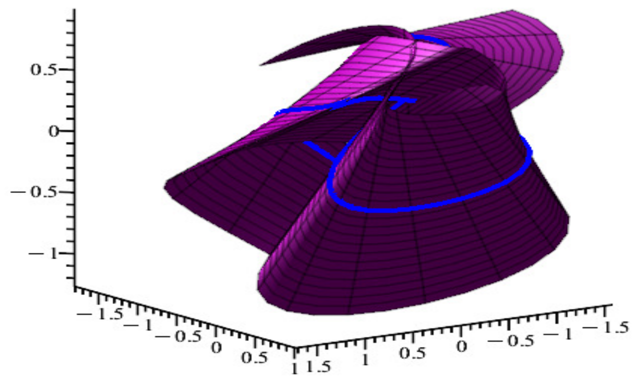

Figure 3.27: $\eta^*(s,v) = \frac{1}{\sqrt{2}}(\overline{u}_1^* + \overline{u}_3^*) + \frac{v}{\sqrt{2}}(\overline{u}_2^* + \overline{u}_3^*)$

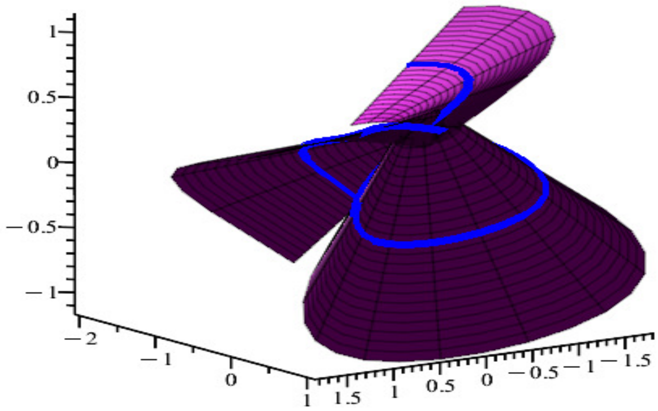

Figure 3.28: $\Gamma^*(s,v) = \frac{1}{\sqrt{2}}(\overline{u}_1^* + \overline{u}_3^*) + \frac{v}{\sqrt{3}}(\overline{u}_1^* + \overline{u}_2^* + \overline{u}_3^*)$

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Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of data and materials: Not applicable

References

[1] H. Pottmann, M. Eigensatz, A. Vaxman, and J. Wallner, *Architectural geometry*, Computers and Graphics, **47** (2015), 145–164.
[2] J. Stillwell, *Mathematics and Its History*, Undergraduate texts in mathematics, Third Edition Springer (2010).
[3] D. J. Struik, *Lectures on Classical Differential Geometry*, Addison-Wesley Publishing Company, (1961).
[4] K. Akutagawa and S. Nishikawa, *The Gauss Map and Space-like surfaces with prescribed mean curvature in Minkowski 3-space*,Tohoku Mathematical Journal Second Series, **42** (1990), 67-82.
[5] L. Grill, S. Şenyurt, and S. G. Mazlum, *Gaussian curvatures of parallel ruled surfaces*, Applied Mathematical Sciences, **14** (2020), 171-183.
[6] S. G. Mazlum, S. Şenyurt, and L. Grill, *The invariants of Dual Parallel equidistant ruled surfaces*, Symmetry, **15** (2023), 206.

[7] S. Şenyurt, D. Canlı, E. Can, and S. G. Mazlum, *Some special smarandache ruled surfaces by Frenet frame in $E^3 - II$*, Honam Mathematical Journal, **44** (2022), 594-617.
[8] S. Şenyurt, D. Canlı, and E. Can, *Some special Smarandache ruled surfaces by Frenet frame in $E^3 - I$*, Turkısh Journal of Science, **7(1)** (2022), 31-42.
[9] W. S. Massey, *Surfaces of gaussian curvature zero in euclidean 3-space*, Tohoku Mathematical Journal Second Series, **14** (1962), 73-79.
[10] A. Pressley, Elementary Differential Geometry, Springer Science and Business Media, (2010).
[11] S. Ouarab, *Smarandache ruled surfaces according to Frenet-serret frame of a regular curve in E^3*, Hindawi Abstract and Applied Analysis (2021), 8 pages.
[12] S. Ouarab, *Smarandache ruled surfaces according to Darboux frame in E^3*, Hindawi Journal of Mathematics (2021), 10 pages.
[13] S. Ouarab, *NC-smarandache ruled surface and NW-Smarandache ruled surface according to Alternative moving frame in E^3*, Hindawi Journal of Mathematics (2021), 6 pages.
[14] E. Kemal and S. Şenyurt, *On ruled surface with Sannia frame in Euclidean 3- Space*, Kyunpook Math. J. **62** (2022), 509-531.
[15] S. Şenyurt, K. H. Ayvacı, and D.Canlı, *Some characterizations of spherical indicatrix curves generated by Sannia frame*, Konuralp Journal of Mathematics, **9(2)** (2021), 222-232.
[16] A. Menninger, *Characterization of the slant helix as Successor curves of the general helix*, International Electronic Journal of Geometry, **2** (2014), 84-91.
[17] M. Masal, *Curves according to the Successor frame in Euclidean 3-space*, Sakarya University Journal of Science, 6 (2018), 1868-1873.
[18] S. Şenyurt and G. Kaya, *Smarandache curves obtained from Frenet vectors of Successor curve (In Turkish)*, G. Gürçay (Ed.), Karadeniz 1. Uluslararası Multidisipliner Çalışmalar Kongresi, Giresun (2019), 318-324
[19] S. Şenyurt and G. Kaya, *Vector moment curves of Frenet vectors of Successor curve (In Turkish)*, G. Gürçay (Ed.), Karadeniz 1. Uluslararası Multidisipliner Çalışmalar Kongresi Giresun 2019 pp.325-330.
[20] T. Erişir and H.K. Öztaş, *Spinor equations of Successor curve*, Universal of Mathematics and Applications, **5(1)** (2022), 32–41.
[21] G. Kaya, *Successor curves and equidistant ruled surfaces on the Dual*, Doctoral Dissertation Ordu University, Ordu, (2023) .
[22] A. T. Ali, *Special Smarandache curves in the Euclidean space*, International Journal of Mathematical Combinatorics, **2** (2010), 30-36.
[23] K. Taşköprü and M. Tosun, *Smarandache curves on S^2*, Bol. Soc. Paran. Mat. **32(1)** (2014), 51-59.
[24] S. Şenyurt and S. Sivas, *An application of Smarandache curve*, Ordu Univ. J. Sci. Tech. **3** (2013), 46-60.
[25] N. Turgut and S. Yılmaz, *Smarandache curves in Minkowski space-time*, International Journal of Mathematical Combinatorics, **3** (2008), 51–55.
[26] J. Monterde, *Salkowski curves revisited: a family of curves with constant curvature and con-constant Torsion*, Computer Aided Geometric Design, **3** (2009), 271-278.