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ON THE EIGENSTRUCTURE OF THE q-STANCU OPERATOR

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Abstract. The main goal of this research is to find the eigenvalues and the corresponding eigenfunctions of the q-Stancu operator, $L_{n,s,q}$, introduced by L. Yun and R. Wang. In this work, an explicit representation for moments of all orders has been derived. Further, it has been proved that $L_{n,s,q}$ possesses $n-s+1$ linearly independent eigenfunctions whose explicit expression and the corresponding eigenvalues are derived. In addition, for special choices of parameters, several eigenfunctions are depicted.

1. Introduction

The discovery of the Bernstein polynomials by S. N. Bernstein in 1912 [2] paved the way for a vast number of studies in the approximation theory. Due to their elegant structure and remarkable properties, these polynomials have formed the basis for research not only in mathematics but also in many other fields such as physics, statistics, engineering (see [6, 8, 16]). The extensive research on the Bernstein operators has enabled the development of various generalizations and modified forms.

In 1981, Stancu proposed a generalization of the Bernstein operator, representing an extension based on the non-negative integer parameter s, of the classical Bernstein operator as follows:

Definition 1. *[17] Let n and s be integers such that $0 \leqslant s < n/2$. Then, for any function $f \in C[0,1]$, the Stancu operator is defined by*

$$L_{n,s}(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) b_{n,k,s}(x), \tag{1}$$

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where

$$b_{n,k,s}(x) = \begin{cases} (1-x)p_{n-s,k}(x), & 0 \leqslant k < s, \\ (1-x)p_{n-s,k}(x) + xp_{n-s,k-s}(x), & s \leqslant k \leqslant n-s, \\ xp_{n-s,k-s}(x), & n-s < k \leqslant n. \end{cases}$$

Here, $p_{n,k}(x)$ *are the Bernstein basis polynomials given by*

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad k = 0, 1, \dots, n.$$

Observe that, for $s = 0, 1$, (1) reveals the classical Bernstein operator.

In [17], Stancu examined the remainder term of the approximation formulas for operator $L_{n,s}$ and established its asymptotic estimate using Voronovskaja-type formula. He also estimated the order of approximation for operator (1) in terms of the modulus of continuity of a function f and its derivative f'. Moreover, he found the eigenvalues of this Bernstein-type operator and proved that the sequence of the eigenvalues is monotonically decreasing. In 2008, L. Yun and X. Xiang delved into the monotonicity-preserving and convexity-preserving properties of the aforementioned operator. They provided a proof regarding the operator's monotonicity for convex functions and gave the theorem about simultaneous approximation [19]. Recently, the Kantorovich extension of Stancu operator was proposed and investigated in [3].

Another way to extend the operator is to obtain a modified version of the classical operator by employing q-calculus. The first steps of this generalization were taken by Lupaş [12] and Phillips [15], who introduced q-generalizations of the Bernstein operator. Owing to their works, the idea of generalizing operator using q-calculus has been extended to many operators and this idea is still fruitful, see for example, [7, 9, 14].

In 2011, L. Yun and R. Wang [20] introduced a q-generalization of the Stancu operator, known as q-Stancu operator. There, they studied shape-preserving and approximation properties of this generalization. A year later, X. Xiang [18] obtained more results pertinent to the q-Stancu operator.

For the convenience of the reader, some notations and definitions related to q-calculus are provided, see [1, Chapter 10], and afterward, the definition of the q-Stancu operator will be given.

Let $q > 0$. For any non-negative integer n, the q-integer $[n]_q$ is defined by

$$[0]_q := 0, \quad [n]_q := 1 + q + \dots + q^{n-1}, \quad n = 1, 2, \dots \tag{2}$$

The expressions below are q-variants of factorials and binomial coefficients known as q-factorials and q-binomial coefficients, respectively,

$$[0]_q! := 1, \quad [n]_q! := [1]_q [2]_q \cdots [n]_q, \quad n = 1, 2, \dots,$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

Also, for each $x \in \mathbb{C}$, the q-analogue of the Pochhammer symbol is defined by

$$(x;q)_0 := 1, \quad (x;q)_n := \prod_{j=0}^{n-1}(1 - xq^j).$$

Definition 2. *[20] Let n and s be integers such that $0 \leqslant s < n/2$. Then, for $0 < q < 1$ and $f \in C[0,1]$, the q-Stancu operator, $L_{n,s,q} : C[0,1] \to C[0,1]$, is given by*

$$L_{n,s,q}(f;x) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) b_{n,k,s}(q;x),$$

where

$$b_{n,k,s}(q;x) = \begin{cases} (1 - q^{n-k-s}x)p_{n-s,k}(q;x), & 0 \leqslant k < s, \\ (1 - q^{n-k-s}x)p_{n-s,k}(q;x) + q^{n-k}xp_{n-s,k-s}(q;x), & s \leqslant k \leqslant n - s, \\ q^{n-k}xp_{n-s,k-s}(q;x), & n - s < k \leqslant n, \end{cases}$$

and

$$p_{n,k}(q;x) = \begin{bmatrix} n \\ k \end{bmatrix}_q x^k (x;q)_{n-k}, \quad k = 0, 1, \ldots, n. \tag{3}$$

The polynomials (3) are known as q-Bernstein basis polynomials.

Along with changing index as $k - s = i$ in the sum and then denoting again the summation index by k, it becomes evident that the operator can be represented for $n = 1, 2, \ldots$, as follows:

$$L_{n,s,q}(f;x) = \sum_{k=0}^{n-s} \left\{ (1 - q^{n-k-s}x) f\left(\frac{[k]_q}{[n]_q}\right) + q^{n-k-s}xf\left(\frac{[k+s]_q}{[n]_q}\right) \right\} p_{n-s,k}(q;x). \tag{4}$$

See [20, formula (1.2)].

Note that, q-Stancu operator reduces to the classical Stancu operator, as introduced in [17], when q is set to 1. Additionally, in the cases where $s = 0, 1$, the operators $L_{n,s,q}$ coincide with the q-Bernstein operators defined by Phillips [15]. Furthermore, this operator possesses some properties of the q-Bernstein polynomials. In the case $0 < q < 1$, q-Stancu operator is a positive linear operator, while in the case $q > 1$, it is not. This operator enjoys the end-point interpolation property, that is, $L_{n,s,q}(f;0) = f(0)$ and $L_{n,s,q}(f;1) = f(1)$ for all $q > 0$. Due to $L_{n,s,q}(1;x) = 1$ and $L_{n,s,q}(t;x) = x$, the q-Stancu operator leaves the linear functions invariant.

The eigenvalues and eigenvectors of linear operators are important issues in the applications of linear algebra to the theory of algorithms, the theory of Markov chains and computer science. The spectral theory of linear operators is also used extensively in other disciplines, like quantum mechanics and the field theory, see, e.g., [11] and [21]. Even though quantum systems are generally described in L^2 spaces of infinite dimensions, the quantum perturbation theory routinely uses their finite-dimensional approximations, see, e.g., [11, Chapter 5]. Apart from that, eigenvalues and eigenvectors are used in the theory of parametric excitation of oscillating systems, see [10, Section 27].

The present paper is devoted to examining the eigenvalues and the eigenfunctions of the q-Stancu operators $L_{n,s,q}$. The structure of this paper is as follows: In Section 1, some preliminary results that will be used through the paper and the explicit formula for the moments of all orders for the q-Stancu operator are provided. Hitherto, only the first three moments have been calculated. Section 2 focuses on the eigenvalues and the corresponding eigenfunctions of the q-Stancu operator. It is demonstrated that while $\xi = 1$ is a double eigenvalue, the others are simple. In the last section, the eigenvectors are graphically illustrated for selected values of parameters.

2. Moments of q-Stancu Operators

The calculation of the moments of linear positive operators plays a significant role when studying their approximation properties. Regarding the q-generalization of the Stancu operator $L_{n,s,q}$, only the first three moments, $L_{n,s,q}(e_i;x)$, where $e_i = t^i$, $i = 0,1,2$ have been found so far, see [20, Proposition 2]. In this section, explicit formulae for all the moments of the q-Stancu operators will be presented through moments of the q-Bernstein operator. To begin with, let us provide the essential details regarding the q-Bernstein operator.

The explicit form of the moments of $B_{n,q}$, mentioned in [5, formula (2.4)], is provided below:

$$B_{n,q}(e_k;x) = \sum_{i=0}^{k} \frac{S_q(k,i)}{[n]_q^{k-i}} \lambda_{i,q}^{(n)} x^i, \tag{5}$$

where $S_q(i,j)$ is defined as the q-Stirling numbers of the second kind [5, formula (2.5)] as follows,

$$S_q(i,j) = \frac{1}{[j]_q!\, q^{j(j-1)/2}} \sum_{r=0}^{j} (-1)^r q^{r(r-1)/2} \begin{bmatrix} j \\ r \end{bmatrix}_q [j-r]_q^i,$$

with $S_q(0,0) = 1$, $S_q(i,0) = 0$ for $i > 0$, $S_q(i,j) = 0$ for $j > i$.

Here are the eigenvalues $\lambda_{m,q}^{(n)}$ of the q-Bernstein operator [13]: for $m = 2,3,\dots,n$,

$$\lambda_{0,q}^{(n)} = \lambda_{1,q}^{(n)} = 1, \quad \lambda_{m,q}^{(n)} = \left(1 - \frac{1}{[n]_q}\right)\left(1 - \frac{[2]_q}{[n]_q}\right)\cdots\left(1 - \frac{[m-1]_q}{[n]_q}\right).$$

Theorem 1. *For $m = 1,2\dots$, there holds*

$$L_{n,s,q}(e_m;x) = \sum_{r=1}^{m} a_{n,s,q}(r,m) x^r, \tag{6}$$

where

$$a_{n,s,q}(r,m) = \frac{[n-s]_q^r}{[n]_q^m} S_q(m,r)\lambda_{r,q}^{(n-s)} + \sum_{j=1}^{m-r+1}\sum_{i=0}^{j-1} A(j,i,r-1) + \sum_{j=m-r+2}^{m}\sum_{i=r-1-m+j}^{j-1} A(j,i,r-1) \quad (7)$$

and

$$A(j,i,r) = \binom{m}{j}\binom{j-1}{i} q^{n-s}(-1)^i(1-q)^i \frac{[n-s]_q^r[s]_q^j}{[n]_q^m} S_q(m-j+i,r)\lambda_{r,q}^{(n-s)}. \quad (8)$$

Proof. From the definition (4), one has

$$\begin{aligned} L_{n,s,q}(e_m;x) &= \sum_{k=0}^{n-s}\left\{(1-q^{n-k-s}x)\left(\frac{[k]_q}{[n]_q}\right)^m + q^{n-k-s}x\left(\frac{[k+s]_q}{[n]_q}\right)^m\right\} p_{n-s,k}(q;x) \\ &= \sum_{k=0}^{n-s}\left(\frac{[k]_q}{[n]_q}\right)^m p_{n-s,k}(q;x) + \frac{x}{[n]_q^m}\sum_{k=0}^{n-s} q^{n-k-s}\left([k+s]_q^m - [k]_q^m\right) p_{n-s,k}(q;x) \\ &= \left(\frac{[n-s]_q}{[n]_q}\right)^m B_{n-s,q}(e_m;x) + \frac{x}{[n]_q^m}\sum_{k=0}^{n-s} q^{n-k-s}\left([k+s]_q^m - [k]_q^m\right) p_{n-s,k}(q;x). \end{aligned}$$

Using the relation $[k+s]_q = [k]_q + q^k[s]_q$ and the binomial expansion formula in the second sum, we get

$$\begin{aligned} &L_{n,s,q}(e_m;x) \\ &= \left(\frac{[n-s]_q}{[n]_q}\right)^m B_{n-s,q}(e_m;x) + \frac{x}{[n]_q^m}\sum_{k=0}^{n-s}\sum_{j=1}^{m}\binom{m}{j}[k]_q^{m-j}q^{n-k-s}\left(q^k[s]_q\right)^j p_{n-s,k}(q;x) \\ &= \left(\frac{[n-s]_q}{[n]_q}\right)^m B_{n-s,q}(e_m;x) + \frac{q^{n-s}x}{[n]_q^m}\sum_{j=1}^{m}\binom{m}{j}[s]_q^j\sum_{k=0}^{n-s}[k]_q^{m-j}q^{k(j-1)}p_{n-s,k}(q;x). \end{aligned}$$

Applying $q^k = 1-(1-q)[k]_q$, one can write

$$L_{n,s,q}(e_m;x) = \left(\frac{[n-s]_q}{[n]_q}\right)^m B_{n-s,q}(e_m;x) + \frac{q^{n-s}x}{[n]_q^m}\sum_{j=1}^{m}\binom{m}{j}[s]_q^j\sum_{k=0}^{n-s}[k]_q^{m-j}(1-(1-q)[k]_q)^{j-1}p_{n-s,k}(q;x).$$

With the use of the binomial theorem,

$$L_{n,s,q}(e_m;x) = \left(\frac{[n-s]_q}{[n]_q}\right)^m B_{n-s,q}(e_m;x) + \frac{q^{n-s}x}{[n]_q^m}\sum_{j=1}^{m}\binom{m}{j}[s]_q^j\sum_{i=0}^{j-1}\binom{j-1}{i}(-1)^i(1-q)^i\sum_{k=0}^{n-s}[k]_q^{m-j+i}p_{n-s,k}(q;x)$$

$$= \left(\frac{[n-s]_q}{[n]_q}\right)^m B_{n-s,q}(e_m;x) + \frac{q^{n-s}x}{[n]_q^m} \sum_{j=1}^{m} \binom{m}{j} [s]_q^j$$
$$\times \sum_{i=0}^{j-1} \binom{j-1}{i} (-1)^i (1-q)^i [n-s]_q^{m-j+i} B_{n-s,q}(e_{m-j+i};x).$$

Employing (5), one obtains

$$L_{n,s,q}(e_m;x) = \frac{1}{[n]_q^m} \sum_{r=0}^{m} [n-s]_q^r S_q(m,r) \lambda_{r,q}^{(n-s)} x^r$$
$$+ \sum_{j=1}^{m} \sum_{i=0}^{j-1} \sum_{r=0}^{m-j+i} \binom{m}{j} \binom{j-1}{i} q^{n-s} (-1)^i (1-q)^i \frac{[n-s]_q^r [s]_q^j}{[n]_q^m} S_q(m-j+i,r) \lambda_{r,q}^{(n-s)} x^{r+1}.$$

Changing the order of triple sums leads to:

$$\sum_{j=1}^{m} \sum_{i=0}^{j-1} \sum_{r=0}^{m-j+i} A(j,i,r) = \sum_{r=0}^{m-1} \sum_{j=1}^{m-r} \sum_{i=0}^{j-1} A(j,i,r) + \sum_{r=1}^{m-1} \sum_{j=m-r+1}^{m} \sum_{i=r-m+j}^{j-1} A(j,i,r),$$

which allows us to write

$$L_{n,s,q}(e_m;x) = \frac{1}{[n]_q^m} \sum_{r=0}^{m} [n-s]_q^r S_q(m,r) \lambda_{r,q}^{(n-s)} x^r$$
$$+ \sum_{r=0}^{m-1} \sum_{j=1}^{m-r} \sum_{i=0}^{j-1} A(j,i,r) x^{r+1} + \sum_{r=1}^{m-1} \sum_{j=m-r+1}^{m} \sum_{i=r-m+j}^{j-1} A(j,i,r) x^{r+1},$$

where $A(j,i,r)$ is given by (8). The first sum can be started from $r=1$ due to the equality $S_q(i,0)=0$ for $i>0$. In the last triple sum, an empty sum is obtained for $r=0$, so it can be started from zero. Additionally, if we make the shift of index $r \mapsto r-1$ in the second and third sums, we arrive at:

$$L_{n,s,q}(e_m;x) = \frac{1}{[n]_q^m} \sum_{r=1}^{m} [n-s]_q^r S_q(m,r) \lambda_{r,q}^{(n-s)} x^r$$
$$+ \sum_{r=1}^{m} \sum_{j=1}^{m-r+1} \sum_{i=0}^{j-1} A(j,i,r-1) x^r + \sum_{r=1}^{m} \sum_{j=m-r+2}^{m} \sum_{i=r-1-m+j}^{j-1} A(j,i,r-1) x^r$$
$$=: \sum_{r=1}^{m} a_{n,s,q}(r,m) x^r,$$

where the coefficients $a_{n,s,q}(r,m)$ are as in (7). This completes the proof. □

Remark 1. *It should be noted that the expression for $m=1,2$ in (6) recovers the same result as the ones in [20, Proposition 2].*

3. Spectrum of the q-Stancu Operator

In this section, we will investigate the spectral properties of the q-Stancu operator, including its eigenvalues and associated eigenvectors. In the next theorem, we will prove that, similar to the q-Bernstein operators, the subsequent eigenvalues, excluding the first two, will be found as simple eigenvalues.

Theorem 2. *For all $0 < q < 1$, the q-Stancu operator owns $n-s+1$ eigenvalues $\xi_{m,q}^{(n,s)}$ expressed as*

$$\xi_{0,q}^{(n,s)} = \xi_{1,q}^{(n,s)} = 1,$$

$$\xi_{m,q}^{(n,s)} = \frac{[n-s]_q^{m-1}}{[n]_q^m}\left([n-s]_q - [m-1]_q + q^{n-s}[ms]_q\right)\lambda_{m-1,q}^{(n-s)}, \quad m = 2,3,\ldots,n-s.$$

Moreover, they obey the following order:

$$1 = \xi_{0,q}^{(n,s)} = \xi_{1,q}^{(n,s)} > \xi_{2,q}^{(n,s)} > \xi_{3,q}^{(n,s)} > \ldots > \xi_{n-s,q}^{(n,s)} > 0.$$

Proof. The polynomial $L_{n,s,q}(e_m;x)$ can be written as

$$L_{n,s,q}(e_m;x) = \xi_{m,q}^{(n,s)} x^m + P_{m-1}(x), \tag{9}$$

where

$$\xi_{m,q}^{(n,s)} = a_{n,s,q}(m,m) = \frac{[n-s]_q^{m-1}}{[n]_q^m}\left([n-s]_q - [m-1]_q + q^{n-s}[ms]_q\right)\lambda_{m-1,q}^{(n-s)},$$

and P_{m-1} is a polynomial of degree at most $m-1$.

By (9), the matrix representation of $L_{n,s,q}$ in the standard basis $\{1, x, x^2, \ldots, x^n\}$ is an upper triangular matrix, whose diagonal entries are $\{\xi_{m,q}^{(n,s)}\}$. Therefore, the numbers $\{\xi_{m,q}^{(n,s)}\}$, $m = 0, \ldots, n-s$ are the eigenvalues of $L_{n,s,q}$.

Next, let us demonstrate that the sequence $\{\xi_{m,q}^{(n,s)}\}_{m\geqslant 1}$ is monotonically decreasing. Obviously,

$$\begin{aligned}
\frac{\xi_{m+1,q}^{(n,s)}}{\xi_{m,q}^{(n,s)}} &= \frac{\frac{[n-s]_q^m}{[n]_q^{m+1}}\left([n-s]_q - [m]_q + q^{n-s}[(m+1)s]_q\right)\lambda_{m,q}^{(n-s)}}{\frac{[n-s]_q^{m-1}}{[n]_q^m}\left([n-s]_q - [m-1]_q + q^{n-s}[ms]_q\right)\lambda_{m-1,q}^{(n-s)}} \\
&= \frac{[n-s]_q}{[n]_q}\left(1 - \frac{[m-1]_q}{[n-s]_q}\right)\frac{[n-s]_q - [m]_q + q^{n-s}[(m+1)s]_q}{[n-s]_q - [m-1]_q + q^{n-s}[ms]_q} \\
&= \frac{[n-s]_q - [m-1]_q}{[n]_q} \cdot \frac{[n-s]_q - [m]_q + q^{n-s}[(m+1)s]_q}{[n-s]_q - [m-1]_q + q^{n-s}[ms]_q} \\
&= \frac{[n-s]_q - [m-1]_q}{[n]_q} \cdot \frac{[n+ms]_q - [m]_q}{[n+ms-s]_q - [m-1]_q} \\
&= \frac{q^{m-1} - q^{n-s}}{1-q^n} \cdot \frac{q^m - q^{n+ms}}{q^{m-1} - q^{n+ms-s}}.
\end{aligned}$$

In order to prove that $\xi_{m,q}^{(n,s)}$ is monotonically decreasing, one needs to show

$$\frac{\xi_{m+1,q}^{(n,s)}}{\xi_{m,q}^{(n,s)}} = \frac{q^{m-1} - q^{n-s}}{1 - q^n} \cdot \frac{q^m - q^{n+ms}}{q^{m-1} - q^{n+ms-s}} < 1,$$

which means that

$$\begin{aligned}(1 - q^n)&(q^{m-1} - q^{n+ms-s}) - (q^{m-1} - q^{n-s})(q^m - q^{n+ms}) \\ &= q^{m-1} - q^{n+ms-s} - q^{n+m-1} - q^{2m-1} + q^{n+ms+m-1} + q^{n+m-s} > 0.\end{aligned}$$

Dividing both sides by q^{m-1}, the latter inequality takes the form

$$1 - q^{n+ms-s-m+1} - q^n - q^m + q^{n+ms} + q^{n-s+1} > 0.$$

Adding and subtracting q^{n+m} on the left side of the inequality and making some simplifications, one gets

$$\begin{aligned}&1 - q^{n+ms-s-m+1} - q^n - q^m + q^{n+ms} + q^{n-s+1} + q^{n+m} - q^{n+m} > 0 \\ \Leftrightarrow&(1 - q^n) - q^m(1 - q^n) + q^{n+m}(-q^{ms-s-2m+1} + q^{ms-m} + q^{-m-s+1} - 1) > 0 \\ \Leftrightarrow&(1 - q^n)(1 - q^m) + q^{n+m}(q^{ms-m}(1 - q^{-m-s+1}) - (1 - q^{-m-s+1}) > 0 \\ \Leftrightarrow&(1 - q^n)(1 - q^m) + q^{n+m}(q^{ms-m} - 1)(1 - q^{-m-s+1}) > 0 \\ \Leftrightarrow&(1 - q^n)(1 - q^m) + q^{n-s+1}(1 - q^{ms-m})(1 - q^{m+s-1}) > 0,\end{aligned}$$

which yields that, for $s \geqslant 1$ and $m \geqslant 1$, the sequence $\{\xi_{m,q}^{(n,s)}\}_{m\geqslant 1}$ is decreasing, implying that the numbers $\xi_{m,q}^{(n,s)}$, $m = 1, \dots, n-s$ are distinct. □

Remark 2. *It is worth mentioning that* $\{\xi_{m,1}^{(n,s)}\}_{m=0}^{n-s}$ *are the eigenvalues of the classical Stancu operator found in [17, Theorem 5.1]. Additionally, when* $s = 0$ *or* $s = 1$*, we obtain the eigenvalues of the q-Bernstein operator defined by Phillips [15], and, accordingly, when q equals 1, we recover the eigenvalues of the classical Bernstein operator given in [4].*

Theorem 3. *For* $n \in \mathbb{N}$ *and* $m = 0, 1, \dots, n-s$*, the monic polynomials* $\varphi_m^{(n,s)}(q;x)$*, which are the eigenfunctions of* $L_{n,s,q}(f;x)$ *associated with the eigenvalues* $\xi_{m,q}^{(n,s)}$*, are given by*

$$\varphi_m^{(n,s)}(q;x) = \sum_{u=0}^{m} d_{n,s,q}(u,m)x^u,$$

where $d_{n,s,q}(m,m) = 1$ *and* $\varphi_0^{(n,s)}(q;x) = 1$*,* $\varphi_1^{(n,s)}(q;x) = x$*, while for* $m > 1$ *and* $v = 1, 2, \dots, m$*,*

$$d_{n,s,q}(m-v,m) = \frac{1}{\xi_{m,q}^{(n,s)} - \xi_{m-v,q}^{(n,s)}} \sum_{u=0}^{v-1} d_{n,s,q}(m-u,m)a_{n,s,q}(m-v,m-u).$$

Proof. Consider the monic eigenfunctions of $L_{n,s,q}(f;x)$:

$$\varphi_m^{(n,s)}(q;x) = \sum_{u=0}^{m} d_{n,s,q}(u,m)x^u, \quad d_{n,s,q}(m,m) := 1, \tag{10}$$

corresponding to the eigenvalue $\xi_{m,q}^{(n,s)}$. Then,

$$L_{n,s,q}(\varphi_m^{(n,s)}(q;x);x) = \xi_{m,q}^{(n,s)}\varphi_m^{(n,s)}(q;x). \tag{11}$$

Taking expression (10) into account, (11) can be written as

$$\begin{aligned}\xi_{m,q}^{(n,s)}\sum_{v=0}^{m} d_{n,s,q}(v,m)x^v &= \sum_{u=0}^{m} d_{n,s,q}(u,m)L_{n,s,q}(t^u;x)\\ &= \sum_{u=0}^{m} d_{n,s,q}(u,m)\sum_{v=1}^{u} a_{n,s,q}(v,u)x^v = \sum_{v=1}^{m}\sum_{u=v}^{m} d_{n,s,q}(u,m)a_{n,s,q}(v,u)x^v.\end{aligned}$$

Comparing the coefficient of x^s in both sides results in

$$\xi_{m,q}^{(n,s)}d_{n,s,q}(v,m) = \sum_{u=v}^{m} d_{n,s,q}(u,m)a_{n,s,q}(v,u).$$

Substituting v with $m-v$ and u with $m-u$ leads to

$$\xi_{m,q}^{(n,s)}d_{n,s,q}(m-v,m) = \sum_{u=0}^{v} d_{n,s,q}(m-u,m)a_{n,s,q}(m-v,m-u),$$

resulting

$$d_{n,s,q}(m-v,m) = \frac{1}{\xi_{m,q}^{(n,s)} - \xi_{m-v,q}^{(n,s)}}\sum_{u=0}^{v-1} d_{n,s,q}(m-u,m)a_{n,s,q}(m-v,m-u),$$

which completes the proof. □

As an application of this theorem, the following result on the convergence of the iterates can be stated.

Corollary 1. *Let $0 < q < 1$, $f \in C[0,1]$ and $L_{n,s,q}^m$ stand for the m-th iterate of $L_{n,s,q}$, which is defined by $L_{n,s,q}^1(f;x) = L_{n,s,q}(f;x)$,*

$$L_{n,s,q}^m(f;x) = L_{n,s,q}\left(L_{n,s,q}^{m-1}(f;x)\right), \quad m = 2,3,\ldots$$

Then, for fixed n and s,

$$\lim_{m\to\infty} L_{n,s,q}^m(f;x) = f(0)(1-x) + f(1)x$$

and the convergence is uniform on $[0,1]$.

4. Numerical Examples

In this part, we will present the visual representation of the eigenfunctions $\varphi_m^{(n,s)}(q;x)$ for some specific parameter values. Figure 1 illustrates the eigenfunctions $\varphi_m^{(9,3)}(q;x)$ for $m = 0, 1, \ldots, 6$ normalized to establish a uniform norm 1. Figure 2 shows how the eigenfunctions $\varphi_3^{(n,4)}(q;x)$ behave as the parameter n varies, whereas Figure 3 displays the eigenfunctions $\varphi_5^{(15,s)}(q;x)$ for different values of s. In Figure 4, while keeping all parameters fixed except for q, the eigenfunctions $\varphi_3^{(10,4)}(q;x)$ are demonstrated with respect to the varying values of q.

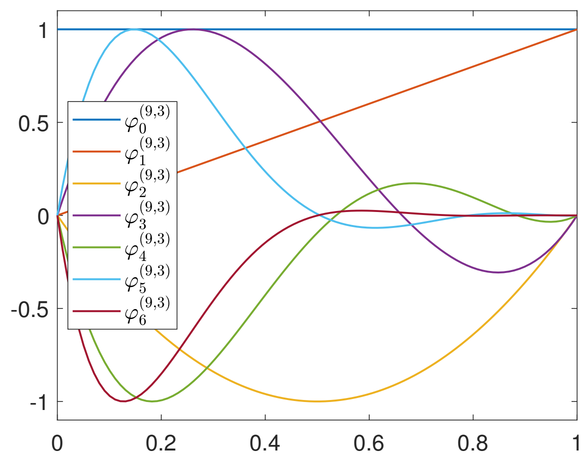

FIGURE 1. The normalized eigenfunctions of $L_{9,3,q}$ for $q = 0.5$.

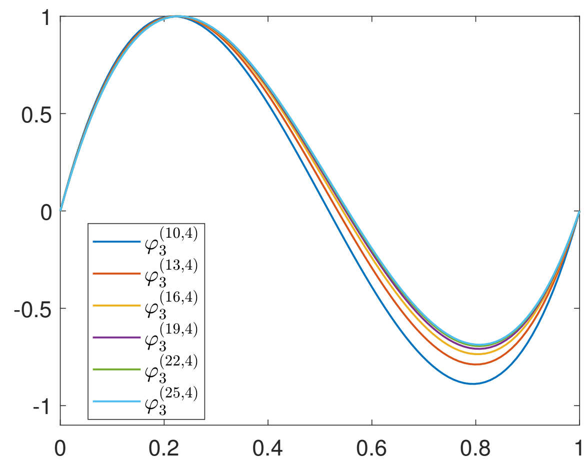

FIGURE 2. The eigenfunctions $\varphi_3^{(n,4)}(q;x)$ for different values of n and $q = 0.8$.

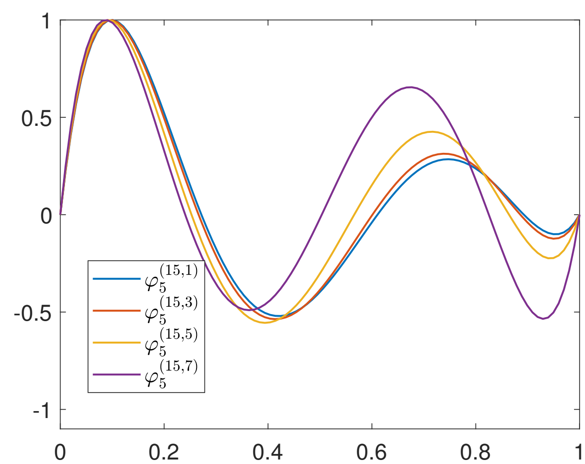

FIGURE 3. The eigenfunctions $\varphi_5^{(15,s)}(q;x)$ for different values of s and $q = 0.8$.

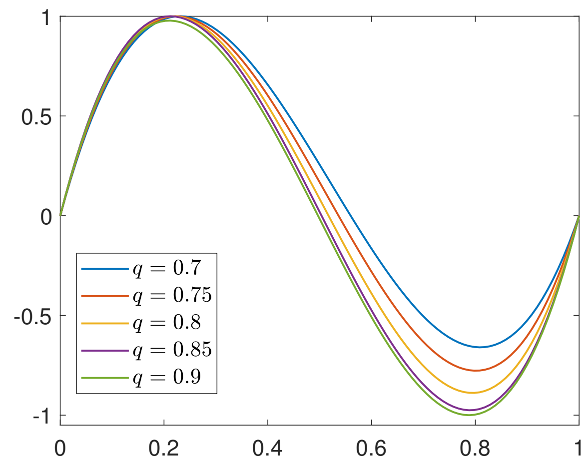

FIGURE 4. The eigenfunctions $\varphi_3^{(10,4)}(q;x)$ for different values of q.

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