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NEUTROSOPHIC SEPERATION AXIOMS

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Abstract. This study is dedicated to make an attempt to define different types of separation axioms in neutrosophic topological spaces. The relationships among them are shown with a diagram and counterexamples. We also introduce some new terms such as introduced neutrosophic topology, neutrosophic regular space, neutrosophic normal space, neutrosophic subspace.

1. Introduction

Undoubtedly, the concept of separation axioms has always been an indispensable character in the world of topology. This concept formed the basis of many valuable researches in general topology. And, these researches played very important roles in many parts of real life and the findings of these researches came to life in many applications. But, as technology advances and the industry evolves, peoples needs have changed and general topology has become inadequate in real life. So, the impact of these findings on real life has diminished. Then,scientists went on to find different types of topological spaces and separation axioms occupied an important place in these topological spaces. In [17], Smarandache offered the concept of neutrosophic set. This idea became the leading actor in numerous studies as in [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 16]. Also, by using this new concept, Salma and Alblowi introduced the theory of neutrosophic topological space in [15]. In this study, we present different types of separation axioms in neutrosophic topological spaces as a new instrument for real life applications and new terms that we think benefit in other investigations.Throughout the paper, without any explanation, we use the symbols and definitions introduced in [13, 15, 17]. For the sake of shortness we use the notation N instead of neutrosophic.

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2. Some Required Definitions

In this section, we give newest predefined definitions that will be required in the next section. Also, some new definitions are given.

Definition 2.1. [4] *An N-point $x_{r,t,s}$ is said to be N-quasi-coincident (N-q-coincident, for short) with F, denoted by $x_{r,t,s}qF$ iff $x_{r,t,s} \subseteq F^c$. If $x_{r,t,s}$ is not N-quasi-coincident with F, we denote by $x_{r,t,s}\tilde{q}F$.*

Definition 2.2. [4] *An N-set F in an N-topological space (X, τ) is said to be an N-q-neighborhood of an N-point $x_{r,t,s}$ iff there exists an N-open set G such that $x_{r,t,s}qG \subset F$.*

Definition 2.3. [4] *An N-set G is said to be N-quasi-coincident (N-q-coincident, for short) with F, denoted by GqF iff $G \nsubseteq F^c$. If G is not N-quasi-coincident with F, we denote by $G\tilde{q}F$.*

Definition 2.4. *Consider that (X, τ) is an N-topological space and $Y \subseteq X$. Let H be an N-set over Y such that*

$$T_H(x) = \begin{cases} 1, & x \in Y \\ 0, & x \notin Y \end{cases}, \quad I_H(x) = \begin{cases} 1, & x \in Y \\ 0, & x \notin Y \end{cases}, \quad F_H(x) = \begin{cases} 0, & x \in Y \\ 1, & x \notin Y \end{cases}$$

Consider that $\tau_Y = \{H \cap F : F \in \tau\}$, then (Y, τ_Y) is called N-subspace of (X, τ). If $H \in \tau$ (resp. $H^c \in \tau$), then (Y, τ_Y) is called N-open (resp. closed) subspace of (X, τ).

Definition 2.5. [4] *An N-point $x_{r,t,s}$ is said to be an N-cluster point of an N-set F iff every N-open q-neighborhood G of $x_{r,t,s}$ is q-coincident with F. The union of all N-cluster points of F is called the N-closure of F and denoted by $\overline{F}$.*

Definition 2.6. [4] *Consider that f is a function from X to Y. Let A be an N-set in X with membership funtion $T_A(x)$, indeterminacy function $I_A(x)$ and non-membership function $F_A(x)$.The image of A under f, written as $f(A)$, is an N-subset of Y whose membership function, indeterminacy function and non-membership function are defined as*

$$T_{f(A)}(y) = \begin{cases} \sup_{z \in f^{-1}(y)}\{T_A(z)\} & \text{, if } f^{-1}(y) \text{ is not empty,} \\ 0 & \text{, if } f^{-1}(y) \text{ is empty,} \end{cases}$$

$$I_{f(A)}(y) = \begin{cases} \sup_{z \in f^{-1}(y)}\{I_A(z)\} & \text{, if } f^{-1}(y) \text{ is not empty,} \\ 0 & \text{, if } f^{-1}(y) \text{ is empty,} \end{cases}$$

$$F_{f(A)}(y) = \begin{cases} \inf_{z \in f^{-1}(y)}\{F_A(z)\} & \text{, if } f^{-1}(y) \text{ is not empty,} \\ 1 & \text{, if } f^{-1}(y) \text{ is empty,} \end{cases}$$

for all y in Y, where $f^{-1}(y) = \{x : f(x) = y\}$, respectively.

If f is a bijective function from X to Y, then it is an invertible N-function. Conversely, consider that B is an N-set in Y with membership funtion $T_B(y)$, indeterminacy function $I_B(y)$ and non-membership function $F_B(y)$. Then, the inverse image of B under f, written as $f^{-1}(B)$, is an N-subset of X whose membership function, indeterminacy function and non-membership function are defined as $T_{f^{-1}(B)}(x) = T_B(f(x))$, $I_{f^{-1}(B)}(x) = I_B(f(x))$ and $F_{f^{-1}(B)}(x) = F_B(f(x))$ for all x in X, respectively.

Definition 2.7. *Consider that (X,τ), (Y,δ) are N-topological spaces and $f : (X,\tau) \to (Y,\delta)$ is an N-function. The function f is said to be N-continuous, if $f^{-1}(G) \in \tau$ for any $G \in \delta$.*

Definition 2.8. *Consider that (X,τ), (Y,δ) are N-topological spaces and $f : (X,\tau) \to (Y,\delta)$ is an N-function. The function f is said to be N-open, if $f(F) \in \delta$ for any $F \in \tau$.*

3. N-T_i-Spaces (i=0, 1, 2)

In this section, we present different types of separation axioms and investigate their properties. Also, the relationships among them are shown with a diagram and counterexamples. Additionally, we analyze their characteristics in N-topological subspaces.

Definition 3.1. *An N-topological space (X,τ) is said to be an N-T_0-space if for every pair of N-points $x_{\alpha,\beta,\gamma}$, $y_{\alpha',\beta',\gamma'}$, whose supports are different, there exist N-open sets F, G such that $x_{\alpha,\beta,\gamma} \in F$, $y_{\alpha',\beta',\gamma'} \in F^c$ or $x_{\alpha,\beta,\gamma} \in G^c$, $y_{\alpha',\beta',\gamma'} \in G$.*

Theorem 3.1. *Consider that (X,τ) is an N-topological space, then (X,τ) is N-T_0-space iff, for any two N-points, $x_{\alpha,\beta,\gamma}$, $y_{\alpha',\beta',\gamma'}$, whose supports are different, $x_{\alpha,\beta,\gamma}\tilde{q}\overline{y_{\alpha',\beta',\gamma'}}$ or $\overline{x_{\alpha,\beta,\gamma}}\tilde{q}y_{\alpha',\beta',\gamma'}$.*

Proof. Consider that (X,τ) is an N-T_0-space and $x_{\alpha,\beta,\gamma}$, $y_{\alpha',\beta',\gamma'}$ are two N-points with different supports. Then, there exist N-open sets F, G such that $x_{\alpha,\beta,\gamma} \in F$, $y_{\alpha',\beta',\gamma'} \in F^c$ or $x_{\alpha,\beta,\gamma} \in G^c$, $y_{\alpha',\beta',\gamma'} \in G$. This implies that $y_{\alpha',\beta',\gamma'}\tilde{q}F$ or $x_{\alpha,\beta,\gamma}\tilde{q}G$. So, $x_{\alpha,\beta,\gamma}\tilde{q}\overline{y_{\alpha',\beta',\gamma'}}$ or $\overline{x_{\alpha,\beta,\gamma}}\tilde{q}y_{\alpha',\beta',\gamma'}$. Let (X,τ) be an N-topological space such that, for any two N-points $x_{\alpha,\beta,\gamma}$, $y_{\alpha',\beta',\gamma'}$ with different supports, $x_{\alpha,\beta,\gamma}\tilde{q}\overline{y_{\alpha',\beta',\gamma'}}$ or $\overline{x_{\alpha,\beta,\gamma}}\tilde{q}y_{\alpha',\beta',\gamma'}$. Then, there exists an N-open set F such that $x_{\alpha,\beta,\gamma} \in F$, $y_{\alpha',\beta',\gamma'}\tilde{q}F$ or there exists an N-open set G such that $y_{\alpha',\beta',\gamma'} \in G$, $x_{\alpha,\beta,\gamma}\tilde{q}G$. This implies that $x_{\alpha,\beta,\gamma} \in F$, $y_{\alpha',\beta',\gamma'} \in F^c$ or $x_{\alpha,\beta,\gamma} \in G^c$, $y_{\alpha',\beta',\gamma'} \in G$. Therefore, (X,τ) is an N-T_0-space. □

Definition 3.2. *An N-topological space (X,τ) is said to be an N-T_1-space if for every pair of N-points $x_{\alpha,\beta,\gamma}$, $y_{\alpha',\beta',\gamma'}$, whose supports are different, there exist N-open sets F, G such that $x_{\alpha,\beta,\gamma} \in F$, $y_{\alpha',\beta',\gamma'} \in F^c$ and $x_{\alpha,\beta,\gamma} \in G^c$, $y_{\alpha',\beta',\gamma'} \in G$.*

Theorem 3.2. *Consider that (X,τ) is an N-topological space, then (X,τ) is N-T_1- space iff, for any two N-points, $x_{\alpha,\beta,\gamma}$, $y_{\alpha',\beta',\gamma'}$, whose supports are different, $x_{\alpha,\beta,\gamma}\tilde{q}\overline{y_{\alpha',\beta',\gamma'}}$ and $\overline{x_{\alpha,\beta,\gamma}}\tilde{q}y_{\alpha',\beta',\gamma'}$.*

Proof. The proof of this theorem is similar to that of above theorem. So, it is omitted. □

Theorem 3.3. *Consider that (X,τ) is an N-topological space. If every N-point $x_{\alpha,\beta,\gamma}$ is N-closed in (X,τ), then (X,τ) is an N-T_1-space.*

Proof. Consider any two N-points $x_{\alpha,\beta,\gamma}$, $y_{\alpha',\beta',\gamma'}$ in (X,τ) such that $x_{\alpha,\beta,\gamma}\tilde{q}y_{\alpha',\beta',\gamma'}$. Then, $x_{\alpha,\beta,\gamma} \subset (y_{\alpha',\beta',\gamma'})^c$ and $y_{\alpha',\beta',\gamma'} \subset (x_{\alpha,\beta,\gamma})^c$ where $(y_{\alpha',\beta',\gamma'})^c$ and $(x_{\alpha,\beta,\gamma})^c$ are N-open sets in (X,τ). Since, $x_{\alpha,\beta,\gamma}\tilde{q}(x_{\alpha,\beta,\gamma})^c$ and $y_{\alpha',\beta',\gamma'}\tilde{q}(y_{\alpha',\beta',\gamma'})^c$, (X,τ) is an N-T_1-space. □

The converse statement is not always true as seen in the example below.

Example 3.1. *Consider the set* $X = \{x, y\}$ *and the family* $\tau = \{\{x_{\alpha,\alpha,1-\alpha}, y_{\beta,\beta,1-\beta}\} : \alpha, \beta \in [0, 1]\}$. *Then,* τ *is an N-topology over* X. *It is easily seen that* (X, τ) *is an N-*T_1*-space. But, the N-point* $x_{0,2,0,2,0,7}$ *is not closed in* τ. *Because,* $x_{0,2,0,2,0,7} \neq \overline{x_{0,2,0,2,0,7}}$.

Definition 3.3. *An N-topological space* τ *is said to be an N-*T_2*-space, if, for every pair of N-points* $x_{\alpha,\beta,\gamma}$, $y_{\alpha',\beta',\gamma'}$, *whose supports are different, there exists N-open sets* F, G *such that* $x_{\alpha,\beta,\gamma} \in F$, $y_{\alpha',\beta',\gamma'} \in F^c$, $y_{\alpha',\beta',\gamma'} \in G$, $x_{\alpha,\beta,\gamma} \in G^c$ *and* $F\tilde{q}G$.

For an N-topological space (X, τ) we have the following diagram:

N-T_2-space
↓
N-T_1-space
↓
N-T_0-space

Converse statements may not be true as shown in the examples below;

Example 3.2. *Consider the set* $X = \{x, y\}$ *and the family*

$$\tau = \{\{x_{\alpha,\alpha,1-\alpha}, y_{\beta,\beta,1-\beta}\} : \alpha \in [0, 1], \beta \in [0, 1)\}.$$

Then, τ *is an N-topology over* X. *It is easily seen that* (X, τ) *is an N-*T_0*-space. But, it is not an N-*T_1*-space. Because,* $x_{1,1,0}$ *and* $y_{1,1,0}$ *are N-points in* (X, τ) *with different supports and the only N-open set that contains* $y_{1,1,0}$ *is* 1_X.

Example 3.3. *Consider that* $X = N$ *is the set of naturel numbers. For any* $n \in N$, $n_{1,1,0}$ *is an N-point. Clearly, there is a one-to-one compatibility between* N *and* $\{n_{1,1,0} : n \in N\}$. *Then, we can define a cofinite topology on* $\{n_{1,1,0} : n \in N\}$. *That is, an N-set* F *is N-open iff it is constituted by discarding a finite number of elements from* $\{n_{1,1,0} : n \in N\}$. *Hence, this cofinite topological space is an N-*T_1*-space. But, it is not an N-*T_2*-space.*

Theorem 3.4. *An N-subspace* (Y, τ_Y) *of an N-*T_i*-space* (X, τ) *is an N-*T_i*-space* $(i = 0, 1, 2)$.

Proof. (Case $i = 0$) Consider that (X, τ) is an N-T_0-space and (Y, τ_Y) is an N-subspace of (X, τ). Take any two N-points $x_{\alpha,\beta,\gamma}$ and $y_{\alpha',\beta',\gamma'}$ in (Y, τ_Y) with different supports. Then, $x_{\alpha,\beta,\gamma}$ and $y_{\alpha',\beta',\gamma'}$ are also N-points in (X, τ). Since (X, τ) is an N-T_0-space, there exist N-sets F and G such that $x_{\alpha,\beta,\gamma} \in F$, $y_{\alpha',\beta',\gamma'} \in F^c$ or $y_{\alpha',\beta',\gamma'} \in G$, $x_{\alpha,\beta,\gamma} \in G^c$. Consider an N-set H as given in Definition 2.4. Then, $F \cap H$ and $G \cap H$ are N-open sets in (Y, τ_Y) such that $x_{\alpha,\beta,\gamma} \in F \cap H$, $y_{\alpha',\beta',\gamma'} \in (F \cap H)^c$ or or $y_{\alpha',\beta',\gamma'} \in G \cap H$, $x_{\alpha,\beta,\gamma} \in (G \cap H)^c$. This implies that (Y, τ_Y) is N-T_0. □

In the other cases in which $i = 1$ and $i = 2$, we can make the proofs in similar ways. So, they are omitted.

4. N-R_i-Spaces (i=0, 1)

In this section, we introduce N-R_0 and N-R_1 spaces. Their connections with N-T_1 and N-T_2 spaces are investigated. Also, we define the concept of N-topological space induced by a topological space and some implications are given in induced N-topological spaces. Additionally, it is shown that inverse statements of these implications are not always true with counter examples.

Definition 4.1. *An N-topological space* (X, τ) *is said to be an N-* R_0*-space iff, for any two N-points* $x_{\alpha,\beta,\gamma}$ *and* $y_{\alpha',\beta',\gamma'}$ *, if* $x_{\alpha,\beta,\gamma}\tilde{q}y_{\alpha',\beta',\gamma'}$ *then* $x_{\alpha,\beta,\gamma}\tilde{q}\overline{y_{\alpha',\beta',\gamma'}}$.

Definition 4.2. *An N-topological space* (X, τ) *is said to be an N-*R_1 *-space iff, for any two N-points* $x_{\alpha,\beta,\gamma}$ *and* $y_{\alpha',\beta',\gamma'}$ *then* $x_{\alpha,\beta,\gamma}$, *if* $x_{\alpha,\beta,\gamma}\tilde{q}\overline{y_{\alpha',\beta',\gamma'}}$ *then there exists two N-open sets F and G in* (X, τ) *such that* $x_{\alpha,\beta,\gamma} \in F$, $y_{\alpha',\beta',\gamma'} \in G$ *and* $F\tilde{q}G$.

Theorem 4.1. *Every N-*T_i*-space* (X, τ) *is an N-*R_{i-1}*-space* $(i = 1, 2)$.

Proof. Obvious. □

Theorem 4.2. *An N-topological space* (X, τ) *is an N-*T_i*-space iff it is N-*T_{i-1} *and N-*R_{i-1} $(i = 1, 2)$.

Proof. (Case $i = 2$) The necessity is obvious from Theorem 4.1 and the diagram given after Definition 3.3. Consider an N-topological space (X, τ) which is N-T_1 and N- R_1. Take two N-points $x_{\alpha,\beta,\gamma}$ and $y_{\alpha',\beta',\gamma'}$ in (X, τ) with different supports. Then, $x_{\alpha,\beta,\gamma}\tilde{q}\overline{y_{\alpha',\beta',\gamma'}}$. Since (X, τ) is N-T_1, $x_{\alpha,\beta,\gamma}\tilde{q}\overline{y_{\alpha',\beta',\gamma'}}$. Then, there exists two N-open sets F and G in (X, τ) such that $x_{\alpha,\beta,\gamma} \in F$, $y_{\alpha',\beta',\gamma'} \in G$ and $F\tilde{q}G$ for (X, τ) is N-R_1. Hence, (X, τ) is N-T_2. In the cases in which $i = 1$, we can make the proof in similar way. So, it is omitted. □

Definition 4.3. *Consider that* (X, τ) *is a topological space and A is a subset of X. Let the N-set* X_A *be whose is with membership function* $T_{X_A}(x)$, *indeterminacy function* $I_{X_A}(x)$ *and non-membership function* $F_{X_A}(x)$ *defined as follows;*

$$T_{X_A}(x) = \begin{cases} 1 & , \text{if } x \in A, \\ 0 & , \text{if } x \notin A, \end{cases}$$

$$I_{X_A}(x) = \begin{cases} 1 & , \text{if } x \in A, \\ 0 & , \text{if } x \notin A, \end{cases}$$

$$F_{X_A}(x) = \begin{cases} 1 & , \text{if } x \in A, \\ 0 & , \text{if } x \notin A, \end{cases}$$

X_A *is called an N-set induced by A and the family* $\delta_\tau = \{X_A : A \in X\}$ *is called an N-topology over X induced by* τ.

Theorem 4.3. *Consider that* (X, τ) *is a topological space and* (X, δ_τ) *is an N-topological space, where* δ_τ *is an N-topology induced by* τ. *If* (X, δ_τ) *is N-*T_0*-space then* (X, τ) *is a* T_0*-space.*

Proof. Take any two distinct points $x, y \in X$. Then, $x_{1,1,0}$ and $y_{1,1,0}$ are two N-points in (X, δ_τ) with different supports. Since (X, δ_τ) is N-T_0-space, there exist N-sets F, G such that $x_{1,1,0} \in F$, $y_{1,1,0} \in F^c$ or $y_{1,1,0} \in G$ and $x_{1,1,0} \in G^c$. Then there exists $0_F \in \tau$ such that $x \in 0_F$, $y \notin 0_F$, where $F = X_{0_F} \in \delta_\tau$ or there exists $0_G \in \tau$ such that $y \in 0_G$,$x \notin 0_G$, where $G = X_{0_G} \in \delta_\tau$. Hence, (X, τ) is a T_0-space. □

Theorem 4.4. *Consider that* (X, τ) *is a topological space and* (X, δ_τ) *is an N-topological space, where* δ_τ *is an N-topology induced by* τ. *If* (X, δ_τ) *is N-*T_1*-space then* (X, τ) *is a* T_1*-space.*

Proof. The proof is similar to that of above theorem. □

The converse statements may not be true as seen in then following examples.

Example 4.1. *Consider that* $X = \{x, y, z\}$. *Then, the family* $\tau = \{\emptyset, X, \{x\}, \{z\}, \{x, z\}\}$ *is a topology over* X *and* $\delta_\tau = \{0_X, 1_X, x_{1,1,0}, z_{1,1,0}, \{x_{1,1,0}, z_{1,1,0}\}\}$ *is an N-topology over* X *induced by* τ. *Then,* (X, τ) *is a* T_0*-space. But,* (X, δ_τ) *is not N-*T_0*-space.*

Example 4.2. *Consider that* $X = \{x\}$. *Then, the family* $\tau = \{\emptyset, X\}$ *is a topology over* X *and* $\delta_\tau = \{0_X, 1_X\}$ *is an N-topology over* X *induced by* τ. *Then,* (X, τ) *is a* T_1*-space. But,* (X, δ_τ) *is not N-*T_1*-space.*

5. N-regular, N-normal and N-T_i-Spaces (i=3, 4)

In this section, we first introduce N-regular spaces and N-normal spaces. Some of their characteristics are given and the relationships with N-R_0 and N-R_1 spaces are investigated. Then, we introduce N-T_3 spaces, N-T_4 spaces and examine their relations.

Definition 5.1. *An N-topological space* (X, τ) *is said to be an N-regular (N-*R_2*-space, for short) space iff, for any N-points* $x_{\alpha,\beta,\gamma}$ *and any N-closed set* H *in* (X, τ) *such that* $x_{\alpha,\beta,\gamma}\tilde{q}H$, *there exists two N-open sets* F *and* G *in* (X, τ) *such that* $x_{\alpha,\beta,\gamma} \in F$, $H \subset G$ *and* $F\tilde{q}G$.

Definition 5.2. *An N-topological space* (X, τ) *is said to be an N-normal (N-*R_3*-space, for short) space iff, for any two N-closed sets* H *and* K *in* (X, τ) *such that* $H\tilde{q}K$, *there exists two N-open sets* F *and* G *in* (X, τ) *such that* $H \subset F$, $K \subset G$ *and* $F\tilde{q}G$.

Theorem 5.1. *Consider that* (X, τ) *and* (Y, δ) *are N-topological spaces and* $f : (X, \tau) \to (Y, \delta)$ *is an N-function which is bijective, N-continuous and N-open. If* (X, τ) *is N-normal, then* (Y, δ) *is also N-normal.*

Proof. Consider that F and G is N-closed sets in (Y, δ) such that $F\tilde{q}G$. Since f is N-continuous, $f^{-1}(F)$ and $f^{-1}(G)$ are also N-closed sets in (X, τ) and $f^{-1}(F)\tilde{q}f^{-1}(G)$. Then, there exists N-open sets K and L such that $f^{-1}(F) \subset K$, $f^{-1}(G) \subset L$ and $K\tilde{q}L$. It follows that $F \subset f(f^{-1}(F)) \subset f(K)$, $G \subset f(f^{-1}(G)) \subset f(L)$ and $f(K)\tilde{q}f(L)$. Since, f is N-open, $f(K)$ and $f(L)$ are N-open sets such that $F \subset f(K)$, $G \subset f(L)$ and $f(K)\tilde{q}f(L)$. Hence, (Y, δ) is N-normal. □

Theorem 5.2. *Consider that* (X, τ) *and* (Y, δ) *are N-topological spaces and* $f : (X, \tau) \to (Y, \delta)$ *is an N-function which is bijective, N-continuous and N-open. If* (X, τ) *is N-regular, then* (Y, δ) *is also N-regular.*

Proof. It is similar. □

Theorem 5.3. *Consider that* (X, τ) *is an N-topological space and* $x_{\alpha,\beta,\gamma}$ *is any N-point in* (X, τ). *Then,* (X, τ) *is an N-*R_2*-space iff, for every N-open set* F *such that* $x_{\alpha,\beta,\gamma} \in F$, *there exists an N-open set* G *such that* $x_{\alpha,\beta,\gamma} \in G$ *and* $\overline{G} \subset F$.

Proof. Consider that (X, τ) is an N-R_2-space and $x_{\alpha,\beta,\gamma}$ is any N-point in (X, τ). Let an N-open set F be in (X, τ) such that $x_{\alpha,\beta,\gamma} \in F$. Then, F^c is an N-closed set in (X, τ). It is clear that $F\tilde{q}F^c$. Since $x_{\alpha,\beta,\gamma} \in F$, $x_{\alpha,\beta,\gamma}\tilde{q}F^c$. There exist N-open sets G and H such that $x_{\alpha,\beta,\gamma} \in G$, $F^c \subset H$ and $G\tilde{q}H$. This implies that $G \subset H^c$. Since H^c is an N-closed set in (X, τ), $G \subset H^c$. Conversely, let $x_{\alpha,\beta,\gamma}$ be an N-point in (X, τ) and F be an N-closed set such that $x_{\alpha,\beta,\gamma}\tilde{q}F$. Then, $x_{\alpha,\beta,\gamma} \in F^c$ and F^c

is an N-open set in (X, τ). From our hypothesis, there exists an N-open set G such that $x_{\alpha,\beta,\gamma} \in G$ and $\overline{G} \subset F^c$. So, $F \subset (\overline{G})^c$ and $G\tilde{q}(\overline{G})^c$. Clearly, $G\tilde{q}(\overline{G})^c$. This implies that (X, τ) is an N-R_2-space. □

Theorem 5.4. *Consider that (X, τ) is an N-topological space. Then, (X, τ) is an N-R_2-space iff, for every N-point $x_{\alpha,\beta,\gamma}$ and N-closed set F in (X, τ) such that $x_{\alpha,\beta,\gamma}\tilde{q}F$, there exist N-open sets G and H such that $x_{\alpha,\beta,\gamma} \in G$, $F \subset H$ and $\overline{G}\tilde{q}\overline{H}$.*

Proof. Consider that (X, τ) is an N-R_2-space. Take an N-point $x_{\alpha,\beta,\gamma}$ and an N-closed set F in (X, τ) such that $x_{\alpha,\beta,\gamma}\tilde{q}F$. Then, there exist N-open sets G and H such that $x_{\alpha,\beta,\gamma} \in G$, $F \subset H$ and $G\tilde{q}H$. So, $H \subset G^c$. Since G^c is N-closed in (X, τ), $\overline{H} \subset G^c$. Clearly, $\overline{H}\tilde{q}G$. It is easily seen that $x_{\alpha,\beta,\gamma}\tilde{q}H$. Since (X, τ) is an N-R_2-space, there exist N-open sets K and L such that $x_{\alpha,\beta,\gamma} \in K$, $\overline{H} \subset L$ and $K\tilde{q}L$. So, $K \subset L^c$. Since L^c is N-closed in (X, τ), $\overline{K} \subset L^c$. Clearly, $\overline{K}\tilde{q}L$. Therefore, $\overline{K}\tilde{q}\overline{H}$.
The proof of the converse statement is obvious. So, it is omitted. □

Theorem 5.5. *Consider that (X, τ) is an N-topological space and F is any N-closed set in (X, τ). Then, (X, τ) is an N-R_3-space iff, for every N-open set G such that $F \subset G$, there exists an N-open set H such that $F \subset H$ and $\overline{H} \subset G$.*

Proof. The proof is analogous to that of Theorem 5.3. □

Theorem 5.6. *Consider that (X, τ) is an N-topological space. Then, (X, τ) is an N-R_3-space iff, for every N-closed sets F, G in (X, τ) such that $F\tilde{q}G$, there exist N-open sets K and H such that $F \subset K$, $G \subset H$ and $\overline{K}\tilde{q}\overline{H}$.*

Proof. The proof is analogous to that of Theorem 5.4. □

Theorem 5.7. *Consider that (X, τ) is an N-topological space. If (X, τ) is an N-R_2 then it is an N-R_1-space.*

Proof. It is obvious. □

Theorem 5.8. *Consider that (X, τ) is an N-topological space. If (X, τ) is an N-R_3 and N-R_0- space then it is an N-R_2-space.*

Proof. Consider that (X, τ) is an N-R_3 and N-R_0-space. Take an N-point $x_{\alpha,\beta,\gamma}$ and an N-closed set F in (X, τ) such that $x_{\alpha,\beta,\gamma}\tilde{q}F$. Since (X, τ) is an N-R_0-space, $\overline{x_{\alpha,\beta,\gamma}}\tilde{q}F$. Since (X, τ) is an N-R_3-space, there exist N-open sets G and H such that $\overline{x_{\alpha,\beta,\gamma}} \subset G$, $F \subset H$ and $G\tilde{q}H$. Hence, (X, τ) is an N-R_2-space. □

Corollary 5.9. *Let (X, τ) be an N-topological space. If (X, τ) is an N-R_3 and N-R_0-space then it is an N-R_1-space.*

Proof. It follows from Theorem 5.7 and Theorem 5.8. □

Definition 5.3. *An N-topological space (X, τ) is said to be an N-T_3-space iff, it is both an N-R_2 and N-T_1-space.*

Definition 5.4. *An N-topological space (X, τ) is said to be an N-T_4-space iff, it is both an N-R_3 and N-T_1-space.*

Theorem 5.10. *Consider that (X, τ) is an N-topological space. If (X, τ) is an N-T_4 - space then it is an N-T_3-space.*

Proof. Consider that (X,τ) is an N-T_4-space. Then, it is both N-R_3 and N-T_1. From Theorem 4.1, it is N-R_0. Take an N-point $x_{\alpha,\beta,\gamma}$ and an N-closed set F such that $x_{\alpha,\beta,\gamma}\tilde{q}F$. This implies that $\overline{x_{\alpha,\beta,\gamma}}\tilde{q}F$. Then, there exist N-open sets G and H such that $\overline{x_{\alpha,\beta,\gamma}} \subset G$, $F \subset H$ and $G\tilde{q}H$. Thus, (X,τ) is N-R_2. Hence, we obtain the result. □

Theorem 5.11. *Consider that (X,τ) is an N-topological space. If (X,τ) is an N-T_3-space then it is an N-T_2-space.*

Proof. It follows from Theorem 5.7 and Theorem 4.2. □

Theorem 5.12. *An N-subspace (Y,τ_Y) of an N-T_3-space (X,τ) is N-T_3.*

Proof. Consider that (X,τ) is an N-T_3-space, $Y \subseteq X$ and (Y,τ_Y) is an N-subspace as described in Definition 2.4. Let $x_{\alpha,\beta,\gamma}$ and $y_{\alpha',\beta',\gamma'}$ in (Y,τ_Y) be N-points in (Y,τ_Y) such that $x_{\alpha,\beta,\gamma}\tilde{q}y_{\alpha',\beta',\gamma'}$. It is obvious that $x_{\alpha,\beta,\gamma}$ and $y_{\alpha',\beta',\gamma'}$ are also N-points in (X,τ). Since (X,τ) is an N-T_1-space, there exists N-open sets F and G in (X,τ) such that $x_{\alpha,\beta,\gamma} \in F$, $y_{\alpha',\beta',\gamma'} \in G$ and $F\tilde{q}G$. Then, there exists N- open sets H and K in (Y,τ_Y) such that $H = F \cap Y$ and $K = G \cap Y$. Clearly, $x_{\alpha,\beta,\gamma} \in H$, $y_{\alpha',\beta',\gamma'} \in K$ and $H\tilde{q}K$. This implies that (Y,τ_Y) is N-T_1. Now, we must show that (Y,τ_Y) is also an N-regular space. Let G be an N-closed set in (Y,τ_Y) and $x_{\alpha,\beta,\gamma}$ be an N-point in (Y,τ_Y) such that $x_{\alpha,\beta,\gamma}\tilde{q}G$. It is obvious that $x_{\alpha,\beta,\gamma}$ is also an N-point in (X,τ) and there exists a N-closed set F in (X,τ), $G = F \cap Y$. It is obvious that $x_{\alpha,\beta,\gamma}\tilde{q}F$. Since (X,τ) is a N-regular space, there exists N-open sets H and L in (X,τ) such that $x_{\alpha,\beta,\gamma} \in H$, $F \subset L$ and $H\tilde{q}L$. Then, there exists N-open sets K and M in (Y,τ_Y) such that $K = H \cap Y$ and $M = L \cap Y$. Clearly, $x_{\alpha,\beta,\gamma} \in K$, $G \subset M$ and $K\tilde{q}M$. This implies that (Y,τ_Y) is N-regular. Hence, (Y,τ_Y) is a N-T_3-space. □

Theorem 5.13. *An N-subspace (Y,τ_Y) of a N-T_4-space (X,τ) is N-T_4.*

Proof. The proof is similar to that of Theorem 5.12. □

6. Conclusion

Thus, we have brought a new perspective to the world of topology on separation axioms in N-topological spaces. In addition, we have given a new definition for N-subspace that we think will benefit the other mathematical studies especially in topology. It is our wish that the new terms and concepts we offer will help other scientists around the world to create new fields of work and make inventions that will benefit people. Besides, among our expectations, this study will pave the way for studies in the fields of statistics, medicine, economics, engineering and many different sciences, and to minimize the problems people face in their daily lives.

Conflict of interests

The authors declare that there is no conflict of interests.

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References

[1] A. Acikgoz and F. Esenbel, An Approach to Pre-Separation Axioms in Neutrosophic Soft Topological Spaces, Commun Fac Sci Univ Ank Ser A1 Math Stat. **69** (2)(2020) 1389-1404.

[2] A. Acikgoz and F. Esenbel, Neutrosophic Soft Semi-Regularization and Neutrosophic SoftSub-Maximality, Filomat. **36** (2) (2022) 651-668.

[3] A. Acikgoz and F. Esenbel, Neutrosophic Soft δ-Topology and Neutrosophic Soft δ-Compactness, Filomat. **34** (10) (2020) 34413458.

[4] A. Acikgoz, H. Çakalli, F. Esenbel and Lj. D.R. Kočinac, A quest of G-continuity in neutrosophic spaces, Math Meth Appl Sci. 44 (9) (2021) 7834-7844.

[5] T. Bera and N.K. Mahapatra, Assignment problem with neutrosophic costs and its solution methodology, Asia Mathematica. **3** (1) (2019) 21-32.

[6] T. Bera and N.K. Mahapatra, To solve assignment problem by centroid method in neutrosophic environment. Book chapter for 'Quadruple Neutrosophic Theory and Applications-Volume I', Neutrosophic Science International Association, Bruselss, Belgium. Chapter 7 (2020) 84-93.

[7] T. Bera and N.K. Mahapatra, Optimisation by dual simplex approach in neutrosophic environment, Int J Fuzzy Comput and Modelling. **2** (4)(2019) 334-352.

[8] T. Bera and N.K. Mahapatra, Generalised single valued neutrosophic number and its application to neutrosophic linear programming, Neutrosophic Sets in Decision Analysis and Operations Research. IGI Global Pennsylvania. (2019) Chapter 9.

[9] T. Bera and N.K. Mahapatra, Neutrosophic linear programming problem and its application to real life. Africa Matematika. 31 (3) (2020) 709-726.

[10] T. Bera and N.K. Mahapatra, On solving Neutrosophic linear programming problem by duality approach in neutrosophic environment, Int J Math Operation Res Inderscience Enterprises Ltd, vol. 18 (3) (2021) 310-335.

[11] A. Chakraborty, S. Broumi and P.K. Singh, Some properties of pentagonal Neutrosophic Numbers and its Applications in Transportation Problem Environment, Neutrosophic Sets and Systems. **28** (2019) 200-215.

[12] A. Chakraborty, S. Mondal and S. Broumi, De-Neutrosophication technique of pentagonal neutrosophic number and application in minimal spanning tree, Neutrosophic Sets and Systems. **29** (2019) 1-18.

[13] R. Dhavaseelan, S. Jafari, C.Ozel and M. A. Al Shumran, Generalized neutrosophic contra-continuity, New Trends in Neutrosophic Theory and Application-Volume II, (2017) 1-17.

[14] M. Parimala, M. Karthika, F. Smarandache and S. Broumi, On $\alpha\omega$-closed sets and its connectedness in terms of neutrosophic topological spaces, Int J Neu Sci. **2** (2) (2020) 82-88.

[15] A.A. Salma and S.A. Alblowi, Neutrosophic set and neutrosophic topological spaces, IOSR J Math. **3** (31) (2012).

[16] M. Saqlain, M.N. Jafar, S. Moin, M. Saeed and S. Broumi, Single and Multi-valued Neutrosophic Hypersoft set and Tangent Similarity Measure of Single valued Neutrosophic Hypersoft Sets, Neutrosophic Sets and Sytems. **32** (2020) 317-329.

[17] F. Smarandache, Neutrosophic set, A generalisation of the intuitionistic fuzzy sets, Int J Pure Appl Math. **24** (2005) 287-297.

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