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On Higher Order Lucas Hybrid Quaternions

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Abstract

In this article, we introduced higher order Lucas hybrid quaternions with the help of higher order Lucas numbers. We also examined some algebraic properties of these quaternions. By obtaining the recurrence relation, we found the Binet formula, the generating function and the exponential generating function. Finally, we calculated the Vajda identity for the higher order Lucas hybrid quaternions and obtained the Catalan, Cassini and d'Ocagne identities with the help of this identity.

Keywords: *Fibonacci number, Lucas number, Hybrid number, Higher order Fibonacci number, Higher order Lucas number, Binet formula, Generating function, Exponential generating function.*

2010 Mathematics Subject Classification: *11R52, 20G20*

1. Introduction

The real quaternions were first described by Irish mathematician William Rowan Hamilton in 1843. Hamilton [1] introduced the set of quaternions which can be represented as

$$H = \{ q = q_0 + i\,q_1 + j\,q_2 + k\,q_3 \mid q_0, q_1, q_2, q_3 \in \mathbb{R} \} \tag{1.1}$$

where

$$i^2 = j^2 = k^2 = -1\,, \;\; i\,j = -j\,i = k\,, \quad j\,k = -k\,j = i\,, \quad k\,i = -i\,k = j\,. \tag{1.2}$$

In 1963, Horadam [2] defined Fibonacci and Lucas quaternions as follows:

$$Q_n = F_n + F_{n+1}\,i + F_{n+2}\,j + F_{n+3}\,k\,, \tag{1.3}$$

$$K_n = L_n + L_{n+1}\,i + L_{n+2}\,j + L_{n+3}\,k\,, \tag{1.4}$$

respectively. In 2012, Halıcı [3] has derived generating functions and some important identities of these quaternions. Other studies on the generalizations of Fibonacci and Lucas quaternions are available in references[4, 5, 6, 7, 8, 9, 10].
The higher order Fibonacci numbers defined by Özvatan [11] in 2018 as follows:

$$F_n^{(s)} = \frac{F_{ns}}{F_s} = \frac{(\alpha^s)^n - (\beta^s)^n}{\alpha^s - \beta^s} \tag{1.5}$$

Since F_{ns} is divisible by F_s , the ratio $\frac{F_{ns}}{F_s}$ is an integer. So, all higher order Fibonacci numbers are integer. Note that for $s = 1$, higher order Fibonacci number $F_n^{(1)}$ is the ordinary Fibonacci numbers.
In 2021, the higher order Fibonacci quaternions defined by Kızılateş [12] as follows:

$$Q_n^{(s)} = F_n^{(s)} + F_{n+1}^{(s)}\,i + F_{n+2}^{(s)}\,j + F_{n+3}^{(s)}\,k. \tag{1.6}$$

In 2022, the higher order Jacobsthal-Lucas quaternions defined by Uysal et al. [13] as follows:

$$j_n^{(s)} = j_n^{(s)} + j_{n+1}^{(s)}\,i + j_{n+2}^{(s)}\,j + j_{n+3}^{(s)}\,k. \tag{1.7}$$

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x	1	i	ε	h
1	1	i	ε	h
i	i	-1	$1-h$	$\varepsilon+i$
ε	ε	1+h	0	$-\varepsilon$
h	h	$-\varepsilon-i$	ε	1

Table 1: Multiplication scheme of hybrid numbers [15]

In 2023, the higher order Jacobsthal quaternions defined by Özkan et al. [14] as follows:

$$J_n^{(s)} = J_n^{(s)} + J_{n+1}^{(s)}\,i + J_{n+2}^{(s)}\,j + J_{n+3}^{(s)}\,k. \tag{1.8}$$

The hybrid number system can be accepted as a generalization of the complex, dual and hyperbolic number systems. In 2018, firstly, set of hybrid numbers was introduced by Özdemir [15] as follows:

$$\mathbb{K} = \{a + b\,i + c\,\varepsilon + d\,h \,|\, a,b,c,d \in \mathbb{R}, i^2 = -1\,, \varepsilon^2 = 0\,, h^2 = 1\,\},$$

where units satisfy the rules

$$i\,h = -h\,i = \varepsilon + i\,.$$

The set $\mathbb{K}$ of hybrid numbers forms non-commutative ring with respect to the addition and multiplication operations. Accordingly, we will use table above (Table 1) for the multiplication of any two hybrid numbers. This table shows us that the multiplication operation in the hybrid numbers is not commutative. But it has the property of associativity.
There are several studies on hybrid quaternions for example Horadam hyrid [16], Leonardo hybrid [17].

The aim of this work is to present new quaternions whose components are higher order Lucas hybrid numbers and derive some algebraic properties of these quaternions. In addition, Binet's Formula, generating function, exponential generating function, Vajda's identity, Catalan's identity, the d'Ocagne's identity, Cassini's identity for higher order Lucas hybrid quaternions are given.

2. Higher order Lucas numbers

Definition 2.1. *The higher order Leonardo numbers described by*

$$L_n^{(s)} = \frac{L_{ns}}{L_s} = \frac{\alpha^{ns} + \beta^{ns}}{\alpha^s + \beta^s}\,. \tag{2.1}$$

Since L_{ns} is divisible by L_s , the ratio $\frac{L_{ns}}{L_s}$ is an integer. So, all higher order Lucas numbers are integer. Note that for $s = 1$, higher order Lucas number $L_n^{(1)}$ is the ordinary Lucas numbers.

Theorem 2.2. *The higher order Lucas numbers provide the following identity.*

$$L_{n+1}^{(s)} = L_s L_n^{(s)} - (-1)^s L_{n-1}^{(s)}\,, \tag{2.2}$$

where $(\alpha\,\beta)^s = (-1)^s$.

Proof.

$$\begin{aligned} L_{n+1}^{(s)} &= \frac{(\alpha^s)^{(n+1)} + (\beta^s)^{(n+1)}}{\alpha^s+\beta^s} \\ &= \frac{\alpha^{(ns+s)} + \alpha^{(s)}\beta^{(ns)} + \beta^{(s)}\alpha^{(ns)} - \alpha^{(ns)}\beta^{(s)} - \beta^{(ns)}\alpha^{(s)} + \beta^{(ns+s)}}{\alpha^s+\beta^s} \\ &= (\alpha^s+\beta^s)\left(\frac{\alpha^{(ns)}+\beta^{(ns)}}{\alpha^s+\beta^s}\right) - (\alpha\beta)^s\left(\frac{\alpha^{(ns-s)}+\beta^{(ns-s)}}{\alpha^s+\beta^s}\right) \\ &= L_s L_n^{(s)} - (-1)^s L_{n-1}^{(s)}\,. \end{aligned}$$

□

Thus, the proof is completed.

3. Higher order Lucas hybrid quaternions

In this section, we we define the higher order Lucas hybrid quaternions and derive some algebraic properties of these quaternions.

Theorem 3.1. *The higher order Lucas hybrid quaternions is denoted by $Q_n^{(s)}$ and defined as follows:*

$$Q_n^{(s)} = L_n^{(s)} + L_{n+1}^{(s)}\, i + L_{n+2}^{(s)}\, \varepsilon + L_{n+3}^{(s)}\, h. \tag{3.1}$$

where $\{i, \varepsilon, h\}$ are hybrid quaternion units and $L_n^{(s)}$ is higher order Lucas numbers. The real and imaginary parts of the higher order Lucas hybrid quaternions in Eq.(3.1) are as follows:

$$Re(Q_n^{(s)}) = L_n^{(s)}\,.$$

and

$$Im(Q_n^{(s)}) = u = L_{n+1}^{(s)}\, i + L_{n+2}^{(s)}\, \varepsilon + Le_{n+3}^{(s)}\, h\,.$$

Thus, we have

$$Q_n^{(s)} = L_n^{(s)} + u\,. \tag{3.2}$$

The conjugate of the higher order Lucas hybrid quaternion $Q_n^{(s)}$ is denoted by $Q_n^{(s)^*}$ as

$$Q_n^{(s)^*} = L_n^{(s)} - u\,. \tag{3.3}$$

Theorem 3.2. *For the higher order Lucas hybrid quaternions, we have*

$$Q_n^{(s)} + Q_n^{(s)^*} = 2\,L_n^{(s)}\,. \tag{3.4}$$

Proof.

$$\begin{aligned} Q_n^{(s)} + Q_n^{(s)^*} &= L_n^{(s)} + L_{n+1}^{(s)}\, i + L_{n+2}^{(s)}\, \varepsilon + L_{n+3}^{(s)}\, h + (L_n^{(s)} - L_{n+1}^{(s)}\, i - L_{n+2}^{(s)}\, \varepsilon - L_{n+3}^{(s)}\, h) \\ &= 2\,L_n^{(s)}\,. \end{aligned}$$

□

Thus, the proof is completed.

Theorem 3.3. *For the higher order Lucas hybrid quaternions, we have*

$$Q_n^{(s)}\,.\,Q_n^{(s)^*} = (L_n^{(s)})^2 + (L_{n+1}^{(s)})^2 - (L_{n+3}^{(s)})^2 - 2\,L_{n+1}^{(s)}\,L_{n+2}^{(s)}\,. \tag{3.5}$$

Proof.

$$\begin{aligned} Q_n^{(s)}\,.\,Q_n^{(s)^*} &= (L_n^{(s)} + L_{n+1}^{(s)}\, i + L_{n+2}^{(s)}\, \varepsilon + L_{n+3}^{(s)}\, h).(L_n^{(s)} - L_{n+1}^{(s)}\, i - L_{n+2}^{(s)}\, \varepsilon - L_{n+3}^{(s)}\, h) \\ &= (L_n^{(s)})^2 + (L_{n+1}^{(s)})^2 - (L_{n+3}^{(s)})^2 - 2\,L_{n+1}^{(s)}\,L_{n+2}^{(s)}\,. \end{aligned}$$

□

Theorem 3.4. *For the higher order Leonardo hybrid quaternions, we have*

$$(Q_n^{(s)})^2 = -Q_n^{(s)}.Q_n^{(s)^*} + 2\,L_n^{(s)}.Q_n^{(s)}\,. \tag{3.6}$$

Proof.

$$\begin{aligned} (Q_n^{(s)})^2 = &\; L_n^{(s)})^2 - (L_{n+1}^{(s)})^2 + (L_{n+3}^{(s)})^2 + L_n^{(s)}\,(L_{n+1}^{(s)}\, i + L_{n+2}^{(s)}\, \varepsilon + L_{n+1}^{(s)}\, h) + 2\,L_{n+1}^{(s)}\,L_{n+2}^{(s)} \\ = &\; -(L_n^{(s)})^2 - (L_{n+1}^{(s)})^2 + (L_{n+3}^{(s)})^2 + 2\,L_n^{(s)}\,(L_n^{(s)} + L_{n+1}^{(s)}\, i + L_{n+2}^{(s)}\, \varepsilon + L_{n+1}^{(s)}\, h) + 2\,L_{n+1}^{(s)}\,L_{n+2}^{(s)} \\ = &\; -Q_n^{(s)}.Q_n^{(s)^*} + 2\,L_n^{(s)}.Q_n^{(s)}\,. \end{aligned}$$

where Eq.(3.5) is used. □

Theorem 3.5. *(Binet's Formula) The Binet formula of the higher order Lucas hybrid quaternions as follows:*

$$Q_n^{(s)} = \frac{\hat{\alpha}\,(\alpha^s)^n + \hat{\beta}\,(\beta^s)^n}{\alpha^s + \beta^s} \tag{3.7}$$

where $\hat{\alpha} = (1 + \alpha^s\, i + \alpha^{2s} \varepsilon + \alpha^{3s}\, h)$ and $\hat{\beta} = (1 + \beta^s\, i + \beta^{2s} \varepsilon + \beta^{3s}\, h)\,.$

Proof.

$$\begin{aligned}
Q_n^{(s)} &= L_n^{(s)} + L_{n+1}^{(s)}\, i + L_{n+2}^{(s)}\, \varepsilon + L_{n+3}^{(s)}\, h \\
&= \Big(\tfrac{(\alpha^s)^n+(\beta^s)^n}{\alpha^s+\beta^s}\Big) + \Big(\tfrac{(\alpha^s)^{(n+1)}+(\beta^s)^{(n+1)}}{\alpha^s+\beta^s}\Big) i + \Big(\tfrac{(\alpha^s)^{(n+2)}+(\beta^s)^{(n+2)}}{\alpha^s+\beta^s}\Big)\varepsilon + \Big(\tfrac{(\alpha^s)^{(n+3)}+(\beta^s)^{(n+3)}}{\alpha^s+\beta^s}\Big) h \\
&= \tfrac{1}{\alpha^s+\beta^s}\left\{(\alpha^s)^n\,(1+\alpha^s\, i+\alpha^{2s}\varepsilon+\alpha^{3s}\, h)+(\beta^s)^n\,(1+\beta^s\, i+\beta^{2s}\varepsilon+\beta^{3s}\, h)\right\} \\
&= \tfrac{\hat{\alpha}\,(\alpha^s)^n+\hat{\beta}\,(\beta^s)^n}{\alpha^s+\beta^s}\,.
\end{aligned}$$

Thus, the proof is completed. □

Theorem 3.6. *There is the following recurrence relation for higher order Lucas hybrid quaternions*

$$Q_{n+1}^{(s)} = L_s\, Q_n^{(s)} + (-1)^{s+1}\, Q_{n-1}^{(s)}\,. \tag{3.8}$$

Proof.

$$\begin{aligned}
Q_{n+1}^{(s)} &= \tfrac{\hat{\alpha}(\alpha^s)^{(n+1)}+\hat{\beta}(\beta^s)^{(n+1)}}{\alpha^s+\beta^s} \\
&= \tfrac{\hat{\alpha}\alpha^{(sn+s)}+\hat{\beta}\beta^{(sn+s)}+\hat{\beta}\beta^{(sn)}\alpha^s+\hat{\alpha}\alpha^{(sn)}\beta^s-\hat{\beta}\beta^{(sn)}\alpha^s-\hat{\alpha}\alpha^{(sn)}\beta^s}{\alpha^s+\beta^s} \\
&= (\alpha^s+\beta^s)\Big(\tfrac{\hat{\alpha}\alpha^{(sn)}+\hat{\beta}\beta^{(sn)}}{\alpha^s+\beta^s}\Big) - (\alpha\beta)^s\Big(\tfrac{\hat{\alpha}\alpha^{(sn-s)}+\hat{\beta}\,\beta^{(sn-s)}}{\alpha^s+\beta^s}\Big) \\
&= (\alpha^s+\beta^s)\, Q_n^{(s)} - (-1)^s\, Q_{n-1}^{(s)} \\
&= L_s\,(Q_n^{(s)}) - (-1)^s\, Q_{n-1}^{(s)}\,.
\end{aligned}$$

Thus, the proof is completed. □

Theorem 3.7. *By extending n and s to negative integer numbers for higher order Lucas hybrid quaternions Q_n^s; the following identities can be derived as*

$$Q_{-n}^{(s)} = (-1)^{(-sn)}\,\frac{\hat{\alpha}(\beta^s)^n+\hat{\beta}(\alpha^s)^n}{\alpha^s+\beta^s}\,, \tag{3.9}$$

$$Q_{-n}^{(-s)} = (-1)^s\, Q_n^{(s)}\,, \tag{3.10}$$

$$Q_n^{(-s)} = (-1)^s\, Q_{-n}^{(s)}\,. \tag{3.11}$$

Proof. By using Eq.(3.7), we have

$$\begin{aligned}
Q_{-n}^{(s)} &= \Big(\tfrac{\hat{\alpha}(\alpha^s)^{(-n)}+\hat{\beta}(\beta^s)^{(-n)}}{\alpha^s+\beta^s}\Big) \\
&= \Big(\tfrac{\frac{\hat{\alpha}}{(\alpha^s)^n}+\frac{\hat{\beta}}{(\beta^s)^n}}{\alpha^s+\beta^s}\Big) \\
&= \Big(\tfrac{\hat{\alpha}(\beta^s)^n+\hat{\beta}(\alpha^s)^n}{(\alpha^s)^n\,(\beta^s)^n(\alpha^s+\beta^s)}\Big) \\
&= (-1)^{-sn}\,\tfrac{\hat{\alpha}(\beta^s)^n+\hat{\beta}(\alpha^s)^n}{\alpha^s+\beta^s}\,.
\end{aligned}$$

$$\begin{aligned}
Q_{-n}^{(-s)} &= \Big(\tfrac{\hat{\alpha}(\alpha^{(-s)})^{(-n)}+\hat{\beta}(\beta^{(-s)})^{(-n)}}{\alpha^{(-s)}+\beta^{(-s)}}\Big) \\
&= \tfrac{\hat{\alpha}(\alpha^s)^n+\hat{\beta}(\beta^s)^n}{\frac{\beta^s+\alpha^s}{\alpha^s\beta^s}} \\
&= (-1)^s\,\tfrac{\hat{\alpha}\alpha^{sn}+\hat{\beta}\beta^{sn}}{\alpha^s+\beta^s} \\
&= (-1)^s\, Q_n^{(s)}\,.
\end{aligned}$$

$$
\begin{aligned}
Q_n^{(-s)} = \ & \left(\frac{\hat{\alpha}\alpha^{(-sn)}+\hat{\beta}\beta^{(-sn)}}{\alpha^{-s}+\beta^{-s}}\right) \\
= \ & \frac{\hat{\alpha}(\alpha^s)^{-n}+\hat{\beta}(\beta^s)^{-n}}{\frac{\beta^s+\alpha^s}{\alpha^s\beta^s}} \\
= \ & (-1)^s \, \frac{\hat{\alpha}(\alpha^s)^{-n}+\hat{\beta}(\beta^s)^{-n}}{\alpha^s+\beta^s} \\
= \ & (-1)^s \, Q_{-n}^{(s)}.
\end{aligned}
$$

□

Lemma 3.8. *The following identities hold:*

$$\hat{\alpha}+\hat{\beta} = 2+L_s\, i+L_2s\, \varepsilon+L_3s\, h, \tag{3.12}$$

$$\hat{\alpha}-\hat{\beta} = (\alpha^s-\beta^s)\,[\, i+L_s\, \varepsilon+(L_{2s}+(-1)^s)\, h\,], \tag{3.13}$$

$$\hat{\alpha}\beta^s-\hat{\beta}\alpha^s = (-1)^s(\alpha^s-\beta^s)\,[(-1)^{s+1}+\varepsilon+(L_s\, h\,], \tag{3.14}$$

$$\hat{\alpha}\beta^s+\hat{\beta}\alpha^s = L_s+2(-1)^s\, i+(-1)^s\, L_s\, \varepsilon+(-1)^s\, L_2s\, h, \tag{3.15}$$

$$\hat{\alpha}\alpha^s-\hat{\beta}\beta^s = \ (\alpha^s-\beta^s)\,[\,1+L_s\, i+(L_{2s}+(-1)^s)\,\varepsilon+(L_{3s}+(-1)^s(\alpha^s+\beta^s)\,)\, h\,]. \tag{3.16}$$

where $\hat{\alpha} = 1+\alpha^s\, i+\alpha^{2s}\varepsilon+\alpha^{3s}\, h$ *and* $\hat{\beta} = 1+\beta^s\, i+\beta^{2s}\varepsilon+\beta^{3s}\, h$.

Proof. From the definitions of $\hat{\alpha}$ and $\hat{\beta}$, we have

$$
\begin{aligned}
\hat{\alpha}+\hat{\beta} = \ & (1+\alpha^s\, i+\alpha^{2s}\varepsilon+\alpha^{3s}\, h)+(1+\beta^s\, i+\beta^{2s}\varepsilon+\beta^{3s}\, h) \\
= \ & 2+(\alpha^s+\beta^s)\, i+(\alpha^{2s}+\beta^{2s})\,\varepsilon+(\alpha^{3s}+\beta^{3s})\, h \\
= \ & 2+L_s\, i+L_2s\,\varepsilon+L_3s\, h,
\end{aligned}
$$

$$
\begin{aligned}
\hat{\alpha}-\hat{\beta} = \ & (1+\alpha^s\, i+\alpha^{2s}\varepsilon+\alpha^{3s}\, h)-(1+\beta^s\, i+\beta^{2s}\varepsilon+\beta^{3s}\, h) \\
= \ & (\alpha^s-\beta^s)\, i+(\alpha^{2s}-\beta^{2s})\,\varepsilon+(\alpha^{3s}-\beta^{3s})\, h \\
= \ & (\alpha^s-\beta^s)[\, i+(\alpha^s+\beta^s)\,\varepsilon+(\alpha^{2s}+\beta^{2s}+\alpha^s\beta^s)\, h] \\
= \ & (\alpha^s-\beta^s)[\, i+L_s\,\varepsilon+(L_{2s}+(-1)^s)\, h\,],
\end{aligned}
$$

$$
\begin{aligned}
\hat{\alpha}\beta^s-\hat{\beta}\alpha^s = \ & (1+\alpha^s\, i+\alpha^{2s}\varepsilon+\alpha^{3s}\, h)\beta^s-(1+\beta^s\, i+\beta^{2s}\varepsilon+\beta^{3s}\, h)\alpha^s \\
= \ & (\beta^s-\alpha^s)+(\alpha^{2s}\beta^s-\beta^{2s}\alpha^s)\,\varepsilon+(\alpha^{3s}\beta^s-\beta^{3s}\alpha^s)\, h \\
= \ & (\beta^s-\alpha^s)+(\alpha\beta)^s\,(\alpha^s-\beta^s)\,\varepsilon+(\alpha\beta)^s\,(\alpha^{2s}-\beta^{2s}\, h \\
= \ & (\beta^s-\alpha^s)+(-1)^s\,(\alpha^s-\beta^s)\,\varepsilon+(-1)^s\,(\alpha^{2s}-\beta^{2s}\, h \\
= \ & (\alpha^s-\beta^s)\,[-1+(-1)^s\,\varepsilon+(-1)^s\, L_s\, h\,],
\end{aligned}
$$

$$
\begin{aligned}
\hat{\alpha}\beta^s+\hat{\beta}\alpha^s = \ & (1+\alpha^s\, i+\alpha^{2s}\varepsilon+\alpha^{3s}\, h)\beta^s+(1+\beta^s\, i+\beta^{2s}\varepsilon+\beta^{3s}\, h)\alpha^s \\
= \ & (\beta^s+\alpha^s)+(\alpha^{2s}\beta^s+\beta^{2s}\alpha^s)\,\varepsilon+(\alpha^{3s}\beta^s+\beta^{3s}\alpha^s)\, h \\
= \ & L_s+2(-1)^s\, i+(-1)^s\, L_s\,\varepsilon+(-1)^s\, L_2s\, h,
\end{aligned}
$$

$$
\begin{aligned}
\hat{\alpha}\alpha^s-\hat{\beta}\beta^s = \ & (1+\alpha^s\, i+\alpha^{2s}\varepsilon+\alpha^{3s}\, h)\alpha^s-(1+\beta^s\, i+\beta^{2s}\varepsilon+\beta^{3s}\, h)\beta^s \\
= \ & (\alpha^s-\beta^s)+(\alpha^{2s}-\beta^{2s}\, i+(\alpha^{3s}+\beta^{3s},\varepsilon+(\alpha^{4s}\beta^{4s}\, h \\
= \ & (\beta^s-\alpha^s)+(\alpha\beta)^s\,(\alpha^s-\beta^s)\,\varepsilon+(\alpha\beta)^s\,(\alpha^{2s}-\beta^{2s}\, h \\
= \ & (\alpha^s-\beta^s)\,[1+(\alpha^s-\beta^s)\, i+(\alpha^{2s}+\beta^{2s}+\alpha^s\beta^s)\,\varepsilon \\
& +(\alpha^{3s}+\beta^{3s}+\alpha^{2s}\beta^s+\alpha^s\beta^{2s})\, h \\
= \ & (\alpha^s-\beta^s)\,[1+L_s\, i+(L_{2s}+(-1)^s)\,\varepsilon \\
& +(L_{3s}+(-1)^s(\alpha^s+\beta^s)\,)\, h\,].
\end{aligned}
$$

□

Theorem 3.9. *(Generating function) The generating function of the higher order Lucas hybrid quaternions $Q_n^{(s)}$ as follows:*

$$G^{(s)}(x)=\sum_{n=0}^{\infty} Q_n^{(s)}\,x^n=\frac{(2+L_s i+L_{2s}\varepsilon+L_{3s}h)-x\,[L_s+(-1)^s(2i+L_s\varepsilon+L_{2s}h)]}{L_s\,(1-L_s x+(-1)^s x^2)}. \tag{3.17}$$

Proof. By using Eq.(3.7), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} Q_n^{(s)}\,x^n &= \sum_{n=0}^{\infty}(L_n^{(s)}+L_{n+1}^{(s)}\,i+L_{n+2}^{(s)}\,\varepsilon+L_{n+3}^{(s)}\,h)\,x^n\\
&= \frac{\hat{\alpha}}{\alpha^s+\beta^s}\sum_{n=0}^{\infty}(\alpha^s)^n x^n+\frac{\hat{\beta}}{\alpha^s+\beta^s}\sum_{n=0}^{\infty}(\beta^s)^n x^n\\
&= \frac{\hat{\alpha}}{\alpha^s+\beta^s}\,\frac{1}{1-\alpha^s x}+\frac{\hat{\beta}}{\alpha^s+\beta^s}\,\frac{1}{1-\beta^s x}\\
&= \frac{\hat{\alpha}(1-\beta^s x)+\hat{\beta}(1-\alpha^s x)}{(\alpha^s+\beta^s)(1-\alpha^s x)(1-\beta^s x)}\\
&= \frac{\hat{\alpha}+\hat{\beta}-(\hat{\alpha}\,\beta^s+\hat{\beta}\,\alpha^s)\,x}{(\alpha^s+\beta^s)(1-L_s x+(-1)^s x^2)}\\
&= \frac{(2+L_s i+L_{2s}\varepsilon+L_{3s}h)-[L_s+(-1)^s(2i+L_s\varepsilon+L_{2s}h)]\,x}{L_s\,(1-L_s x+(-1)^s x^2)}.
\end{aligned}$$

□

Theorem 3.10. *(Exponential generating function) The exponential generating function of the higher order Lucas hybrid quaternions $Q_n^{(s)}$ as follows:*

$$\sum_{n=0}^{\infty} Q_n^{(s)}\,\frac{x^n}{n!}=\frac{\hat{\alpha}\,e^{\alpha^s x}+\hat{\beta}\,e^{\beta^s x}}{\alpha^s+\beta^s}. \tag{3.18}$$

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} Q_n^{(s)}\,\frac{x^n}{n!} &= \sum_{n=0}^{\infty}\Big(\frac{\hat{\alpha}(\alpha^s)^n+\hat{\beta}(\beta^s)^n}{\alpha^s+\beta^s}\Big)\frac{x^n}{n!}\\
&= \frac{\hat{\alpha}}{\alpha^s+\beta^s}\Big(\sum_{n=0}^{\infty}\frac{(\alpha^s)^n x^n}{n!}\Big)+\frac{\hat{\beta}}{\alpha^s+\beta^s}\Big(\sum_{n=0}^{\infty}\frac{(\beta^s)^n x^n}{n!}\Big)\\
&= \frac{\hat{\alpha}\,e^{\alpha^s x}+\hat{\beta}\,e^{\beta^s x}}{\alpha^s+\beta^s}.
\end{aligned}$$

Thus, the proof is completed. □

Theorem 3.11. *For $m,n\in\mathbb{Z}$, the generating function of the higher order Lucas hybrid quaternions Q_n^s is*

$$\sum_{n=0}^{\infty} Q_{m+n}^{(s)}\,x^n=\frac{(Q_m^{(s)})+(-1)^{s+1}\,Q_{m-1}^{(s)}\,x}{1-L_s x+(-1)^s x^2}. \tag{3.19}$$

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} Q_{m+n}^{(s)}\,x^n &= \sum_{n=0}^{\infty}\Big(\frac{\hat{\alpha}(\alpha^s)^{(m+n)}+\hat{\beta}(\beta^s)^{(m+n)}}{\alpha^s+\beta^s}\Big)x^n\\
&= \frac{1}{\alpha^s+\beta^s}\Big[\hat{\alpha}\sum_{n=0}^{\infty}(\alpha^s)^{(m+n)}x^n+\hat{\beta}\sum_{n=0}^{\infty}(\beta^s)^{(m+n)}x^n\Big]\\
&= \frac{1}{\alpha^s+\beta^s}\Big[\hat{\alpha}(\alpha^s)^m\sum_{n=0}^{\infty}(\alpha^s)^n x^n+\hat{\beta}(\beta^s)^m\sum_{n=0}^{\infty}(\beta^s)^n x^n\Big]\\
&= \frac{1}{\alpha^s+\beta^s}\Big[\hat{\alpha}(\alpha^s)^m\frac{1}{1-\alpha^s x}+\hat{\beta}(\beta^s)^m\frac{1}{1-\beta^s x}\Big]\\
&= \frac{1}{\alpha^s+\beta^s}\Big[\frac{\hat{\alpha}(\alpha^s)^m(1-\beta^s x)+\hat{\beta}(\beta^s)^m(1-\alpha^s x)}{(1-\alpha^s x)(1-\beta^s x)}\Big]\\
&= \frac{\hat{\alpha}(\alpha^s)^m+\hat{\beta}(\beta^s)^m}{(\alpha^s+\beta^s)(1-L_s x+(-1)^s x^2)}-\frac{\alpha^s\beta^s\,(\hat{\alpha}(\alpha^s)^{(m-1)}+\hat{\beta}(\beta^s)^{(m-1)})\,x}{(\alpha^s+\beta^s)(1-L_s x+(-1)^s x^2)}\\
&= \frac{Q_m^{(s)}}{1-L_s x+(-1)^s x^2)}-\frac{(\hat{\alpha}(\alpha^s)^{(m-1)}+\hat{\beta}(\beta^s)^{(m-1)})\,x}{\alpha^s+\beta^s}\,\frac{(-1)^s}{(1-L_s x+(-1)^s x^2)}\\
&= \frac{(Q_m^{(s)})}{(1-L_s x+(-1)^s x^2)}-\frac{(Q_{m-1}^{(s)})(-1)^s\,x}{(1-L_s x+(-1)^s x^2)}\\
&= \frac{Q_m^{(s)}+(-1)^{s+1}\,Q_{m-1}^{(s)}\,x}{1-L_s x+(-1)^s x^2}.
\end{aligned}$$

Thus, the proof is completed. □

4. Some Identities for Higher Order Lucas Hybrid Quaternions

In this section, we derive some identities of higher order Lucas hybrid quaternions. We will give the following Lemma to calculate our next result.

Lemma 4.1. *The following equations hold:*

$$\hat{\alpha}\,\hat{\beta} = A - \Delta B, \tag{4.1}$$

and

$$\hat{\beta}\,\hat{\alpha} = A + \Delta B. \tag{4.2}$$

where

$$A = [1 + (-1)^s L_s + L_s\, i + L_{2s}\,\varepsilon + L_{3s}\, h],$$
$$B = [(-1)^s (L_s\, i + (L_s - 1)\varepsilon - h),$$
$$\Delta = (\alpha^s - \beta^s).$$

Proof.

$$\begin{aligned}
\hat{\alpha}\,\hat{\beta} =\ & (1 + \alpha^s i + \alpha^{2s}\varepsilon + \alpha^{3s} h)(1 + \beta^s i + \beta^{2s}\varepsilon + \beta^{3s} h) \\
=\ & [1 - (\alpha\beta)^s + (\alpha\beta)^{3s} + (\alpha\beta)^s (\beta^s + \alpha^s)] \\
& + i[\alpha^s + \beta^s + (\alpha\beta)^s (\beta^{2s} - \alpha^{2s}) + \varepsilon[\alpha^{2s} + \beta^{2s} + (\alpha\beta)^s (\beta^{2s} - \alpha^{2s} - \alpha^s\beta^{2s} + \alpha^{2s}\beta^s)] + h[\alpha^{3s} + \beta^{3s} - (\alpha\beta)^s (\alpha^s - \beta^s)] \\
=\ & [1 - (-1)^s + (-1)^{3s} + (-1)^s L_s] + i[L_s + (-1)^{s+1}(\alpha^s - \beta^s) L_s] + \varepsilon[L_{2s} + (-1)^{s+1}(\alpha^s - \beta^s)(\alpha^s + \beta^s) - (\alpha\beta)^s] \\
& + h[L_{3s} + (-1)^s (\alpha^s - \beta^s)] \\
=\ & [1 + (-1)^s L_s + L_s\, i + L_{2s}\,\varepsilon + L_{3s}\, h] + (-1)^{s+1}(\alpha^s - \beta^s)[L_s\, i + (L_s + 1)\,\varepsilon - h] \\
=\ & [1 + (-1)^s L_s + L_s\, i + L_{2s}\,\varepsilon + L_{3s}\, h] - \Delta\{(-1)^s [L_s\, i + (L_s + 1)\,\varepsilon - h]\} \\
=\ & A - \Delta B.
\end{aligned}$$

$$\begin{aligned}
\hat{\beta}\,\hat{\alpha} =\ & (1 + \beta^s i + \beta^{2s}\varepsilon + \beta^{3s} h)(1 + \alpha^s i + \alpha^{2s}\varepsilon + \alpha^{3s} h) \\
=\ & (1 - \beta^s\alpha^s + \beta^{3s}\alpha^{3s} + (\alpha\beta)^s (\alpha^s + \beta^s)) \\
& + i(\alpha^s + \beta^s + (\alpha\beta)^s (\alpha^{2s} - \beta^{2s})) + \varepsilon(\alpha^{2s} + \beta^{2s} + \beta^s\alpha^{3s} - \beta^{2s}\alpha^{3s} - \beta^{3s}\alpha^s + \beta^{3s}\alpha^{2s}) + h(\alpha^{3s} + \beta^{3s} - \beta^s\alpha^{2s} + \beta^{2s}\alpha^s) \\
=\ & (1 + (-1)^{s+1} + (-1)^s + (-1)^s L_s) + i[L_s + (-1)^s (\alpha^s - \beta^s)(\alpha^s + \beta^s] + \varepsilon[L_{2s} + (-1)^s (\alpha^s - \beta^s)(\alpha^s + \beta^s - (-1)^s] \\
& + h(L_{3s} + (-1)^{s+1}(\alpha^s - \beta^s)) \\
=\ & [1 + (-1)^s L_s + (L_s\, i + L_{2s}\,\varepsilon + L_{3s}\, h)] + (-1)^s (\alpha^s - \beta^s [L_s\, i + (L_s + 1)\,\varepsilon - h] \\
=\ & [1 + (-1)^s L_s + (L_s\, i + L_{2s}\varepsilon + L_{3s}\, h)] + \Delta\{(-1)^s [L_s\, i + (L_s + 1)\,\varepsilon - h]\} \\
=\ & A + \Delta B.
\end{aligned}$$

Thus, the proof is completed. □

Theorem 4.2. *(Vajda's Identity) For $n, m, r \in \mathbb{Z}$, we have*

$$Q_{n+m}^{(s)} Q_{n+r}^{(s)} - Q_n^{(s)} Q_{n+m+r}^{(s)} = -(-1)^{(sn)} \Delta^2 F_m{}^{(s)} (L_s)^{-2} (A F_r{}^{(s)} + B L_{rs}). \tag{4.3}$$

Proof.

$$\begin{aligned}
Q_{n+m}^{(s)} Q_{n+r}^{(s)} - Q_n^{(s)} Q_{n+m+r}^{(s)} &= \left(\tfrac{\hat{\alpha}(\alpha^s)^{(n+m)} + \hat{\beta}(\beta^s)^{(n+m)}}{\alpha^s + \beta^s}\right)\left(\tfrac{\hat{\alpha}(\alpha^s)^{(n+r)} + \hat{\beta}(\beta^s)^{(n+r)}}{\alpha^s + \beta^s}\right) - \left(\tfrac{\hat{\alpha}(\alpha^s)^{(n)} + \hat{\beta}(\beta^s)^{(n)}}{\alpha^s + \beta^s}\right)\left(\tfrac{\hat{\alpha}(\alpha^s)^{(n+m+r)} + \hat{\beta}(\beta^s)^{(n+m+r)}}{\alpha^s + \beta^s}\right) \\
&= \tfrac{1}{(\alpha^s + \beta^s)^2}\left[\hat{\alpha}\hat{\beta}(\alpha^s)^{(n+m)}(\beta^s)^{(n+r)} - \hat{\alpha}\hat{\beta}(\alpha^s)^{(n)}(\beta^s)^{(n+m+r)} + \hat{\beta}\hat{\alpha}(\beta^s)^{(n+m)}(\alpha^s)^{(n+r)} - \hat{\beta}\hat{\alpha}(\beta^s)^{(n)}(\alpha^s)^{(n+m+r)}\right] \\
&= \tfrac{1}{(\alpha^s + \beta^s)^2}\left[\hat{\alpha}\hat{\beta}\,\alpha^{(ns)}\beta^{(ns+rs)}((\alpha^s)^m - (\beta^s)^m) + \hat{\beta}\hat{\alpha}\beta^{(ns)}\alpha^{(ns+rs)}((\beta^s)^m - (\alpha^s)^m)\right] \\
&= \tfrac{1}{(\alpha^s + \beta^s)^2}\left[\hat{\alpha}\hat{\beta}\,(\alpha\beta)^{(ns)}(\beta^{(rs)}((\alpha^s)^m - (\beta^s)^m) - \hat{\beta}\hat{\alpha}(\alpha\beta)^{(ns)}\alpha^{(rs)}((\alpha^s)^m - (\beta^s)^m)\right) \\
&= \tfrac{1}{(\alpha^s + \beta^s)^2}\left[(-1)^{(ns)}((\alpha^s)^m - (\beta^s)^m)(\hat{\alpha}\hat{\beta}\,\beta^{(rs)} - \hat{\beta}\hat{\alpha}\,\alpha^{(rs)})\right] \\
&= \tfrac{(-1)^{(ns)}((\alpha^s)^m - (\beta^s)^m)}{(\alpha^s + \beta^s)^2}\left[(A - \Delta B)\,\beta^{(rs)} - (A + \Delta B)\,\alpha^{(rs)}\right] \\
&= (-1)^{(sn)}\Delta F_m^{(s)}(L_s)^{-2}(-A\Delta F_r^{(s)} - \Delta B L_{rs}) \\
&= -(-1)^{(sn)}\Delta^2 F_m^{(s)}(L_s)^{-2}(A F_r^{(s)} + B L_{rs}).
\end{aligned}$$

□

where $(\alpha^s)^m-(\beta^s)^m=\Delta F_m^{(s)}$ is used.
Thus, the proof is completed.

Now, we have the following identities from the Vajda's identity:

Corollary 4.3. *(Catalan's identity) For $n,r\in\mathbb{Z}$, we obtain*

$$Q_{n-r}^{(s)}Q_{n+r}^{(s)}-(Q_n^{(s)})^2= \quad -(-1)^{(sn)}\Delta^2F_{-r}^{(s)}(L_s)^{-2}(AF_r^{(s)}+BL_{rs}). \tag{4.4}$$

Proof. For $m=-r$, we have

$$Q_{n-r}^{(s)}Q_{n+r}^{(s)}-(Q_n^{(s)})^2= \quad -(-1)^{(sn)}\Delta^2F_{-r}^{(s)}(L_s)^{-2}(AF_r^{(s)}+BL_{rs}).$$

Now let's obtain the Catalan identity using the Binet formula:

$$\begin{aligned}
Q_{n-r}^{(s)}Q_{n+r}^{(s)}-(Q_n^{(s)})^2= \quad & (\tfrac{\hat\alpha(\alpha^s)^{(n-r)}+\hat\beta(\beta^s)^{(n-r)}}{\alpha^s+\beta^s})(\tfrac{\hat\alpha(\alpha^s)^{(n+r)}+\hat\beta(\beta^s)^{(n+r)}}{\alpha^s+\beta^s})-(\tfrac{\hat\alpha(\alpha^s)^{(n)}+\hat\beta(\beta^s)^{(n)}}{\alpha^s+\beta^s})(\tfrac{\hat\alpha(\alpha^s)^{(n)}+\hat\beta(\beta^s)^{(n)}}{\alpha^s+\beta^s})\\
&=\tfrac{1}{(\alpha^s+\beta^s)^2}[\hat\alpha\hat\beta(\alpha^s)^{(n-r)}(\beta^s)^{(n+r)}-\hat\alpha\hat\beta(\alpha^s)^{(n)}(\beta^s)^{(n)}+\hat\beta\hat\alpha(\beta^s)^{(n-r)}(\alpha^s)^{(n+r)}-\hat\beta\hat\alpha(\beta^s)^{(n)}(\alpha^s)^{(n)}]\\
&=\tfrac{1}{(\alpha^s+\beta^s)^2}(\alpha\beta)^{sn}[\hat\alpha\hat\beta(\alpha^{(-sr)}\beta^{(sr)}-1)+\hat\beta\hat\alpha(\beta^{(-sr)}\alpha^{(sr)}-1)]\\
&=(L_s)^{-2}(-1)^{sn}(\beta^{(sr)}-\alpha^{(sr)})[\tfrac{\hat\alpha\hat\beta}{\alpha^{sr}}-\tfrac{\hat\beta\hat\alpha}{\beta^{sr}}]\\
&=(L_s)^{-2}(-1)^{sn}(-1)^{(sr)}\Delta F_{-r}^{(s)}[\tfrac{\hat\alpha\hat\beta\beta^{sr}-\hat\beta\hat\alpha}{(\alpha\beta)^{sr}}]\\
&=(L_s)^{-2}(-1)^{sn}\Delta F_{-r}^{(s)}[(A-\Delta B)\beta^{sr}-(A+\Delta B)\alpha^{sr}]\\
&=(L_s)^{-2}(-1)^{sn}\Delta F_{-r}^{(s)}[-A(\alpha^{sr}-\beta^{sr})-\Delta B(\alpha^{sr}+\beta^{sr})]\\
&=(-1)^{(sn+1)}(L_s)^{-2}\Delta^2F_{-r}^{(s)}[AF_r^{(s)}+BL_{sr}].
\end{aligned}$$

□

where $(\alpha^s)^r-(\beta^s)^r=\Delta F_r^{(s)}$ and $(\beta^s)^r-(\alpha^s)^r=(-1)^{sr}\Delta F_{-r}^{(s)}$ is used.
Thus, the proof is completed.

Corollary 4.4. *(Cassini's identity) For $n\in\mathbb{Z}$, we obtain*

$$Q_{n-1}^{(s)}Q_{n+1}^{(s)}-(Q_n^{(s)})^2= \quad -(-1)^{(sn)}\Delta^2F_{-1}^{(s)}(L_s)^{-2}(AF_1^{(s)}+BL_s). \tag{4.5}$$

Proof. For $r=-m=1$, we have

$$Q_{n-1}^{(s)}Q_{n+1}^{(s)}-(Q_n^{(s)})^2= \quad -(-1)^{(sn)}\Delta^2F_{-1}^{(s)}(L_s)^{-2}(AF_1^{(s)}+BL_s).$$

Thus, the proof is completed. □

Corollary 4.5. *(d'Ocagne's identity) For $n\in\mathbb{Z}$, we obtain*

$$Q_k^{(s)}Q_{n+1}^{(s)}-Q_{k+1}^{(s)}Q_n^{(s)}= \quad -(-1)^{(sn)}\Delta^2F_{k-n}^{(s)}(L_s)^{-2}(AF_1^{(s)}+BL_s). \tag{4.6}$$

Proof. For $m+n=k$ and $r=1$, we have

$$Q_k^{(s)}Q_{n+1}^{(s)}-Q_{k+1}^{(s)}Q_n^{(s)}= \quad -(-1)^{(sn)}\Delta^2F_{k-n}^{(s)}(L_s)^{-2}(AF_1^{(s)}+BL_s).$$

□

Thus, the proof is completed.

5. Conclusion

In this paper, we introduced higher order Lucas hybrid quaternions. We proved some propositions for these quaternions. Also, we obtained the recurrence relation, the Binet formula, the generating function and exponential generating function which are basic concepts in number sequences for these quaternions. Additionally, we gave the Vajda's identity, which is important for higher order Lucas hybrid quaternions, and used this identity to obtain the Catalan's, Cassini's and d'Ocagne's identities.

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