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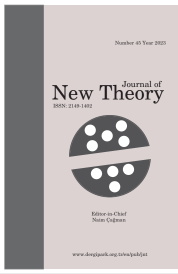

Crossed Corner and Reduced Simplicial Commutative Algebras

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Abstract — In this paper, we describe the crossed corner of commutative algebras and present the relation between the category of crossed corners of commutative algebras and the category of reduced simplicial commutative algebras with Moore complex of length 2. We provide a passage from crossed corners to bisimplicial algebras. In this construction, we utilize the Artin-Mazur codiagonal functor from reduced bisimplicial algebras to simplicial algebras and the hypercrossed complex pairings in the Moore complex of a simplicial algebra. Using the coskeleton functor from the category of k-truncated simplicial algebras to the category simplicial algebras with Moore complex of length k, we see that the length of Moore complex of the reduced simplicial algebra obtained from a crossed corner is 2.

Keywords *Crossed modules, simplicial algebras, crossed corner*

Mathematics Subject Classification (2020) 18N50, 55U10

1. Introduction

Whitehead [1] introduced the concept of crossed modules of groups as an algebraic model of connected homotopy 2-types of topological spaces. As a 2-dimensional crossed module, or a crossed module of crossed modules, the notion of crossed square has been introduced by Guin-Waléry and Loday [2]. Another 2-dimensional crossed modules of groups is the quadratic module was introduced by Baues as an algebraic model for 3-types in [3]. The commutative algebra and the Lie algebra versions of quadratic modules were introduced by Arvasi and Ulualan [4] and Ulualan and Uslu [5], respectively. The quasi quadratic modules over Lie algebras has been studied in [6]. For further work about the 2-dimensional crossed modules, see [7].

Alp [8] has defined crossed corners of groups, closely associated with crossed squares, and studied relationships between them. The commutative algebra analogue of crossed modules has been studied by Porter [9]. Moreover, the commutative, associative, and Lie algebra versions of crossed squares has been defined by Ellis [10], as higher dimensional versions of crossed modules of algebras. The equivalence between simplicial algebras and these crossed structures was proven in [4, 10–12]. In this paper, our first aim is to achieve the definition of a crossed corner over commutative algebras. We investigate the close relationship between the categories of crossed corners of commutative algebras and reduced simplicial algebras with Moore complex of length 2 in terms of Peiffer pairings in the Moore complex. Throughout this paper, an algebra action of $r \in R$ on $s \in S$ will be denoted by $r \cdot s$ or $s \cdot r$. Since all algebras in this work are commutative algebras, we can write $r \cdot s = s \cdot r$. Recall from [13]

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that a crossed module of algebras is a homomorphism of R-algebras $\partial : S \to R$ with the algebra action of R on S such that the following axioms are satisfied: $CM1$. $\partial(s \cdot r) = \partial(s)r$, $\partial(r \cdot s) = r\partial(s)$, and $CM2$. $\partial(s) \cdot s' = ss' = s \cdot \partial(s')$, for all $r \in R$, and $s, s' \in S$. It is well known that a crossed module is equivalent to a simplicial algebra with Moore complex of length 1. For the connection between crossed modules of Lie algebras and simplicial Lie algebras and for the Lie-Rinehart version of this connection, see [14, 15].

A crossed corner can be regarded as a 2-dimensional crossed module. By giving the definition of a crossed corner of commutative algebras, we will prove that the category of crossed corners is equivalent to the category of reduced simplicial commutative algebras with Moore complex of length 2. In this equivalence, we will define a passage from the crossed corners to reduced bisimplicial algebras and Artin-Mazur codioganal functor from bisimplicial algebras to simplicial algebras. In this construction, we see that the length of this reduced simplicial algebras is 2.

2. Crossed Corner of Commutative Algebras

Suppose that k is a fixed commutative ring. All of the k-algebras studied in this work are assumed to be commutative and associative. We will denote the category of commutative algebras by Alg_k. In this section, we provide the commutative algebra version of a crossed corner of groups, presented by Alp [8, 16, 17].

Definition 2.1. A crossed corner of algebras is a diagram of commutative algebras

$$\begin{array}{ccc} K_1 & \xrightarrow{\partial} & K_2 \\ {\scriptstyle \partial'}\downarrow & & \\ K_3 & & \end{array}$$

together with algebra actions of K_2 on K_1 and K_3 on K_1 and homomorphisms $\partial : K_1 \to K_2$ and $\partial' : K_1 \to K_3$ of algebras with a map $h : K_2 \otimes K_3 \to K_1$ satisfying the following axioms:

CC1. ∂ and ∂' are crossed modules of algebras

CC2. $h((k_2 + k_2{}') \otimes k_3) = h(k_2 \otimes k_3) + h(k_2{}' \otimes k_3)$ and $h(k_2 \otimes (k_3 + k_3{}')) = h(k_2 \otimes k_3) + h(k_2 \otimes k_3{}')$

CC3. $h(\partial(k_1) \otimes k_3) = k_3 \cdot k_1$ and $h(k_2 \otimes \partial'(k_1)) = k_2 \cdot k_1$

CC4. $(k_2 \cdot k_3) \cdot k_1 = (k_2k_3) \cdot k_1$ and $(k_3 \cdot k_2) \cdot k_1 = (k_3k_2) \cdot k_1$

where the actions

$$k_3 \cdot k_2 = \partial' h(k_2 \otimes k_3)$$

and

$$k_2 \cdot k_3 = \partial h(k_2 \otimes k_3)$$

These two actions are commutative algebra actions, for all $k_1 \in K_1$, $k_2, k_2' \in K_2$, $k_3, k_3' \in K_3$.

Example 2.2. Let I_1 and I_2 be two ideals of a k-algebra I. The following diagram of inclusions

$$\begin{array}{ccc} I_1 \cap I_2 & \xrightarrow{\partial} & I_1 \\ {\scriptstyle \partial'}\downarrow & & \\ I_2 & & \end{array}$$

together with the actions of I_1, I_2 on $I_1 \cap I_2$ given by multiplication and the function $h : I_1 \otimes I_2 \to I_1 \cap I_2$, $h(i_1 \otimes i_2) = i_1 i_2$ is a crossed corner. It can be observed that this is a crossed corner of commutative algebras.

2.1. Morphisms of Crossed Corners

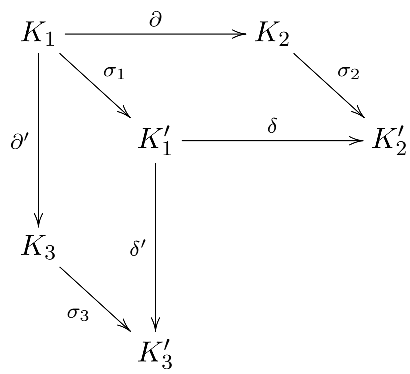
In this section, we define the morphism between two crossed corners. Let

$$\begin{array}{ccc} \mathcal{K}: & K_1 & \xrightarrow{\partial} K_2 \\ & \partial' \downarrow & \\ & K_3 & \end{array}$$

and

$$\begin{array}{ccc} \mathcal{K}': & K_1' & \xrightarrow{\delta} K_2' \\ & \delta' \downarrow & \\ & K_3' & \end{array}$$

be crossed corners together with maps $h : K_2 \otimes K_3 \to K_1$ and $h' : K_2' \otimes K_3' \to K_1'$. The morphism $\sigma = (\sigma_1, \sigma_2, \sigma_3) : \mathcal{K} \to \mathcal{K}'$ is provided by the following commutative diagram



where $\delta'\sigma_1 = \sigma_3\partial'$ and $\delta\sigma_1 = \sigma_2\partial$ and for $k_2 \in K_2, k_3 \in K_3$

$$\sigma_1 h(k_2 \otimes k_3) = h'(\sigma_2(k_2) \otimes \sigma_3(k_3))$$

Furthermore, for $k_2 \in K_2, k_1 \in K_1$,

$$\sigma_1(k_2 \cdot k_1) = \sigma_2(k_2) \cdot \sigma_1(k_1)$$

and for $k_3 \in K_3$

$$\sigma_1(k_3 \cdot k_1) = \sigma_3(k_3) \cdot \sigma_1(k_1)$$

and where $\sigma_1, \sigma_2, \sigma_3$ are k-algebra homomorphism.

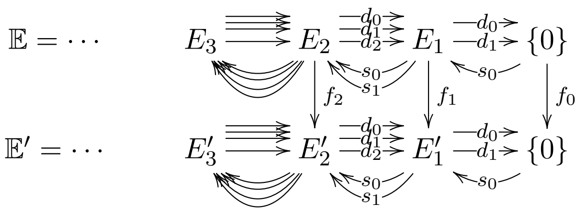
Thus, we can define the category of crossed corners of algebras denoting it as CC.

3. (Bi)simplicial Algebras

Recall from [12] that a simplicial algebra $\mathbb{E}$ consists of k-algebras E_n, for $n \in Z^+ \cup \{0\}$, together with the homomorphisms $d_i^n : E_n \to E_{n-1}$, $0 \leq i \leq n$, and $s_j^n : E_n \to E_{n+1}$, $0 \leq j \leq n$, called faces and degeneracies, respectively, satisfying the usual simplicial identities given in [4]. As an alternative description of a simplicial algebra, we can say that a simplicial algebra $\mathbb{E}$ can be regarded as a functor from the opposite category of finite ordinals $\Delta^{op}[n]$, for $n \in Z^+ \cup \{0\}$ to the category of algebras. That is, $\mathbb{E}$ is simplicial object in the category of commutative algebras. For each $k \geq 0$, it is obtained a subcategory $\Delta[n]_{\leq k}$ of $\Delta[n]$ whose objects are $[j] = \{0 < 1 < \cdots < j\}$ of $\Delta[n]$ with $j \leq k$. Then, for each $k \geq 0$, we can obtain a k-truncated simplicial algebra by defining the functor $\mathbb{E} : \Delta[n]_{\leq k} \to Alg$. Let $\mathbb{E}$ be a simplicial algebra. Then, its Moore complex (NE, ∂) is a chain complex defined on each level by $NE_n = \bigcap\limits_{i=0}^{n-1} \mathrm{Ker} d_i^n = \mathrm{Ker} d_0^n \cap \mathrm{Ker} d_1^n \cap \ldots \cap \mathrm{Ker} d_{n-1}^n$ with the boundary morphism $\partial_n : NE_n \to NE_{n-1}$ restricted to NE_n of the morphism $d_n^n : E_n \to E_{n-1}$. Thus, we can illustrate the Moore (chain) complex by

$$(NE, \partial) : \cdots \xrightarrow{\partial_3} NE_2 \xrightarrow{\partial_2} NE_1 \xrightarrow{\partial_1} NE_0$$

If $NE_n = \{0\}$, for $n \geq k+1$, then the Moore complex NE is of length k. We will denote the category of simplicial algebras with Moore complex of length k by $SimpAlg_{\leq k}$. If the first component E_0 of a simplicial algebra $\mathbb{E}$ is zero, that is $E_0 = \{0\}$, then $\mathbb{E}$ is called a reduced simplicial algebra. We denote the category of reduced simplicial algebras with Moore complex of length k by $ReSimpAlg_{\leq k}$. A morphism between reduced simplicial algebras in this category is given by the following diagram



in which $f : \mathbb{E} \to \mathbb{E}'$ consists of k-algebra homomorphisms $f_i : E_i \to E'_i$, $i \in \mathbb{Z}^+ \cup \{0\}$, commuting with all the face and degeneracy operators.

Arvasi and Porter [12] have defined the functions $C_{\alpha,\beta}$ in the Moore complex of a simplicial algebra $\mathbb{E}$. We recall these functions to use them in the connection between reduced simplicial algebras and crossed corners. We only use these functions in dimension 3. These functions are

$$C_{(0),(2,1)}(x, z) = (s_2 s_1(x))(-s_0(z) + s_1(z) + s_2(z))$$

and

$$C_{(2,0),(1)}(x, z) = (s_2 s_0(x) - s_2 s_1(x))(s_1(z) - s_2(z))$$

For the images of these functions under the boundary map ∂_3, see [12].

Now consider the product category $\Delta[n] \times \Delta[n]$ whose objects are the pairs $([p], [q])$ and whose morphisms between objects are the pairs of non-decreasing maps. Then, the functor $\mathbb{E}_{.,.}$ from $(\Delta \times \Delta)^{op}$ to Alg can be regarded as a bisimplicial algebra. Thus, we can give the definition of a bisimplicial algebra equivalently as follows. For each object (p, q) of $(\Delta \times \Delta)^{op}$, there is an k-algebra $E_{p,q}$ and for each morphism between the pairs (p, q), there are homomorphisms of algebras

$$\begin{array}{lll} d_i^h : E_{p,q} \to E_{p-1,q}; & s_i^h : E_{p,q} \to E_{p+1,q}, & p \geq i \geq 0 \\ d_j^v : E_{p,q} \to E_{p,q-1}; & s_j^v : E_{p,q} \to E_{p,q+1}, & q \geq j \geq 0 \end{array}$$

such that morphisms d_j^v, s_j^v commute with d_i^h, s_i^h. Furthermore, these morphisms satisfy the usual simplicial identities. The Moore bicomplex of a bisimplicial algebra $\mathbb{E}_{.,.}$ is given by

$$NE_{n,m} = \bigcap_{(i,j)=(0,0)}^{(n-1,m-1)} \mathrm{Ker} d_i^h \cap \mathrm{Ker} d_j^v$$

with the boundary homomorphisms $\partial_i^h : NE_{n,m} \to NE_{n-1,m}$ and $\partial_j^v : NE_{n,m} \to NE_{n,m-1}$ obtained by the restriction to d_i^h and d_j^v, respectively. Thus, we can show pictorially a Moore bicomplex of a bisimplicial algebra by the following diagram

$$\begin{array}{ccccc} & & \downarrow & & \downarrow \\ \cdots & \xrightarrow{\vdots} & NE_{1,2} & \xrightarrow{\vdots} & NE_{0,2} \\ & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & NE_{1,1} & \longrightarrow & NE_{0,1} \\ & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & NE_{1,0} & \longrightarrow & NE_{0,0} \end{array}$$

If $E_{0,0}$ is a zero in a bisimplicial algebra $\mathbb{E}_{.,.}$, then it is called a reduced bisimplicial algebra. If $NE_{p,q} = \{0\}$, for $p+q \geq k+1$, then the Moore bicomplex is of length k. For 2-dimensional version of $C_{\alpha,\beta}$ functions for bisimplicial algebras, see [18].

4. From Reduced Simplicial Algebras to Crossed Corners

In this section, we investigate the relation between the categories of crossed corners and reduced simplicial algebras. Suppose that $\mathbb{E}$ is a reduced simplicial algebra with $E_0 = \{0\}$. We will construct a crossed corner of commutative algebras as

$$\begin{array}{ccc} K_1 & \xrightarrow{\partial} & K_2 \\ {\scriptstyle \partial'}\downarrow & & \\ K_3 & & \end{array}$$

with the h-map $h : K_2 \otimes K_3 \to K_1$.

Suppose $K_2 = NE_1 = \mathrm{Ker} d_0^1$ and $K_3 = NE_1^* = \mathrm{Ker} d_1^1$. Let $K_1 = \overline{NE_2} = NE_2/_{\partial_3(NE_3 \cap I_3)}$, where I_3 is the ideal of E_3 generated by the degeneracy elements given in [12]. Then, the action of K_2 on K_1 is given by $k_2 \in K_2$ and $\overline{k_1} = k_1 + \partial_3(NE_3 \cap I_3) \in K_1$, $k_2 \cdot \overline{k_1} = \overline{s_1(k_2)k_1}$, and $\overline{k_1} \cdot k_2 = \overline{k_1 s_1(k_2)}$. The action of $k_3 \in K_3$ on K_1 is given by $k_3 \cdot \overline{k_1} = \overline{s_1(k_3)k_1} = \overline{k_1 s_1(k_3)} = \overline{k_1} \cdot k_3$. The homomorphism $\partial : K_1 \to K_2$ is given by the restriction of $d_2^2 : E_2 \to E_1$ on $\mathrm{Ker} d_0^1$ and similarly $\partial' : K_1 \to K_3$ is given by the restriction of d_2^2 on $\mathrm{Ker} d_1^1$. Then, we obtain the following diagram:

$$\begin{array}{ccc} NE_2/_{\partial_3(NE_3 \cap I_3)} & \xrightarrow{\partial_2} & NE_1 \\ {\scriptstyle \partial_2^*}\downarrow & & \\ NE_1^* & & \end{array}$$

where $x \in NE_1 = \mathrm{Ker} d_0^1$ and $y \in NE_1^* = \mathrm{Ker} d_1^1$ and h map is provided by

$$\begin{array}{rcl} h : NE_1 \otimes NE_1^* & \longrightarrow & NE_2/_{\partial_3(NE_3 \cap I_3)} \\ (x \otimes y) & \longmapsto & \overline{s_1(x)s_1(y) - s_0(x)s_1(y)} = (s_1(x) - s_0(x))s_1(y) + \partial_3(NE_3 \cap I_3) \end{array}$$

We will show that all axioms of crossed corner are verified.

CC1. ∂_2 and ∂_2^* are crossed modules. Because, there are actions of NE_1^* on $\overline{NE_2} = NE_2/_{\partial_3(NE_3 \cap I_3)}$ and NE_1 via s_1 and NE_1 acts on $NE_2/_{\partial_3(NE_3 \cap I_3)}$and NE_1^* via s_1. For $x \in NE_1$ and $\overline{y} = y + \partial_3 NE_3 \in \overline{NE_2}$,

$$\partial_2(x \cdot \overline{y}) = \partial_2(\overline{x \cdot y}) = \partial_2(\overline{s_1 x y}) = x\partial_2(\overline{y})$$

and for $\overline{y}, \overline{y'} \in \overline{NE_2}$,

$$\partial_2(\overline{y}) \cdot \overline{y'} = s_1 d_2 y.y' + \partial_3(NE_3 \cap I_3)$$

We know from the $C_{\alpha,\beta}$ functions from [12] that

$$yy' - s_1d_2y \cdot y' = d_2(s_1ys_1y' - s_0ys_1y') \in \partial_3(NE_3 \cap I_3)$$

Thus,

$$\partial_2(\overline{y}) \cdot \overline{y'} = \overline{yy'}$$

and then ∂_2 is a crossed module of algebras. Similarly ∂_2^* is a crossed module of algebras.

CC2. For $x_1, x_2 \in NE_1$ and $y \in NE_1^*$, it must be $h((x_1+x_2)\otimes y) = h(x_1 \otimes y) + h(x_2 \otimes y)$.

$$\begin{aligned}
h((x_1+x_2)\otimes y) &= s_1(x_1+x_2)s_1(y) - s_0(x_1+x_2)s_1(y) + \partial_3(NE_3 \cap I_3)\\
&= (s_1(x_1)+s_1(x_2))s_1(y) - (s_0(x_1)+s_0(x_2))s_1(y) + \partial_3(NE_3 \cap I_3)\\
&= (s_1(x_1)s_1(y) + s_1(x_2)s_1(y)) - (s_0(x_1)s_1(y) + s_0(x_2)s_1(y)) + \partial_3(NE_3 \cap I_3)\\
&= (s_1(x_1)s_1(y) - s_0(x_1)s_1(y)) + (s_1(x_2)s_1(y) - s_0(x_2)s_1(y)) + \partial_3(NE_3 \cap I_3)\\
&= h(x_1 \otimes y) + h(x_2 \otimes y)
\end{aligned}$$

Similarly, for $x \in NE_1$ and $y_1, y_2 \in NE_1^*$, $h(x \otimes (y_1+y_2)) = h(x\otimes y_1) + h(x \otimes y_2)$ is satisfied.

CC3. For $\overline{z} = z + \partial_3(NE_3 \cap I_3) \in NE_2/_{\partial_3(NE_3\cap I_3)}$, $x \in NE_1, y \in NE_1^*$ it must be $h(\partial_2(\overline{z})\otimes y) = y \cdot \overline{z}$ and $h(x \otimes \partial_2^*(\overline{z})) = x \cdot \overline{z}$. We will use the image of the $C_{\alpha,\beta}$ pairings in the Moore complex of a simplicial commutative algebra. For the image of these elements, see [12].

Firstly, $h(\partial_2(\overline{z})\otimes y) = s_1d_2(z)s_1(y) - s_0d_2(z)s_1(y) + \partial_3(NE_3 \cap I_3)$. For $\alpha = (0)$ and $\beta = (2,1)$, from [12],

$$\begin{aligned}
d_3(C_{(0),(2,1)}(y,z)) &= d_3[(s_2s_1(y))(-s_0(z) + s_1(z) + s_2(z))]\\
&= d_3(s_2s_1(y))(-d_3s_0(z) + d_3s_1(z) + d_3s_2(z))\\
&= s_1(y)(-s_0d_2(z) + s_1d_2(z) + z) \qquad (\because d_3s_2 = id, d_3s_0 = s_0d_2, d_3s_1 = s_1d_2)\\
&= s_1d_2(z)s_1(y) - s_0d_2(z)s_1(y) + s_1(y)z \in \partial_3(NE_3 \cap I_3)
\end{aligned}$$

Thus,

$$\begin{aligned}
h(\partial_2(\overline{z})\otimes y) &= s_1(y)\overline{z} \qquad (\text{mod } \partial_3(NE_3 \cap I_3))\\
&= y \cdot \overline{z}
\end{aligned}$$

Similarly, $h(x \otimes d_2^*(\overline{z})) = s_1(x)s_1d_2(z) - s_0(x)s_1d_2(z) + \partial_3(NE_3 \cap I_3)$. For $\alpha = (2,0)$ and $\beta = (1)$, from [12],

$$\begin{aligned}
d_3(C_{(2,0),(1)}(x,z)) &= d_3[(s_2s_0(x) - s_2s_1(x))(s_1(z) - s_2(z))]\\
&= d_3s_2s_0(x)d_3s_1(z) - d_3s_2s_0(x)d_3s_2(z) - d_3s_2s_1(x)d_3s_1(z) + d_3s_2s_1(x)d_3s_2(z)\\
&= s_0(x)s_1d_2(z) - s_0(x)z - s_1(x)s_1d_2(z) + s_1(x)z \in \partial_3(NE_3 \cap I_3)
\end{aligned}$$

Then,

$$\begin{aligned}
h(x \otimes \partial_2^*(\overline{z})) &= s_1(x)\overline{z} - s_0(x)\overline{z} \qquad (\text{mod } \partial_3(NE_3 \cap I_3))\\
&= s_1(x)\overline{z} \qquad (\because \partial_1(x) = 0)\\
&= x \cdot \overline{z}
\end{aligned}$$

CC4. We show $(x \cdot y) \cdot \overline{z} = (xy) \cdot \overline{z}$, for $x \in NE_1, y \in NE_1^*$, and $\overline{z} \in NE_2/_{\partial_3(NE_3\cap I_3)}$:

$$
\begin{aligned}
(x \cdot y) \cdot \overline{z} &= (\partial_2 h(x \otimes y)) \cdot \overline{z} \qquad (\because x \cdot y = \partial_2 h(x \otimes y)) \\
&= d_2(s_1(x)s_1(y) - s_0(x)s_1(y)) \cdot \overline{z} \\
&= (d_2 s_1(x) d_2 s_1(y) - d_2 s_0(x) d_2 s_1(y)) \cdot \overline{z} \\
&= (xy - d_2 s_0(x) y) \cdot \overline{z} \\
&= (xy - s_0 d_1(x) y) \cdot \overline{z} \\
&= (xy) \cdot \overline{z} \qquad (\because\ \partial_1(x) = 0)
\end{aligned}
$$

Similarly, the axiom $(y \cdot x) \cdot \overline{z} = (yx) \cdot \overline{z}$ is satisfied.

Thus, we obtained a crossed corner of a reduced simplicial algebra. If the length of the Moore complex of given reduced simplicial algebra $\mathbb{E}$ is 2, then $NE_3 = \{0\}$ and thus $\partial_3(NE_3 \cap I_3) = \{0\}$. Therefore, the equivalence between cosets becomes equality. Thus, we have defined a functor from the category of reduced simplicial algebras to the category of crossed corners,

$$N : ReSimpAlg_{\leq 2} \to CC$$

5. From Crossed Corners to Reduced Simplicial Algebras

In this section, we will construct a reduced simplicial algebra with Moore complex of length ≤ 2 from a crossed corner

$$
\begin{array}{ccc}
K_1 & \xrightarrow{\partial} & K_2 \\
{\scriptstyle \partial'} \downarrow & & \\
K_3 & &
\end{array}
$$

together with the h-map $h : K_2 \otimes K_3 \to K_1$. We can consider this crossed corner as a crossed square

$$
\begin{array}{ccc}
K_1 & \xrightarrow{\partial} & K_2 \\
{\scriptstyle \partial'} \downarrow & & \downarrow {\scriptstyle \zeta} \\
K_3 & \xrightarrow[\zeta']{} & K_0 = \{0\}
\end{array}
$$

with the h-map $h : K_2 \otimes K_3 \to K_1$. Since $\zeta : K_2 \to \{0\}$ is the zero morphism, then we obtain a diagonal simplicial algebra

$$\cdots K_2 \ltimes (K_2 \ltimes \{0\}) \begin{array}{c} \longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longleftarrow \\ \longleftarrow \end{array} K_2 \ltimes \{0\} \begin{array}{c} \longrightarrow \\ \longrightarrow \\ \longleftarrow \end{array} \{0\}$$

and then we can say that this is a reduced simplicial algebra. In this structure, the face and degeneracy maps are given by

$$d_0^1(k_2, 0) = d_1^1(k_2, 0) = 0 \qquad s_0^0(0) = (0, 0)$$

and

$$
\begin{aligned}
d_0^1(k_2, k_2', 0) &= (k_2 k_2', 0) \\
d_1^1(k_2, k_2', 0) &= (k_2, 0) \\
d_2^1(k_2, k_2', 0) &= (k_2', 0) \\
s_0^1(k_2, 0) &= (0, k_2, 0) \\
s_1^1(k_2, 0) &= (k_2, 0, 0)
\end{aligned}
$$

Since $K_2 \times \{0\} \cong K_2$, we can write it as

$$\cdots K_2 \ltimes K_2 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longleftarrow\\ \longleftarrow}\; K_2 \;\substack{\longrightarrow\\ \longrightarrow\\ \longleftarrow}\; \{0\}$$

Similarly, since $\zeta' : K_3 \to \{0\}$ is a crossed module, we obtain a reduced simplicial algebra as

$$\cdots K_3 \ltimes K_3 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longleftarrow\\ \longleftarrow}\; K_3 \;\substack{\longrightarrow\\ \longrightarrow\\ \longleftarrow}\; \{0\}$$

Using the actions of K_2 on K_1 and of K_3 on K_1, we have a reduced bisimplicial algebra.

$$\begin{array}{ccccc}
\cdots & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longleftarrow\\ \longleftarrow} & (K_1 \ltimes K_3) \ltimes ((K_1 \ltimes K_3) \ltimes K_2) & \substack{\longrightarrow\\ \longrightarrow\\ \longleftarrow} & K_2 \ltimes K_2 \\
\uparrow\uparrow\downarrow\downarrow\downarrow & & \uparrow\uparrow\downarrow\downarrow\downarrow & & \uparrow\uparrow\downarrow\downarrow\downarrow \\
((K_1 \ltimes K_3) \ltimes K_2) \ltimes (K_2 \ltimes K_2) & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longleftarrow\\ \longleftarrow} & (K_1 \ltimes K_3) \ltimes K_2 & \substack{\longrightarrow\\ \longrightarrow\\ \longleftarrow} & K_2 \\
\uparrow\downarrow\downarrow & & \uparrow\downarrow\downarrow & & \uparrow\downarrow\downarrow \\
K_3 \ltimes K_3 & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longleftarrow\\ \longleftarrow} & K_3 & \substack{\longrightarrow\\ \longrightarrow\\ \longleftarrow} & \{0\}
\end{array}$$

We will use the way from bisimplicial algebras to simplicial algebras with the help of the functor defined by Artin Mazur [19]. The subset of the algebra $E_{1,0} \times E_{0,1} = (K_2 \ltimes \{0\}) \times (K_3 \ltimes \{0\})$ is

$$E_1 = \{((k_2, 0), (k_3, 0)) | d_0^v(k_2, 0) = d_1^h(k_3, 0) = 0\}$$

where $\{0\} \cong E_0$. The isomorphism between E_1 and $E_{1,0} \times E_{0,1}$ can be defined by

$$\begin{array}{rcl} \eta : E_1 & \longrightarrow & K_3 \ltimes K_2 \ltimes \{0\} \\ ((k_2, 0), (k_3, 0)) & \longmapsto & (k_3, k_2, 0) \end{array}$$

Thus, we can write $E_1 \cong K_3 \ltimes K_2 \ltimes \{0\}$. Then, we have the structural homomorphisms between E_0 and E_1 obviously

$$d_0(k_3, k_2, 0) = 0$$

and

$$d_1(k_3, k_2, 0) = 0$$

Hence, we obtain $\{E_1, E_0\}$ as a reduced 1-truncated simplicial algebra together with these zero homomorphisms. Moreover, the elements of the subalgebra

$$E_{2,0} \times E_{1,1} \times E_{0,2} = (K_2 \ltimes (K_2 \ltimes \{0\})) \times ((K_1 \ltimes K_3) \ltimes (K_2 \ltimes \{0\})) \times (K_3 \ltimes (K_3 \ltimes \{0\}))$$

can be written by

$$((k_2', k_2'', 0), ((k_1, k_3), (k_2, 0)), (k_3', k_3'', 0))$$

Moreover, the vertical face maps are

$$d_0^v(k_2', k_2'', 0) = d_1^h(k_1, k_3, k_2, 0)$$

and

$$d_1^v(k_1, k_3, k_2, 0) = d_2^h(k_3', k_3'', 0)$$

Then, $k_2'' = k_2$ and $\partial'(k_1)k_3 = k_3'$. Thus, it can be written by

$$((k_2', k_2'', 0), ((k_1, \partial'(k_1)k_3'), (k_2, 0)), (\partial'(k_1)k_3, k_3'', 0))$$

as elements of E_2. We see that the map

$$\begin{array}{rccl} \eta' : & E_2 & \longrightarrow & (K_1 \ltimes (K_3 \ltimes K_2)) \ltimes (K_3 \ltimes (K_2 \ltimes \{0\})) \\ & ((k_2', k_2'', 0), ((k_1, k_3), (k_2'', 0)), (\partial'(k_1)k_3, k_3'', 0)) & \longmapsto & ((k_1, (k_3, k_2'')), (k_3'', (k_2, 0))) \end{array}$$

is an isomorphism. Consequently, using the Artin-Mazur codiagonal functor, we obtain the following reduced simplicial algebra.

$$\mathbb{E} : (K_1 \ltimes (K_3 \ltimes K_2)) \ltimes (K_3 \ltimes (K_2 \ltimes \{0\})) \rightrightarrows (K_3 \ltimes (K_2 \ltimes \{0\})) \rightrightarrows \{0\}$$

The faces and degeneracies maps are defined as follows:

$$\begin{aligned} d_0^1(k_3, k_2, 0) &= 0 \\ d_1^1(k_3, k_2, 0) &= 0 \\ s_0^0(0) &= (0, 0, 0) \end{aligned}$$

and

$$\begin{aligned} d_0^2((k_1, (k_3, k_2^{''})), (k_3^{''}, (k_2^{'}, 0))) &= (k_3^{''}, \partial(k_1)k_2^{''}, 0) \\ d_1^2((k_1, (k_3, k_2^{''})), (k_3^{''}, (k_2^{'}, 0))) &= (k_3^{''}, \partial^{'}(k_1)k_3, k_2^{''}k_2^{'}, 0) \\ d_2^2((k_1, (k_3, k_2^{''})), (k_3^{''}, (k_2^{'}, 0))) &= (k_3, k_2^{'}, 0) \\ s_0^1(k_3, k_2, 0) &= ((0, (0, k_2)), (k_3, (0, 0))) \\ s_1^1(k_3, k_2, 0) &= ((0, (k_3, 0)), (0, (k_2, 0))) \end{aligned}$$

We can get a 2-truncated reduced simplicial algebra. Using the coskeleton functor from k-truncated simplicial algebras to simplicial algebras with Moore complex of length k given in [12], we can see that $\mathbb{E}$ is a reduced simplicial algebra with Moore complex of length 2. Therefore, we obtained the following functor

$$\Delta : CC \to ReSimpAlg_{\leq 2}$$

We can provide the following result:

Theorem 5.1. The category of reduced simplicial algebras with Moore complex of length 2 is equivalent to that of crossed corners of commutative algebras.

6. Conclusion

In this paper, the commutative algebra analog of crossed corners has been introduced. We have obtained that the category of reduced simplicial algebras with Moore complex of length 2 is equivalent to that of crossed corners of commutative algebras. We know a categorical equivalence exists between braided crossed modules, reduced quadratic modules, and reduced simplicial groups with Moore complex of length 2. This result establishes the equivalence between crossed corners and braided crossed modules or reduced quadratic modules. Furthermore, this idea can be extended to the Lie algebra case for further research.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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