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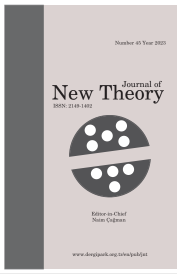

An Application of Nonstandard Finite Difference Method to a Model Describing Diabetes Mellitus and Its Complications

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Abstract — In this study, a mathematical model describing diabetes mellitus and its complications in a population is considered. Since standard numerical methods can lead to numerical instabilities, it aims to solve the problem using a nonstandard method. Among the nonstandard methods, nonstandard finite difference (NSFD) schemes that satisfy dynamical consistency are preferred to make the model discrete. Both continuous and discrete models are analyzed to show the stability of the model at the equilibrium points. The Schur-Cohn criterion is used to perform stability analysis at the equilibrium point of the discretized model. Thus, asymptotically stability of the model is presented. Moreover, the advantages of the NSFD method are emphasized by comparing the stability for different step sizes with classical methods, such as Euler and Runge-Kutta. It has been observed that the NSFD method is convergence for larger step sizes. In addition, the numerical results obtained by NSFD schemes are compared with the Runge–Kutta–Fehlberg (RKF45) method in graphical forms. The accuracy of the NSFD method is observed.

Keywords *Diabetes mellitus, nonstandard finite difference scheme, stability analysis*

Mathematics Subject Classification (2020) 34A30, 65L07

1. Introduction

Many biological problems can be modeled by using differential equations. As is known, diabetes has become a very common disease recently. Many important studies about diabetes have been performed. In epidemic models, stability analysis has an important role. Some of the studies about diabetes can be summarized as follows:

Boutayeb et al. [1] propose a mathematical model of diabetes to present a better quality of line for humans. The numerical solution and the stability analysis for the linear model in which the unknowns are numbers of diabetics with and without complications are presented. Akinsola and Oluyo [2–4] obtain the numerical and analytical solution of the model of complications and control of diabetes mellitus in their studies with different methods. Moreover, the linear diabetes mellitus model is considered by AlShurbaji et al. [5]. The numerical comparison of the solution of a system of linear differential equations by numerical methods such as Euler, Heun, Runge-Kutta, and Adams-Moulton is presented. Stability analysis is given. Furthermore, Vanitha and Porchelvi [6] consider the linear mathematical model of diabetes mellitus. A numerical solution by the Euler-Cauchy method is

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presented. Besides, Boutayeb et al. [7] present a nonlinear mathematical model of diabetes mellitus by applying appropriate parameters. Stability analysis and numerical experiments are presented. de Oliveira et al. [8] consider the model proposed in [7] and global asymptotic stability is studied. The stability is verified through numerical simulations. Boutayeb et al. [9] present the dynamics of a population of healthy people, pre-diabetics, and diabetics with and without complications. Optimal control theory is used. Permatasari et al. [10] considered the model developed by [9]. Global stability and controllability for linear and nonlinear systems are presented, respectively. Widyaningsih et al. [11] consider the nonlinear diabetes mellitus model in terms of lifestyle and genetic factors. Fourth-order Runge-Kutta method is applied to predict the number of deaths due to diabetes, recently. Aye [12] presents stability analysis of a linear model describing diabetes mellitus and its complications. The stability is tested by using the Bellman and Coke theorem. Aye et al. [13] solve a similar model using the Homotopy Perturbation Method. Aye [14] investigates the effect of control on the same model.

It is known that stability analysis of mathematical models plays an important role in the disciplines of applied mathematics. Since, in real-life problems, the points are discrete, discretizing the models is very important in stability analysis. Likewise, in solving such problems, standard numerical methods can lead to numerical instabilities. Hence, nonstandard methods are important. Among the discrete methods, the Nonstandard Finite Difference (NSFD) method developed by Mickens [15–20] is very effective and easy to apply. Moreover, it provides convergence results in even bigger step sizes. The detailed literature survey about NSFD schemes is presented in the studies of Patidar [21, 22]. There are many studies about NSFD schemes in many disciplines of applied mathematics. Some of the recent studies about NSFD schemes and stability analysis can be listed as follows:

Adekanye and Washington [23] consider a mathematical model presented by the collapse of the Tacoma Narrows Bridge. Two NSFD schemes are constructed for the vertical and torsional models. Graphics present vertical and torsional motions. An application of NSFD schemes to a model of the Ebola virus in Africa is presented in [24] by Anguelov et al. Epidemic fractional models about susceptible-infected (SI) and susceptible-infected-recovered (SIR) are proposed by Arenas et al. in [25]. NSFD schemes are applied, and some comparisons with classical methods are given. Baleanu et al. [26] analyze a novel fractional chaotic system for integer and fractional order cases. Stability analysis is presented for both cases. Numerical simulations are presented with the help of NSFD schemes. Dang and Hoang [27] construct NSFD schemes for two metapopulation models. Stability analysis and other properties, such as positivity, boundedness, and monotone convergence, are presented. Numerical calculations are given to support the theoretical study. Dang and Hoang [28], and Kocabıyık et al. [29] approximate a computer virus model with the NSFD method. Ozdogan and Ongun [30] solve a mathematical model describing the Michaelis-Menten harvesting rate with the help of NSFD schemes. NSFD discretization of a distributed order smoking model is presented to determine the effects of smoking on humans by Kocabiyik and Ongun [31]. A comparison of two different NSFD schemes for the fractional order Hantavirus model is given in the study of Ongun and Arslan [32]. A predator-prey model is constructed by NSFD schemes by Shabbir et al. in [33]. Stability analysis and other properties such as positivity, boundedness, and persistence of solutions are investigated. Vaz and Torres [34] proposed an NSFD scheme for the Susceptible–Infected–Chronic–AIDS (SICA) model. Elementary and global stability are studied. A linear mathematical model of pharmacokinetics is considered by Egbelowo et al. in [35]. The Standard Finite Difference method, NSFD method, and analytical solution are presented. More recent studies about the stability analysis of the mathematical models are presented in [36–39].

In this study, a system of linear ordinary differential equations led from diabetes mellitus and its complications given in [13] is considered. The second section defines the mathematical model and its parameters and variables. The stability of the continuous model is given. The third section is devoted to discretizing the model by the NSFD method. The fourth section includes the stability analysis of the discrete model. The fifth section is the numerical simulation section. Finally, the last section is the conclusion section.

2. The Continuous Model Describing Diabetes Mellitus and Its Complications

This section presents the definition of a mathematical model of diabetes mellitus and its complications provided in [13]. The model consists of a system of linear ordinary differential equations and is defined as

$$\begin{aligned}
\frac{dH}{dt} &= \beta\theta - (\mu+\tau)H + \sigma S \\
\frac{dS}{dt} &= \beta(1-\theta) + \tau H - (\mu+\alpha+\sigma)S \\
\frac{dD}{dt} &= \alpha S - (\mu+\lambda)D + \omega T \\
\frac{dC}{dt} &= \lambda D - (\mu+\delta+\gamma)C \\
\frac{dT}{dt} &= \gamma C - (\mu+\omega)T
\end{aligned} \tag{1}$$

with the initial conditions $H(0) = H_0$, $S(0) = S_0$, $D(0) = D_0$, $C(0) = C_0$, and $T(0) = T_0$, where the variables $H(t)$, $S(t)$, $D(t)$, $C(t)$, and $T(t)$ denote to the healthy, susceptible, diabetics without complication, diabetics with complication and diabetics with complications receiving a cure, respectively. The parameters β, θ, μ, τ, σ, α, λ, ω, δ, and γ denote rate of birth, rate of children born healthy, rate of natural mortality death, the rate at which healthy individual become susceptible, the rate at which susceptible individual become healthy, probability rate of incidence of diabetes, rate of a diabetic person developing complications, rate at which diabetic with complications after cured return diabetic without complications, rate of mortality due to complications and rate at which diabetic with complications are cured.

Hereinafter, the asymptotic stability of the continuous model described by Equation 1 will be presented. Thus, we first give some basic preliminaries. For a general autonomous vector field

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n \tag{2}$$

the linearized system can be defined as

$$\frac{dy}{dt} = J(E)y$$

where E and $J(E)$ denotes the equilibrium point of the Equation 2 and Jacobian matrix of the Equation 2 at the equilibrium point E, respectively.

Theorem 2.1. [40] Suppose all the $J(E)$ have negative reel parts. Then, the equilibrium solution of the nonlinear vector field defined by Equation 2 is asymptotically stable.

The equilibrium point of Equation 1 is obtained as $E^* = (H^*, S^*, D^*, C^*, T^*)$, where

$$H^* = \frac{\beta(\sigma+\theta\mu+\theta\alpha)}{\chi}$$

$$S^* = \frac{-\beta(-\mu-\tau+\theta\mu)}{\chi}$$

$$D^* = \frac{-\alpha\beta(\mu+\omega)(-\mu-\tau+\theta\mu)(\mu+\delta+\gamma)}{\varpi}$$
$$C^* = \frac{-\alpha\beta\lambda(\mu+\omega)(-\mu-\tau+\theta\mu)}{\varpi}$$
$$T^* = \frac{-\alpha\beta\lambda\gamma(-\mu-\tau+\theta\mu)}{\varpi}$$
$$\chi = \mu^2 + (\sigma+\tau+\alpha)\mu + \alpha\tau$$

and

$$\varpi = (\mu^3 + (\omega+\delta+\lambda+\gamma)\mu^2 + (\omega(\gamma+\delta+\lambda) + \lambda(\delta+\gamma))\mu + \omega\lambda\delta)\chi$$

The Jacobian matrix of the continuous model at the equilibrium point $E^* = (H^*, S^*, D^*, C^*, T^*)$ is determined as

$$J(H^*, S^*, D^*, C^*, T^*) = \begin{pmatrix} -\tau-\mu & \sigma & 0 & 0 & 0 \\ \tau & -\mu-\alpha-\sigma & 0 & 0 & 0 \\ 0 & \alpha & -\mu-\lambda & 0 & \omega \\ 0 & 0 & \lambda & -\mu-\delta-\lambda & 0 \\ 0 & 0 & 0 & \gamma & -\mu-\omega \end{pmatrix}$$

Thus, considering Theorem 2.1, the continuous system defined by Equation 1 is asymptotically stable if all the eigenvalues of $J(H^*, S^*, D^*, C^*, T^*)$ have negative reel parts. A detailed analysis of the asymptotic stability of the continuous model will be given in Section 5.

3. Discretization of the Model by NSFD Schemes

In this section, the model defined by Equation 1 is discretized by using NSFD schemes, an effective method. Some advantages of the proposed method are that it removes the numerical instabilities obtained by standard finite difference procedures, gives more approximate results compared to classical methods such as Runge-Kutta and Euler methods, and is converged for bigger step sizes compared to classical methods.

The rules for constructing NSFD schemes and determination of denominator function can be summarized as follows [16]:

i. To avoid numerical instabilities, the order of discrete derivatives should be equal to the derivatives in the differential equations.

ii. The discretization of first-order derivatives is usually in the following general form:

$$\frac{dx}{dt} \to \frac{x_{n+1} - \psi(h)x_n}{\phi(h)}$$

where $\psi(h)$ and $\phi(h)$ are called numerator and denominator functions, respectively.

iii. Nonlinear terms should be replaced by nonlocal discrete terms such as $x^2 \to x_{k+1}x_k$ and $x^2 \to x_k^2$.

iv. Additional conditions for the differential equations should be satisfied for difference equations.

In the view of the procedure given above, the model is discretized by using the following steps to satisfy the positivity conditions:

In the first equation of Equation 1, the replacements $H(t) \to H(n+1)$ and $S(t) \to S(n)$ are used. Similarly, in the second equation of Equation 1, the replacements $H(t) \to H(n)$ and $S(t) \to S(n+1)$; in the third equation of Equation 1, the replacements $S(t) \to S(n)$, $D(t) \to D(n+1)$, and $T(t) \to T(n)$; in the fourth equation of Equation 1, the replacements $D(t) \to D(n)$ and $C(t) \to C(n+1)$; and finally, in the last equation of Equation 1, the replacements $C(t) \to C(n)$ and $T(t) \to T(n+1)$ are implemented. Thus, the following discrete model is obtained:

$$\begin{aligned}
H(n+1) &= \frac{H(n) + (\beta\theta + \sigma S(n))\phi_1}{1 + (\mu + \tau)\phi_1} \\
S(n+1) &= \frac{S(n) + (\beta(1-\theta) + \tau H(n))\phi_2}{1 + (\mu + \alpha + \sigma)\phi_2} \\
D(n+1) &= \frac{D(n) + (\alpha S(n) + \omega T(n))\phi_3}{1 + (\mu + \lambda)\phi_3} \\
C(n+1) &= \frac{C(n) + \lambda\phi_4 D(n)}{1 + (\mu + \delta + \gamma)\phi_4} \\
T(n+1) &= \frac{T(n) + \gamma\phi_5 C(n)}{1 + (\mu + \omega)\phi_5}
\end{aligned} \tag{3}$$

where $\phi_i,\ i = \overline{1,5}$ are denominator functions and determined as

$$\phi_1 = \frac{e^{h(\mu+\tau)} - 1}{\mu + \tau}$$

$$\phi_2 = \frac{e^{h(\mu+\alpha+\sigma)} - 1}{\mu + \alpha + \sigma}$$

$$\phi_3 = \frac{e^{h(\mu+\lambda)} - 1}{\mu + \lambda}$$

$$\phi_4 = \frac{e^{h(\mu+\delta+\gamma)} - 1}{\mu + \delta + \gamma}$$

and

$$\phi_5 = \frac{e^{h(\mu+\omega)} - 1}{\mu + \omega}$$

4. Stability Analysis of the Discretized Model

In this section, the stability analysis of the discretized model is performed. Some theorems and lemmas about the stability and properties such as positivity, and permanence of the solutions of the discrete system given by Equation 3 are presented.

Lemma 4.1. All solutions of discrete system given in Equation 3 are positive with positive initial conditions and positive parameters $\beta, \theta, \mu, \tau, \sigma, \alpha, \lambda, \omega, \delta, \gamma$, and h under the assumption of

$$\frac{S(n)}{\phi_2} > -(\beta(1-\theta) + \tau H(n))$$

PROOF.
Assume that the parameters $\beta, \theta, \mu, \tau, \sigma, \alpha, \lambda, \omega, \delta, \gamma$, and h are positive. Moreover, assume that the initial conditions $H(0) = H_0,\ S(0) = S_0,\ D(0) = D_0,\ C(0) = C_0$, and $T(0) = T_0$ are positive. Then, it is obvious that the denominator functions are all positive, i.e.,

$$\phi_1 = \frac{e^{h(\mu+\tau)} - 1}{\mu + \tau} > 0, \qquad \phi_2 = \frac{e^{h(\mu+\alpha+\sigma)} - 1}{\mu + \alpha + \sigma} > 0,$$

$$\phi_3 = \frac{e^{h(\mu+\lambda)} - 1}{\mu + \lambda} > 0, \qquad \phi_4 = \frac{e^{h(\mu+\delta+\gamma)} - 1}{\mu + \delta + \gamma} > 0,$$

and

$$\phi_5 = \frac{e^{h(\mu+\omega)} - 1}{\mu+\omega} > 0$$

Therefore, for the positive parameters, it is obvious that

$$H(n+1) = \frac{H(n) + (\beta\theta + \sigma S(n))\phi_1}{1 + (\mu+\tau)\phi_1} > 0$$

$$D(n+1) = \frac{D(n) + (\alpha S(n) + \omega T(n))\phi_3}{1 + (\mu+\lambda)\phi_3} > 0$$

$$C(n+1) = \frac{C(n) + \lambda\phi_4 D(n)}{1 + (\mu+\delta+\gamma)\phi_4} > 0$$

and

$$T(n+1) = \frac{T(n) + \gamma\phi_5 C(n)}{1 + (\mu+\omega)\phi_5} > 0$$

As well, assuming $\frac{S(n)}{\phi_2} > -(\beta(1-\theta) + \tau H(n))$, it can be concluded that

$$S(n+1) = \frac{S(n) + (\beta(1-\theta) + \tau H(n))\phi_2}{1 + (\mu+\alpha+\sigma)\phi_2}$$

Thus, the discrete system is positive for all the positive parameters and initial conditions. □

Locally asymptotic stability of the model can be analyzed by obtaining the eigenvalues of the Jacobian matrix at equilibrium points. Local asymptotic stability of the system depends on the eigenvalues of the Jacobian matrix at the equilibrium points.

Theorem 4.2 (Schur-Cohn Criterion). [41] Consider the characteristic polynomial

$$p(\lambda) \quad = \quad \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n \tag{4}$$

where $a_1, a_2, \cdots, a_n$ are constants. The zeros of the characteristic polynomial defined by Equation 4 lie inside the unit disk if and only if the following conditions hold:

i. $p(1) > 0$

ii. $(-1)^n p(-1) = 1 - a_1 + a_2 - \cdots + (-1)^n a_n > 0$

iii. The matrices

$$B^{\pm}_{n-1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n-3} & a_{n-4} & \cdots & 1 & 0 \\ a_{n-2} & a_{n-3} & \cdots & a_1 & 1 \end{pmatrix} \pm \begin{pmatrix} 0 & 0 & \cdots & 0 & a_n \\ 0 & 0 & \cdots & a_n & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & a_n & \cdots & a_4 & a_3 \\ a_n & a_{n-1} & \cdots & a_3 & a_2 \end{pmatrix}$$

are positive innerwise.

Hence, one can conclude that if the Schur-Cohn criterion is satisfied, then the discrete system is asymptotically stable. Note that the equilibrium point of the discrete system given by Equation 3 is the same as the continuous model. Therefore, the Jacobian matrix at the equilibrium point $E^* = (H^*, S^*, D^*, C^*, T^*)$ can be written as

$$J^* = J(H^*, S^*, D^*, C^*, T^*) = \begin{pmatrix} j_1 & \phi_1\sigma j_1 & 0 & 0 & 0 \\ \phi_2\tau j_2 & j_2 & 0 & 0 & 0 \\ 0 & \phi_3\alpha j_3 & j_3 & 0 & \phi_3\omega j_3 \\ 0 & 0 & \phi_4\lambda j_4 & j_4 & 0 \\ 0 & 0 & 0 & \phi_5\gamma j_5 & j_5 \end{pmatrix}$$

where

$$j_1 = \frac{1}{1+\phi_1(\mu+\tau)}$$

$$j_2 = \frac{1}{1+\phi_2(\mu+\alpha+\sigma)}$$

$$j_3 = \frac{1}{1+\phi_3(\mu+\lambda)}$$

$$j_4 = \frac{1}{1+\phi_4(\mu+\delta+\gamma)}$$

and

$$j_5 = \frac{1}{1+\phi_5(\mu+\omega)}$$

The characteristic equation is as follows:

$$p(\lambda) = \lambda^5 + a_1\lambda^4 + a_2\lambda^3 + a_3\lambda^2 + a_4\lambda + a_5 \tag{5}$$

where the coefficients of Equation 5 are determined as

$$\begin{aligned} a_1 &= -(j_1 + j_2 + j_3 + j_4 + j_5) \\ a_2 &= (j_3 + j_4)j_5 + j_3j_4 + (j_1 + j_2)(j_3 + j_4 + j_5) - (1 - \tau\sigma\phi_1\phi_2)j_1j_2 \\ a_3 &= -(1 + \gamma\lambda\omega\phi_3\phi_4\phi_5)j_5j_4j_3 + (\tau\sigma\phi_1\phi_2 - 1)j_2j_1(j_5 + j_4 + j_3) - (j_2 + j_1)[j_5(j_4 + j_3) + j_3j_4] \\ a_4 &= (j_2 + j_1)(1 + \phi_3\phi_4\phi_5\lambda\omega\gamma) + j_2j_1(1 - \sigma\tau\phi_1\phi_2)[j_5(j_4 + j_3) + j_4j_3] \\ a_5 &= j_5j_4j_3j_2j_1(\sigma\tau\phi_1\phi_2 - 1)(1 + \phi_3\phi_4\phi_5\lambda\omega\gamma) \end{aligned} \tag{6}$$

To analyze the stability of the model at the equilibrium point, the following theorem for the discrete system given by Equation 3 is presented.

Theorem 4.3. The discrete system in Equation 3 is locally asymptotically stable at the equilibrium point $E^* = (H^*, S^*, D^*, C^*, T^*)$ if the following conditions are satisfied:

i. $[-1 + \phi_3\phi_4\phi_5\gamma\omega\lambda j_3j_4j_5 + j_5 + (j_5 - 1)(j_4j_3 - j_4 - j_3)][\phi_1\phi_2\tau\sigma j_1j_2 + (1 - j_1)(j_2 - 1)] > 0$

ii. $[1 + \phi_3\phi_4\phi_5\gamma\omega\lambda j_3j_4j_5 + (1 + j_3)(j_4 + j_5(1 + j_4)) + j_3][-\phi_1\phi_2\tau\sigma j_1j_2 + (j_2 + 1)(j_1 + 1)] > 0$

iii. $1 - a_1a_5 + a_4 - a_5^2 > 0$

$$\begin{aligned} &[1 + (a_1a_2a_5 - a_1a_5(1 - a_4))(1 + a_4) + a_4a_5(2a_1 + a_3) - a_4^2(a_4 + a_2 + 1) + (a_5 + a_3 + a_1)(a_5^3 - a_3) \\ &-a_5^2(a_4 + 2 + a_2 + a_1^2 + a_2^2) - a_5a_1^2(a_1 + a_3) + a_4(1 + a_1(a_3 + a_1)) + a_2(1 + a_5(2a_3 - a_5a_4))] > 0 \\ &[1 + a_4a_5(2a_1 - 3a_3) + a_4^2(a_4 - 1 - a_2) + a_3(a_5 - a_3 + a_1(1 + a_4)) - a_1^2(a_4 + a_5(a_3 - a_1 + a_5)) \\ &+a_1a_5(1 - 3a_2 + a_4(a_2 - a_4)) + a_5^2(a_5(a_5 - a_3 - a_1) + a_2(1 - a_2) + a_4(1 + a_2) - 2(1 + a_1a_3)) \\ &-a_4(1 - 2a_2) - a_2(1 - 2a_2a_3a_5)] > 0 \end{aligned}$$

where a_1, a_2, a_3, a_4, and a_5 are denoted by Equation 6.

Proof.
Considering Theorem 4.2, it is obvious that if the conditions *i-iii* in Theorem 4.3 are satisfied, then the discrete system given by Equation 3 is locally asymptotically stable at the equilibrium point $E^* = (H^*, S^*, D^*, C^*, T^*)$. □

5. Numerical Results

This section presents the stability analysis for the parameters given in [13]. The parameters are taken into consideration as

$$\begin{aligned} &\beta = 0.038, \quad \theta = 0.923, \quad \mu = 0.118, \quad \tau = 0.04, \quad \sigma = 0.08, \\ &\alpha = 0.02, \quad \lambda = 0.05, \quad \omega = 0.08, \quad \delta = 0.02, \quad \text{and} \quad \gamma = 0.08 \end{aligned} \tag{7}$$

Under the given parameters above, the stability of the continuous model and discrete model will be analyzed in the view of Theorems 2.1 and 4.2.

5.1. Stability Analysis of the Continuous Model

The characteristic polynomial of the Jacobian matrix at the equilibrium point

$$E^* = (0.2522152093, 0.0597000384, 0.007435253915, 0.001705333467, 0.000689023623)$$

is determined as

$$J(E^*) = \begin{pmatrix} -0.158 & 0.08 & 0 & 0 & 0 \\ 0.04 & -0.218 & 0 & 0 & 0 \\ 0 & 0.02 & -0.168 & 0 & 0.08 \\ 0 & 0 & 0.05 & -0.218 & 0 \\ 0 & 0 & 0 & 0.08 & -0.198 \end{pmatrix}$$

The eigenvalues of $J(E^*)$ is as follows:

$$\lambda_1 = -0.123042327249903$$

$$\lambda_2 = -0.123968757625671$$

$$\lambda_3 = -0.252031242374328$$

and

$$\lambda_{4,5} = -0.230478836375048 \mp 0.0566939252859813i$$

Since all the eigenvalues of $J(E^*)$ have negative reel parts, according to Theorem 2.1, the continuous model defined by Equation 1 is asymptotically stable at the equilibrium point E^*.

5.2. Stability Analysis of the Discrete Model

In addition to the parameters given in Equation 7, the step size is chosen as $h = 0.01$. The Jacobian matrix at the equilibrium point

$$E^* = (0.2522152093, 0.0597000384, 0.007435253915, 0.001705333467, 0.000689023623)$$

is determined as

$$J(E^*) = \begin{pmatrix} 0.9984212475 & 0.0007993683 & 0 & 0 & 0 \\ 0.0003995643 & 0.9978223744 & 0 & 0 & 0 \\ 0 & 0.00019983209 & 0.9983214104 & 0 & 0.0007993283 \\ 0 & 0 & 0.0004994553 & 0.9978223744 & 0 \\ 0 & 0 & 0 & 0.0007992085 & 0.9980219589 \end{pmatrix} \tag{8}$$

The characteristic polynomial of the Jacobian matrix defined by Equation 8 is as

$$p(\lambda) = \lambda^5 + a_1\lambda^4 + a_2\lambda^3 + a_3\lambda^2 + a_4\lambda + a_5$$

where the constants of characteristic polynomials are

$$a_1 = -4.990409365, \quad a_2 = 9.9616737805, \quad a_3 = -9.9425650792, \quad a_4 = 4.9617462802,$$

and

$$a_5 = -0.99044561567$$

In the view of Theorem 4.2, since

i. $p(1) = 1 + a_1 + a_2 + a_3 + a_4 + a_5 = 0.215 \times 10^{-13} > 0$

ii. $-p(-1) = 1 - a_1 + a_2 - a_3 + a_4 - a_5 = 31.84684012 > 0$

iii. The inners of the matrices

$$B_4^{\pm} = \begin{pmatrix} 1 & 0 & 0 & \pm a_5 \\ a_1 & 1 & \pm a_5 & \pm a_4 \\ a_2 & a_1 \pm a_5 & 1 \pm a_4 & \pm a_3 \\ a_3 \pm a_5 & a_2 \pm a_4 & a_1 \pm a_3 & 1 \pm a_2 \end{pmatrix}$$

are the matrices $B_4^{\pm}$ itself and the matrice

$$IB^{\pm} = \begin{pmatrix} 1 & \pm a_5 \\ a_1 \pm a_5 & 1 \pm a_4 \end{pmatrix}$$

Since the determinants of the inners of the matrice $B_4^{\pm}$

$$\det(B_4^{+}) = 0.213 \times 10^{-12}$$

$$\det(B_4^{-}) = 0.614 \times 10^{-24}$$

$$\det(IB_4^{+}) = 0.038034685$$

and

$$\det(IB_4^{-}) = 0.278 \times 10^{-6}$$

are positive, the matrices $B_4^{\pm}$ are positive innerwise. Thus, since all the conditions of the Schur-Cohn criterion are satisfied, the discrete system given in Equation 3 is locally asymptotically stable for the estimated parameters.

A numerical solution obtained by NSFD schemes is presented in the figures to support the stability of the discrete model. Moreover, the Runge-Kutta-Fehlberg (RKF45) method is applied for the estimated parameters. Therefore, the accuracy of the results obtained by the NSFD method is shown.

The estimated parameters defined in Equation 7, the step size $h = 0.01$, and the positive initial condition

$$H(0) = 198195839, \quad S(0) = 101535728, \quad D(0) = 940000, \quad C(0) = 3760000,$$

and

$$T(0) = 1193250$$

are used during the calculations.

Figures 1-5 present the numerical comparison of the results obtained by the NSFD method with the RKF45 method. It can be observed from Figures 1-5 that the results approach the equilibrium point

$$E^* = (0.2522152093, 0.0597000384, 0.007435253915, 0.001705333467, 0.000689023623)$$

The compatibility of the results can also be observed in Figures 1-5.

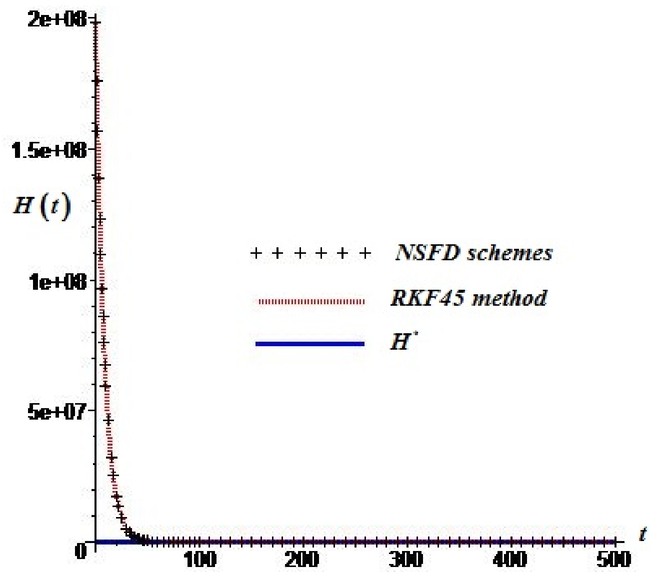

Figure 1. Variation of healthy class $H(t)$

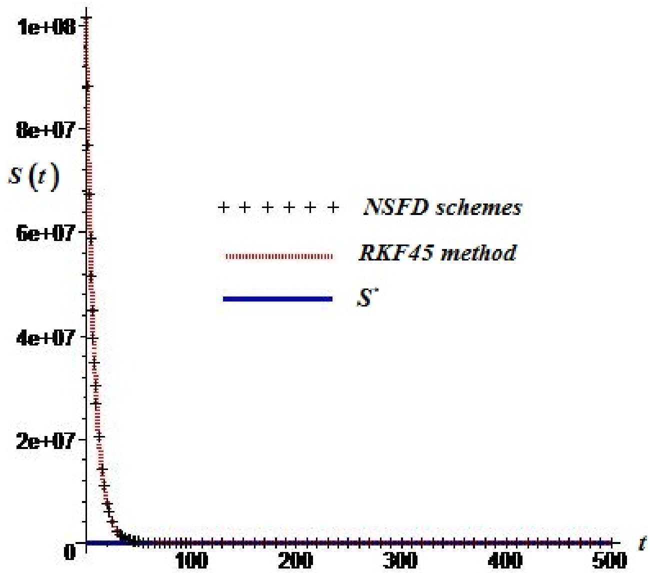

Figure 2. Variation of susceptible class $S(t)$

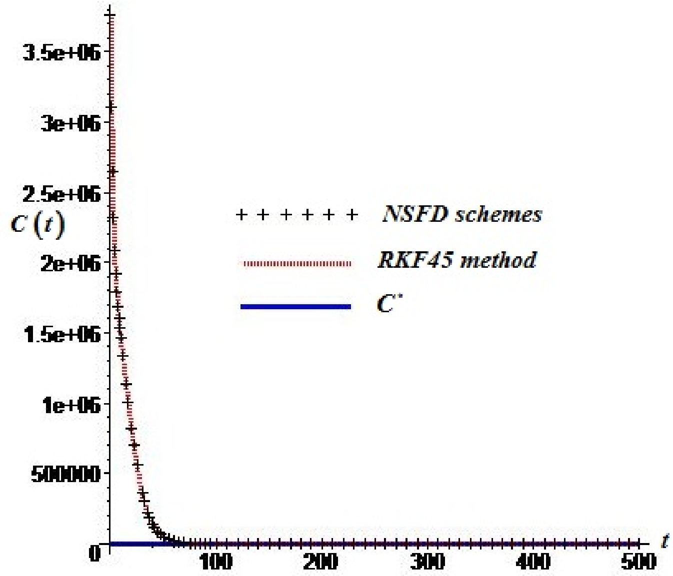

Figure 3. Variation of diabetics without complication $C(t)$

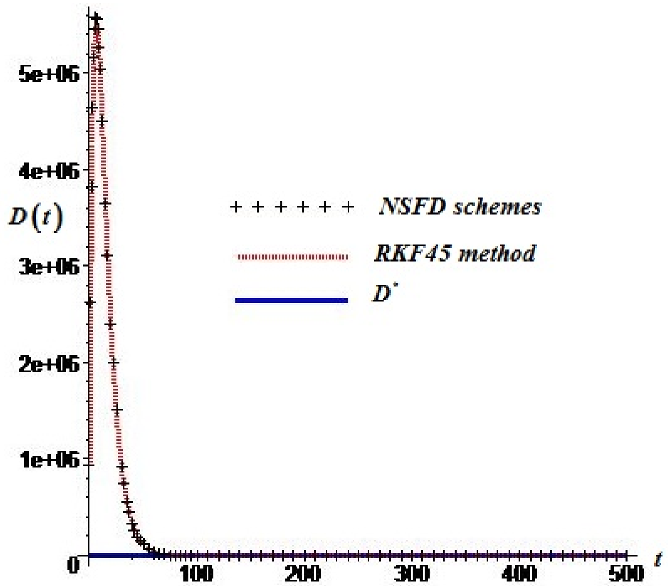

Figure 4. Variation of diabetics with complication $D(t)$

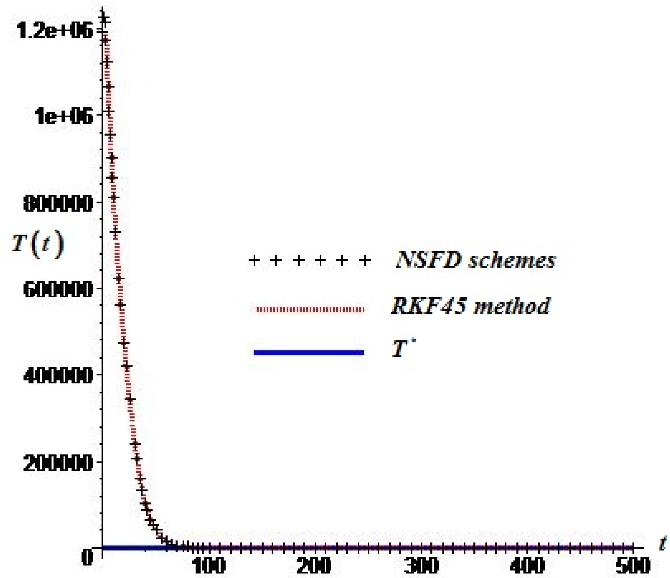

Figure 5. Variation of diabetics with complications receiving a treatment $T(t)$

NSFD method is a very effective method for the bigger step size. Table 1 compares the convergence of the methods. One can see the effectiveness of NSFD schemes from Table 1.

Table 1. Stability of equilibrium point E^* under application of different methods for different step size h

h	**Euler Method**	**Fourth Order Runge-Kutta Method**	**NSFD Schemes**
0.01	Convergence	Convergence	Convergence
0.1	Convergence	Convergence	Convergence
0.5	Convergence	Convergence	Convergence
1	Convergence	Convergence	Convergence
5	Convergence	Convergence	Convergence
7	Divergence	Convergence	Convergence
10	Divergence	Divergence	Convergence
25	Divergence	Divergence	Convergence
50	Divergence	Divergence	Convergence
100	Divergence	Divergence	Convergence

6. Conclusion

This paper presents the stability analysis of a mathematical model describing diabetes mellitus and its complications. The main aims of the study are to analyze the stability of the model and show the advantages of the NSFD method. Thus, the stability of the continuous model is analyzed, and it is concluded that the model is asymptotically stable. Moreover, the continuous model is discretized with the help of the NSFD method. Considering the Schur-Cohn criterion, it is concluded that the discrete model is asymptotically stable, too. The accuracy of the NSFD scheme is supported by comparing the numerical results with the RKF45 method. The compatibility of the numerical results can be seen through graphics. One of the advantages of the NSFD method is to be convergence for the bigger step sizes. The efficiency of the NSFD method for the bigger step size is presented in tabular form.

In future studies, the NSFD schemes for the linear and nonlinear models can be constructed, and their stability analysis can be performed using a similar technique. Moreover, since the NSFD method can be applied to the fractional order differential equations, fractional diabetes models can be solved by the NSFD method. In addition, stability analysis can be given.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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