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A Receding Contact Problem of Two Layers one of Functionally Graded, Loaded by Circular Rigid Block and Resting on a Pasternak Foundation

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ABSTRACT

In this study, frictionless receding contact problem of two elastic layers which one is functionally graded material (FGM) resting on a Pasternak foundation is considered. The external load is applied to the homogeneous elastic layer by means of a circular rigid block and the functionally graded layer rests on a Pasternak foundation. The effect of gravity forces is neglected, and only compressive normal tractions can be transmitted through the interfaces. Displacement and stress expressions for the layers are obtained using the theory of elasticity and integral transformation technique. By applying the boundary conditions for the problem, reduced to two integral equations in which the contact stresses and contact lengths are unknown. The system of integral equations is numerically solved by making use of appropriate Gauss Chebyshev integration formulas. The equilibrium conditions are satisfied in the solution and the contact stresses and contact distances related to the problem are obtained for various dimensionless quantities.

Keywords: Contact problem, functionally graded material, Pasternak foundation.

1. INTRODUCTION

In everyday life, objects are commonly in contact with one other, resulting in displacements and deformations within these objects. To calculate these displacements and deformations, the field of mechanics has transformed such objects into ideal problems that can be solved.

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In the literature, these issues are commonly referred to as contact problems. Throughout history, numerous contact problems requiring complex and lengthy procedures in various geometries have been studied by numerous researchers.

The contact of a layer resting on an elastic half-plane and loaded by a rigid circular block was examined by Dhaliwal [1]. The problem was reduced to a Fredholm integral equation and was solved using power series and numerical methods. The frictionless contact problem of an elastic sheet resting on an elastic half-plane with a spring load applied to the entire surface except for a specific length of the sheet was investigated by Keer and Chantarammungkorn [2]. The contact situation of the elastic layer pressed against a block with different profiles was addressed by Ratwani and Erdogan [4]. Both frictional and frictionless contact of a layer resting on an elastic half-plane and loaded from above by a rigid flat block were explored by Cakıroglu [5]. A contact problem for an elastic layer supported by two elastic quarter planes was analyzed by Aksogan [6]. The contact problem of a composite layer consisting of two elastic beams resting on an elastic base according to the theory of elasticity was solved by Birinci and Erdöl [7]. The contact problem involving two infinite elastic layers resting on the Winkler foundation and subjected to symmetrical distributed loads was studied by Birinci and Erdöl [8]. Frictionless double contact problem for two elastic layers which the lower one is entirely bonded to a rigid support was investigated by Comez [9]. A double receding contact problem involving a rigid stamp and two layers was explored by Comez *et al.* [10]. The plane problem of a frictionless receding contact between an elastic functionally graded (FG) layer and a homogeneous half-space when the two bodies were pressed together was investigated by El-Borgi *et al.* [11]. A frictionless receding contact problem between an anisotropic elastic layer and an anisotropic elastic half plane, with both bodies pressed together by means of a rigid circular stamp, was analyzed by Kahya *et al.* [12]. The frictionless contact problem of two elastic layers with different elastic properties and heights, resting on two homogeneous, isotropic, and symmetric elastic quarter planes according to the theory of elasticity, was examined by Yaylacı and Birinci [13]. A contact model between a homogeneous half-space with a linearly graded layer and a rigid punch was examined by Chen and Chen [14]. The plane problem of a smooth double receding contact between an FG layer and an elastic layer when pressed together was studied by Yan and Li [15]. The moving contact problem for a rigid cylindrical punch and a functionally graded layer was solved by Comez [16]. Continuous contact on a functionally graded layer was analyzed using the finite element method by Polat *et al.* [17]. The plane contact problem based on the theory of elasticity following indentation into an elastic transversely isotropic half-plane with a transversely isotropic FG coating using a non-deformable punch with a flat base was examined by Vasiliev *et al.* [18]. The discontinuous contact problem between a functionally graded (FG) layer, loaded symmetrically with a point load P through a rigid block, and a homogeneous half-space was addressed by Oner and Birinci [19]. The quasi-static frictional contact problem of a rigid rounded punch sliding over a bi-directionally graded half-plane within the framework of plane strain elasticity was examined by Arslan [20]. A method for determining the parameters of the exponentially varying shear modulus of an FG half-space was proposed by Zelentsov *et al.* [21]. The effect of punch speed on frictional contact mechanics of finite-thickness graded layer resting on the rigid foundation researched by Balcı [22]. A double receding contact problem of a functionally graded layer and a homogeneous elastic layer resting on a Winkler foundation was investigated by Birinci *et al.* [23]. Contact analysis of an elastic layer supported by a wedge in plane was investigated by Bakioğlu *et al.* [24]. The problem was formulated with closed-form integral equations. An analysis of a contact problem using analytical, finite element method (FEM), and multilayer perceptron (MLP) approaches was conducted by Yaylacı *et al.* [25]. The main objective of this study was to assess the applicability of MLP analysis for the frictionless contact problem of an FG layer bonded to a rigid foundation. An analytical approach to solving the continuous and discontinuous contact problems of a functionally graded (FG) layer subjected to a distributed load was proposed by Adıyaman *et al.* [26]. The problem was analytically solved by applying boundary conditions for both continuous and discontinuous contact cases.

In this paper, the frictionless contact problem of two infinite layers resting on a Pasternak foundation and loaded by a circular rigid block is considered according to theory of Elasticity. The layers have different material properties and heights, with the upper layer being homogeneous and the lower layer functionally graded. The problem is assumed that contact along the interface is frictionless, and the effect of gravity force is neglected. By applying the boundary conditions for the problem, reduced to two integral equations in which the contact stresses and contact lengths are unknown. Solving the integral equations numerically by using appropriate Gauss Chebyshev integration formulation, and the contact stresses and contact lengths are investigated for various dimensionless values in the problem.

1.1. Formulation of the Problem

Consider the receding contact problem between a homogeneous elastic layer and an FG layer resting on a Pasternak foundation and loaded by a circular rigid block, as shown in Fig.1.

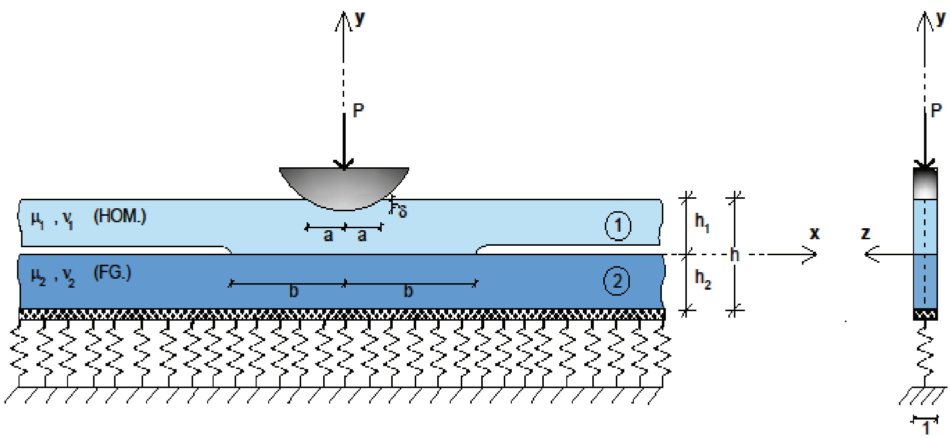


Fig. 1 - Geometry and loading of contact problem.

Observing that a circular rigid block with radius R transmits a concentrated normal force to a homogeneous layer of thickness h_1, and an FG layer of thickness h_2 rests on a Pasternak foundation having an elastic spring constant k_0 and *G* Pasternak foundation moduli.

The circular rigid block and homogeneous layer are in contact with each other on the interval *(-a, a),* and the homogeneous layer and FG are in contact with each other on the interval *(-b, b).* The thickness in the z-direction is unity. For the graded layer, the material was modeled as an inhomogeneous isotropic material with a gradient oriented along the y direction. The Poisson's ratio, υ_2 , was assumed constant, and the shear modulus, μ_2, depended on the y-coordinate only and is expressed as the following exponential function:

$$\mu_2 = \mu_0 . e^{\beta y}, (-h_2 \leq y \leq 0), \tag{1}$$

where μ_0 is the shear modulus on the top surface of the FG layer (for $y = 0$), and β is the stiffness parameter controlling the shear modulus variations in the graded medium. For the homogeneous layer, the shear modulus and Poisson's ratio are, μ_1 and υ_1, respectively.

It is assumed that the contact surfaces are frictionless and only compressive normal tractions can be transmitted through the contact interfaces. Observing that $x = 0$ is a plane symmetry, it is sufficient to consider the problem in the regions $(0 \leq x < \infty)$. The displacement $u(x, y), v(x, y)$ and stress components $\sigma_x(x, y), \sigma_y(x, y)$ and $\tau_{xy}(x, y)$ for the layers have been obtained using the basic equations of Elasticity and the Fourier transform techniques as follows [27]:

For the homogeneous layer;

$$u_1(x, y) = \frac{2}{\pi} \int_0^\infty [(A_1 + A_2 y)^{-\xi y} + (A_3 + A_4 y)^{\xi y} \sin(\xi x)\, d\xi \tag{2a}$$

$$v_1(x, y) = \frac{2}{\pi} \int_0^\infty \left\{ [A_1 + (\frac{\kappa_1}{\xi} + y) A_2] e^{-\xi y} + [-A_3 + (\frac{\kappa_1}{\xi} - y) A_4] e^{\xi y} \right\} \cos(\xi x)\, d\xi \tag{2b}$$

$$\frac{1}{2\mu_1} \sigma_{x_1}(x, y) = \frac{2}{\pi} \int_0^\infty \left\{ \left[\xi (A_1 + A_2 y) - \left(\frac{3-\kappa_1}{2}\right) A_2 \right] e^{-\xi y} + \left[\xi (A_3 + A_4 y) + \left(\frac{3-\kappa_1}{2}\right) A_4 \right] e^{\xi y} \right\} cos(\xi x) d\xi \tag{2c}$$

$$\frac{1}{2\mu_1} \sigma_{y_1}(x, y) = \frac{2}{\pi} \int_0^\infty \left\{ \left[-\xi (A_1 + A_2 y) - \left(\frac{\kappa_1+1}{2}\right) A_2 \right] e^{-\xi y} + \left[-\xi (A_3 + A_4 y) + \left(\frac{\kappa_1+1}{2}\right) A_4 \right] e^{\xi y} \right\} cos(\xi x) d\xi \tag{2d}$$

$$\frac{1}{2\mu_1} \tau_{xy_1}(x, y) = \frac{2}{\pi} \int_0^\infty \left\{ \left[-\xi (A_1 + A_2 y) - \left(\frac{\kappa_1-1}{2}\right) A_2 \right] e^{-\xi y} + \left[\xi (A_3 + A_4 y) - \left(\frac{\kappa_1-1}{2}\right) A_4 \right] e^{\xi y} \right\} sin(\xi x) d\xi \tag{2e}$$

For the FG layer;

$$u_2(x, y) = \frac{2}{\pi} \int_0^\infty \sum_{j=1}^4 B_j e^{s_j y} \sin(\xi x)\, d\xi \tag{3a}$$

$$v_2(x, y) = \frac{2}{\pi} \int_0^\infty \sum_{j=1}^4 B_j m_j e^{s_j y} \cos(\xi x)\, d\xi \tag{3b}$$

$$\sigma_{x_2}(x,y) = \frac{2\mu_0 e^{\beta y}}{\pi(\kappa_2 - 1)} \int_0^\infty \sum_{j=1}^4 B_j\left[(3-\kappa_1)m_j n_j + \xi(\kappa_1+1)\right] e^{s_j y} cos(\xi x) d\xi \tag{3c}$$

$$\sigma_{y_2}(x,y) = \frac{2\mu_0 e^{\beta y}}{\pi(\kappa_2 - 1)} \int_0^\infty \sum_{j=1}^4 B_j C_j e^{s_j y} cos(\xi x) d\xi \tag{3d}$$

$$\tau_{xy_2}(x,y) = \frac{2\mu_0 e^{\beta y}}{\pi} \int_0^\infty \sum_{j=1}^4 B_j D_j e^{s_j y} sin(\xi x) d\xi \tag{3e}$$

where $A_j, B_j (j = 1, \ldots .4)$ are the unknown functions which will be determined from boundary conditions prescribed on $y = h_1, y = 0$ and $y = -h_2$. $\kappa_i = \frac{3-v_i}{1+v_i}$ for plane stress and $\kappa_i = 3 - 4v_i$ for plane strain. C_j, D_j, m_j and s_j $(j = 1, \ldots 4)$ in the equations for the FG layer are as follows:

$$C_j = \left[(\kappa_2 + 1)m_j s_j + \xi(3 - \kappa_2)\right] \tag{4}$$

$$D_j = \left[s_j - \xi m_j\right] \tag{5}$$

$$m_j = \frac{(3\beta + 2s_j - \beta\kappa_2)\left[s_j(\beta + s_j)(\kappa_2 + 1) - \xi^2(\kappa_2 + 3)\right]}{\xi\left[4\xi^2 - \beta^2(\kappa_2 - 3)(\kappa_2 + 1)\right]} \tag{6}$$

$$s_1 = -\frac{1}{2}\left(\beta + \sqrt{4\xi^2 + \beta^2 - 4\xi\beta i\sqrt{\frac{3-\kappa_i}{\kappa_2+1}}}\right), s_2 = -\frac{1}{2}\left(\beta - \sqrt{4\xi^2 + \beta^2 - 4\xi\beta i\sqrt{\frac{3-\kappa_i}{\kappa_2+1}}}\right) \tag{7a}$$

$$s_3 = -\frac{1}{2}\left(\beta + \sqrt{4\xi^2 + \beta^2 + 4\xi\beta i\sqrt{\frac{3-\kappa_i}{\kappa_2+1}}}\right), s_4 = -\frac{1}{2}\left(\beta - \sqrt{4\xi^2 + \beta^2 + 4\xi\beta i\sqrt{\frac{3-\kappa_i}{\kappa_2+1}}}\right) \tag{7b}$$

1.2. Boundary Conditions and Integral Equations

The contact problem must be solved under the following boundary conditions:

$$\sigma_{y_1}(x,h_1) = \begin{cases} -p_1(x) & ; \quad (0 \le x \le a) \\ 0 & ; \quad (a \le x < \infty) \end{cases} \tag{8a}$$

$$\tau_{xy_1}(x,h_1) = 0, \qquad (0 \le x < \infty) \tag{8b}$$

$$\sigma_{y_2}(x,0) = \begin{cases} -p_2(x) & ; \quad (0 \le x \le b) \\ 0 & ; \quad (b \le x < \infty) \end{cases} \tag{8c}$$

$$\sigma_{y_1}(x,0) = \sigma_{y_2}(x,0), \qquad (0 \le x < \infty) \tag{8d}$$

$$\tau_{xy_1}(x,0) = 0, \qquad (0 \le x < \infty) \tag{8e}$$

$$\tau_{xy_2}(x,0) = 0, \qquad (0 \le x < \infty) \tag{8f}$$

$$\sigma_{y_2}(x,-h_2) = k_0 v_2(x,-h_2) - G\frac{\partial^2 v_2(x,-h_2)}{\partial x^2}, \quad (0 \le x \le \infty) \tag{8g}$$

$$\tau_{xy_2}(x,-h_2) = 0, \qquad (0 \le x \le \infty) \tag{8h}$$

$$\frac{\partial}{\partial x}[v_1(x,h_1)] = \frac{\partial F(x)}{\partial x} \qquad (0 \le x \le a) \tag{8i}$$

$$\frac{\partial}{\partial x}[v_1(x,0) - v_2(x,0)] = 0, \qquad (0 \le x \le b) \tag{8j}$$

The equilibrium conditions of the problem are written as follows:

$$\int_{-a}^{+a} p_1(x)dx = P, \int_{-b}^{+b} p_2(x)dx = P \tag{9}$$

Here, a and b are the half contact lengths between the circular rigid block and homogeneous layer, and between the homogeneous layer and FG layer, respectively; $p_1(x)$ and $p_2(x)$ are the primary unknown contact pressures on the contact surfaces; k_0 and G are the Winkler and Pasternak foundation moduli, P represents the singular load applied through the circular rigid block. *F(x)* is a known function denoting the profile of the circular rigid block and is expressed as follows:

$$F(x) = R + h_2 + \delta - (R^2 - x^2)^{0.5} \tag{10}$$

Derivative of this function with respect to x,

$$\frac{\partial F(x)}{\partial x} = \frac{x}{(R^2-x^2)^{0.5}} \cong \frac{x}{R} \qquad R \gg x \tag{11}$$

In these expressions, δ is the maximum vertical displacement of the layer on the symmetry axis and R is the radius of the circular block. Using the boundary conditions in Eqs. (8a-8h), the unknown coefficients $A_j, B_j (j = 1, \ldots .4)$ appearing in equations (2a-3e) are determined in the terms of the primary unknown contact pressures, $p_1(x)$ and $p_2(x)$. Thus, the stresses and displacements can be expressed depending on the unknown contact pressures.

The unknown contact pressures, $p_1(x)$ and $p_2(x)$ are determined from the mixed conditions in Eqs. (8i-8j), which have not yet been satisfied. After some routine manipulations and using the symmetry property, these mixed conditions gave the following system of integral equations depending on $p_1(x)$ and $p_2(x)$.

$$\frac{2}{\pi}\int_{-a}^{a} p_1(t_1)\left\{F_1(x_1,t_1) - \frac{\kappa_1+1}{8}\left[\frac{1}{t_1-x_1}\right]\right\}dt_1 + \frac{2}{\pi}\int_{-b}^{b} p_2(t_2)\,F_2(x_1,t_2)dt_2 = \frac{x}{R} \tag{11a}$$

$$\frac{2}{\pi}\int_{-a}^{a} p_1(t_1)T_1(x_2,t_1)\,dt_1 + \frac{2}{\pi}\int_{-b}^{b} p_2(t_2)\left\{T_2(x_2,t_2) - T_3(x_2,t_2) + \left[\left(\frac{\kappa_1+1}{8\mu_1} + \frac{\kappa_2+1}{8\mu_0}\right)\left(\frac{1}{t_2-x_2}\right)\right]\right\}dt_2 = 0 \tag{11b}$$

Integral kernels as given F_1, F_2, T_1, T_2 and T_3 have not been included in the paper due to the extensive and intricate nature of their expressions.

1.3. Numerical Solution of the System of Integral Equations

Dimensionless quantities can be introduced to simplify the numerical solution to integral equations,

$$s_1 = x_1/a \quad s_2 = b/x_2 \tag{12a}$$

$$r_1 = t_1/a, \quad r_2 = t_2/b \tag{12b}$$

$$G_1(r_1) = \frac{h_1}{P} p_1(t_1), \quad G_2(r_2) = \frac{h_1}{P} p_2(t_2) \tag{12c}$$

$$k_w = \frac{k_0}{\mu_0}, \quad k_p = \frac{G}{\mu_0} \tag{12d}$$

where k_w and k_p non-dimensional foundation moduli.

The normalized form of the integral Equations (11a-11b) and equilibrium conditions (9) may be written as follows:

$$\int_{-1}^{1} G_1(r_1) \left\{ \frac{a}{h_1} \overline{F_1}(s_1, r_1) - \frac{\kappa_1 + 1}{8\mu_1} \left[\frac{1}{r_1 - s_1} \right] \right\} dr_1 \ldots$$

$$\ldots + \frac{b}{h_1} \int_{-1}^{1} G_2(r_2)\, \overline{F_2}(s_1, r_2) dr_2 = \frac{\pi}{2} \mu_1 \frac{a/h_1}{P/h_1} \tag{13a}$$

$$\frac{a}{h_1} \int_{-1}^{1} G_1(r_1) \overline{T_1}(s_2, r_1)\, dr_1 \ldots$$

$$\ldots + \int_{-1}^{1} G_2(r_2) \left\{ \frac{b}{h_1} \left(\overline{T_2}(s_2, r_2) + \overline{T_3}(s_2, r_2) \right) + \left[\left(\frac{\kappa_1 + 1}{8\mu_1} + \frac{\kappa_2 + 1}{8\mu_0} \right) \left(\frac{1}{r_2 - s_2} \right) \right] \right\} dr_2 = 0 \tag{13b}$$

$$\frac{a}{h_1} \int_{-1}^{1} G_1(r_1) dr_1 = 1, \quad \frac{b}{h_1} \int_{-1}^{1} G_2(r_2) dr_2 = 1 \tag{13c}$$

Because of the smooth contact at the end points a and b, the contact pressures $p_1(x)$ and $p_2(x)$ are zero at the edges. Thus, the integral equation (13) has an index of -1, and the following expressions can be obtained [3]:

$$G_1(r_{1i}) = w_1(r_{1i}) g_1(r_{1i}), \quad w_1(r_{1i}) = (1 - r_{1i})^{0.5} (1 + r_{1i})^{0.5}, \quad (\mathrm{I} = 1, \ldots\ldots, \mathrm{N}) \tag{14a}$$

$$G_2(r_{2i}) = w_2(r_{2i}) g_2(r_{2i}), \quad w_2(r_{2i}) = (1 - r_{2i})^{0.5} (1 + r_{2i})^{0.5}, \quad (\mathrm{I} = 1, \ldots\ldots \mathrm{N}) \tag{14b}$$

where $g_1(r_1)$ and $g_2(r_2)$ are continuous and bounded functions in the interval $[-1, 1]$, respectively. Using Gauss-Chebyshev integration formulas [3], equation (13) can be converted to a system of algebraic equations as follows:

$$\sum_{i=1}^{N} w_{1i} g_{1i}(r_{1i}) \left\{ \frac{a}{h_1} \overline{F_1}(n_{1k}, r_{1i}) - \frac{\kappa_1+1}{8} \left[\frac{1}{r_{1i}-n_{1k}} \right] \right\} \dots$$

$$\dots + \frac{b}{h_1} \sum_{i=1}^{N} w_{2i} g_{2i}(r_{2i}) \overline{F_1}(n_{1k}, r_{2i}) = \frac{\pi}{2} \frac{\mu_1}{P/h_1} \frac{a/h_1}{R/h_1} n_{1k} \quad \text{(15a)}$$

$$\frac{a}{h_1} \sum_{i=1}^{N} W_{1i} g_{1i}(r_{1i}) \overline{T_1}(s_{2k}, r_{1i}) \dots$$

$$+ \sum_{i=1}^{N} W_{2i} g_{2i}(r_{2i}) \left[\frac{b}{h_1} \left(\overline{T_2}(s_{2k}, r_{2i}) - \overline{T_3}(s_{2k}, r_{2i}) \right) + \left(\left(\frac{\kappa_1+1}{8\mu_1} + \frac{\kappa_2+1}{8\mu_0} \right) \left(\frac{1}{r_{2i}-s_{2k}} \right) \right) \right] = 0$$

$$(k = 1, \dots\dots, N+1) \quad \text{(15b)}$$

$$\frac{a}{h_1} \sum_{i=1}^{N} W_{1i} g_1(r_{1i}) = 1 \; (i = 1, \dots., N) \quad \text{(15c)}$$

$$\frac{b}{h_1} \sum_{i=1}^{N} W_{2i} g_2(r_{2i}) = 1 \; (i = 1, \dots., N) \quad \text{(15d)}$$

where r_i and s_k are the zeros of the related Chebyshev polynomials and W_i^N is the weighting constant, expressed as follows:

$$W_i^N = \pi \left(\frac{1-r_i^2}{N+1} \right) (i = 1, \dots\dots, N) \quad \text{(16a)}$$

$$r_i = cos \left(\frac{i\pi}{N+1} \right) (i = 1, \dots\dots, N) \quad \text{(16b)}$$

$$s_k = cos \left(\frac{\pi}{2} \frac{2k-1}{N+1} \right) (k = 1, \dots\dots, N+1) \quad \text{(16c)}$$

It can be demonstrated that the $\left(\frac{N}{2} + 1 \right)$-th equations in (15a) and (15b) are automatically satisfied. Thus equation (15) *2N+2* algebraic equations to determine the *2N+2* unknowns, namely $g_1(r_{1i})$, $g_2(r_{2i})$ $(i = 1, ., N)$, *a* and *b*. The system of equations is linear in $g_1(r_{1i})$ and $g_2(r_{2i})$, but nonlinear in *a* and *b*. Therefore, an iteration scheme should be used to determine these two unknowns. This iteration scheme for the problem was created with a code by using MATLAB program. Twenty Gauss points and Chebyshev function *(N=20)* are used for the solution in the MATLAB code. Until the equilibrium conditions are satisfied, iteration continue to verify the obtained contact stresses. Once the equilibrium conditions are met with the desired precision, a solution is considered achieved. Figure 2 shows the flowchart of the iterative MATLAB code.

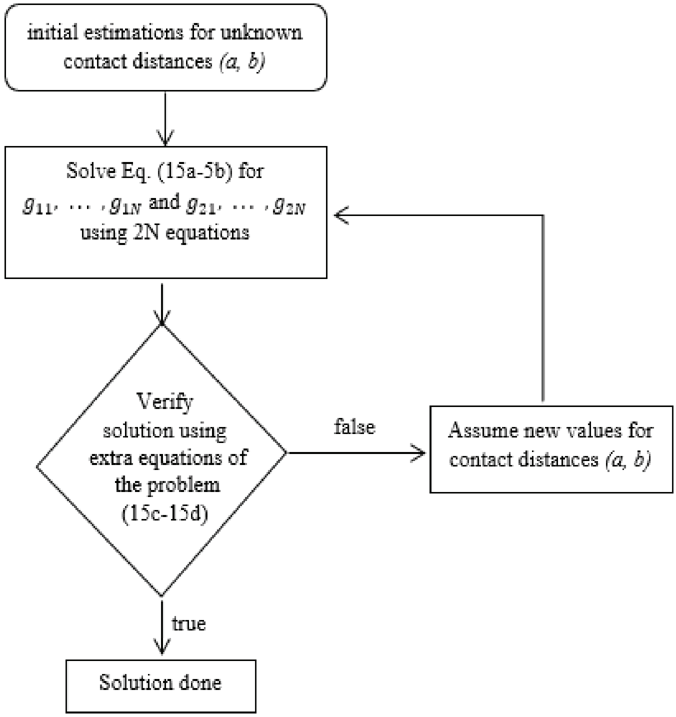


Fig. 2 - Flowchart of the iterative algorithm

1.4. Numerical Results

This section presents numerical results and discussion for the contact pressures and contact lengths at both interfaces of contact. Therefore, the effects of the different shear modulus ratios of the layers (μ_0/μ_1) different radius of circular rigid block (R/h_1), non-dimensional Pasternak foundation parameter (k_p) and different material properties (κ_i) the data obtained from numerical applications are presented with the help of graphs.

Table 1 presents the number of iterations involving the equilibrium condition conducted to achieve a solution for the receding contact problem. Iteration resultant the equilibrium conditions should be equal to 1. Columns (1)–(4) of Table 1 correspond, respectively, to the iteration number, value of the equilibrium condition $(\frac{a}{h_1}\int_{-1}^{1} G_1(r_1)dr_1, \frac{b}{h_1}\int_{-1}^{1} G_2(r_2)dr_2)$, the normalized receding contact length $(a/h_1,\ b/h_1)$ and the relative error of the normalized receding contact length between two successive iterations $(E(\%))$.

Table 2-3 illustrates that for significantly high values of the spring constant in our study, the foundation was assumed to be rigid and compared with a prior study conducted by Comez [9]. As shown in the table, the results are consistent.

Table 1 - Solution iteration of the receding contact problem including the equilibrium condition for initial value $a/h_1 = b/h_1 = 1$.

Iteration no.	$\boldsymbol{\frac{a}{h_1}\int_{-1}^{1} G_1(r_1)dr_1}$	$\boldsymbol{a/h_1}$	$\boldsymbol{E}$**(%)**	$\boldsymbol{\frac{b}{h_1}\int_{-1}^{1} G_2(r_2)dr_2}$	$\boldsymbol{b/h_1}$	$\boldsymbol{E}$**(%)**
(1)	**(2)**	**(3)**	**(4)**	**(2)**	**(3)**	**(4)**
	1.0	**1.0**		**1.0**	**1.0**	
1	-0.4556759	1.0000	…	-0.4583063	1.0000	…
2	-0.3024829	0.8922	-12.08	-0.3025341	1.5572	35.78
3	0.0331262	1.0986	18.78	0.0331247	1.7018	8.49
4	0.0004239	1.0802	-1.70	0.0004238	1.6904	-0.67
5	0.0000011	1.0799	-0.03	0.0000011	1.6903	0
6	**0.9999999**	**1.0799**	**0**	**0.9999999**	**1.6903**	**0**

Table 2 - Comparison of the half contact length (a/h_1) between the values reported in the literature and obtained in this study for various.
$(h_1/h_2 = 2,\ \mu_1/\mu_2 = 1,\ \kappa_1 = \kappa_2 = 2,\ k_w = \infty)$

$\frac{\mu_1}{P/h_1}$	**100**	**250**	**500**
	$R/h_1 = 10$	$R/h_1 = 10$	$R/h_1 = 10$
Comez (2003)	0.2202	0.1386	0.0978
This study	0.2204	0.1386	0.0978

Table 3 - Comparison of the half contact length (b/h_1) between the values reported in the literature and obtained in this study for various
. $(h_1/h_2 = 2,\ \mu_1/\mu_2 = 1,\ \kappa_1 = \kappa_2 = 2, k_w = \infty)$

$\frac{\mu_1}{P/h_1}$	**100**	**250**	**500**
	$R/h_1 = 10$	$R/h_1 = 10$	$R/h_1 = 10$
Comez (2003)	1.2534	1.2446	1.2421
This study	1.2691	1.2599	1.2568

The reason for the 0.01 error rate in Table 3 is attributed to the stiffness parameter controlling the shear modulus variations in the graded medium (βh_1) in the FG layer.

Table 4-5 is compared with a previous study by Eyüpoğlu [28], where the stiffness parameter of the FG layer is assumed to be homogeneous for a value of βh_1=0.01 in our study. As shown in the tables, the results agree.

Table 4 - Comparison of the half contact length (a/h_1) *between the values reported in the literature and obtained in this study for various.*
$(h_1/h_2 = 1,\ \mu_0/\mu_1 = 1,\ \kappa_1 = \kappa_2 = 2,\ k_w = 1)$

$\beta h_1 = 0.01$	$R/h_1 = 10$	$R/h_1 = 50$	$R/h_1 = 100$
Eyüpoğlu [28]	0.0975	0.2210	0.3173
This study	0.0980	0.2219	0.3177

Table 5 - Comparison of the half contact length (b/h_1) *between the values reported in the literature and obtained in this study for various.* $(h_1/h_2 = 1,\ \mu_0/\mu_1 = 1,\ \kappa_1 = \kappa_2 = 2,\ k_w = 1)$

$\beta h_1 = 0.01$	$R/h_1 = 10$	$R/h_1 = 50$	$R/h_1 = 100$
Eyüpoğlu [28]	1.4454	1.4553	1.4684
This study	1.4450	1.4550	1.4680

Table 6, values of the half contact lengths are provided for various dimensionless values of β and the ratio of the circular rigid block radius to (R/h_1). In Figure 3, the values are presented graphically.

Table 6 and figure 3 illustrates that as the radius of the circular rigid block increases, the contact distances between the two contact surfaces also increase. Due to the FGM layer being located at the lower part of the x-axis, an increase in the stiffness of the FGM layer leads to an increase in the contact distance between the rigid block and the homogeneous layer, as well as between the homogeneous layer and the FGM layer.

Table 6 - The variation of half-contact lengths with respect to values of circular rigid block and βh_1. $(h_1/h_2 = 1,\ \mu_1/\mu_0 = 1,\ \kappa_1 = \kappa_2 = 2,\ k_w = 1,\ k_p = 1)$

βh_1	$R/h_1 = 10$		$R/h_1 = 50$		$R/h_1 = 100$	
	a/h_1	b/h_1	a/h_1	b/h_1	a/h_1	b/h_1
-1	0.219854	1.251549	0.500749	1.323122	0.717755	1.413195
0.01	0.220851	1.332804	0.510805	1.397999	0.741781	1.485439
1	0.222252	1.437649	0.525736	1.496970	0.778251	1.584113

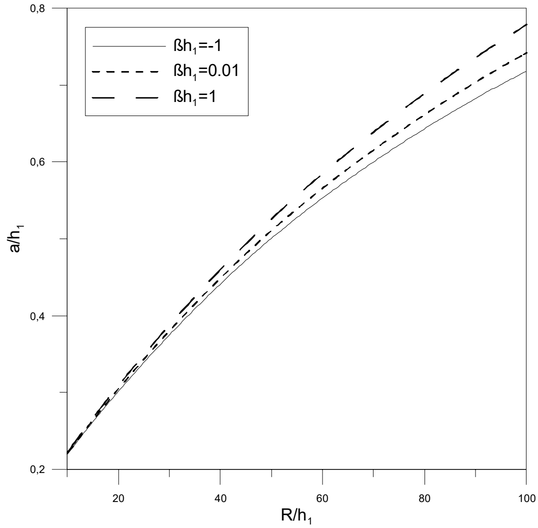


Fig. 3a - The variation of half-contact lengths (a/h_1) *with respect to values of circular rigid block and* βh_1. $(h_1/h_2 = 1,\ \mu_1/\mu_0 = 1,\ \kappa_1 = \kappa_2 = 2,\ k_w = 1,\ k_p = 1)$

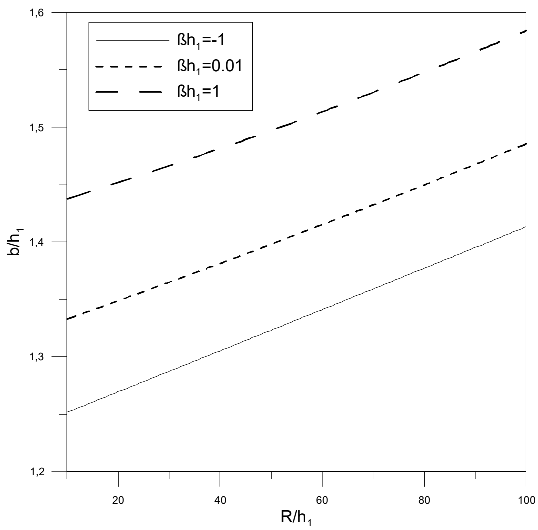


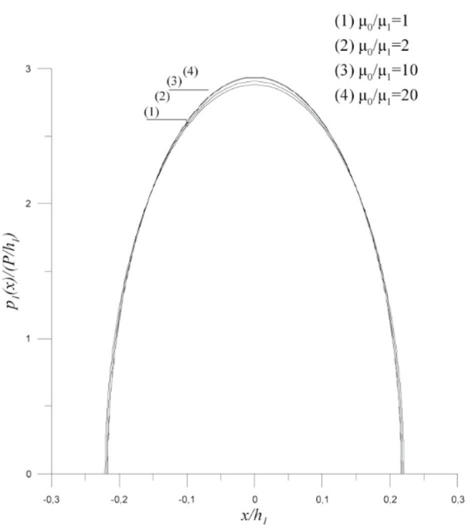
Fig. 3b - The variation of half-contact lengths (b/h_1) *with respect to values of circular rigid block and* βh_1. $(h_1/h_2 = 1,\ \mu_1/\mu_0 = 1,\ \kappa_1 = \kappa_2 = 2,\ k_w = 1,\ k_p = 1)$

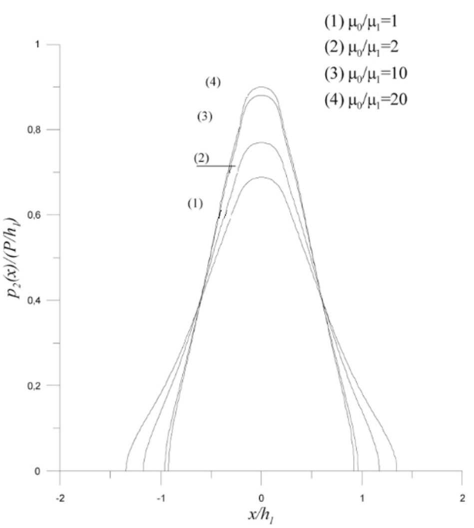
Table 7 show the variation of half contact lengths for various non-dimensional Pasternak foundation spring constant k_w and shear layer k_p. For very large values of the Pasternak basic modulus (G), k_p is taken to infinity and the contact lengths are shown in Table 7 for both contact surfaces.

Table 7 - The variation of half-contact lengths with respect to dimensionless Pasternak foundation spring constant kw and shear layer k_p values.
$(h_1/h_2 = 1,\ \mu_1/\mu_0 = 1, \quad \kappa_1 = \kappa_2 = 2, \beta h_1 = 0.01)$

k_p	$k_w = 1$		$k_w = 5$		$k_w = 10$	
	a/h_1	b/h_1	a/h_1	b/h_1	a/h_1	b/h_1
10	0.219914	1.209245	0.219833	1.196974	0.219794	1.191304
100	0.219693	1.179251	0.219689	1.178665	0.219687	1.178266
∞	0.219661	1.174866	0.219661	1.174865	0.219661	1.174868

Figure 4a illustrates the distribution of contact stress along the contact distance between the circular rigid block and the homogeneous layer for different dimensionless μ_0/μ_1 values. In Figure 4b, similar stress distributions are presented along the contact distance between the homogeneous layer and the FG layer. Figure 5 illustrates the distribution of contact stresses along the contact distance between the circular rigid block and the homogeneous layer, as well as between the homogeneous layer and the FG layer. These distributions are observed with respect to the increasing radius of various circular blocks (R/h_1).

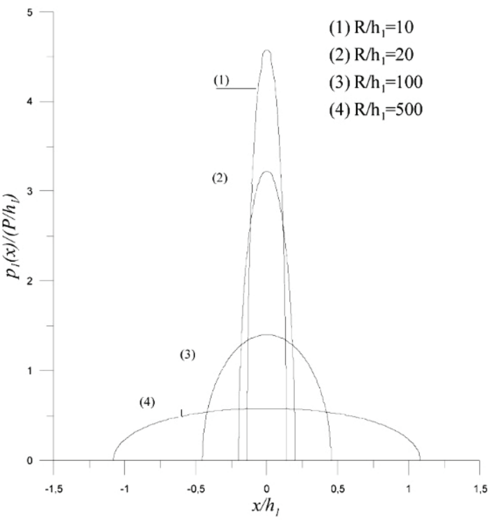


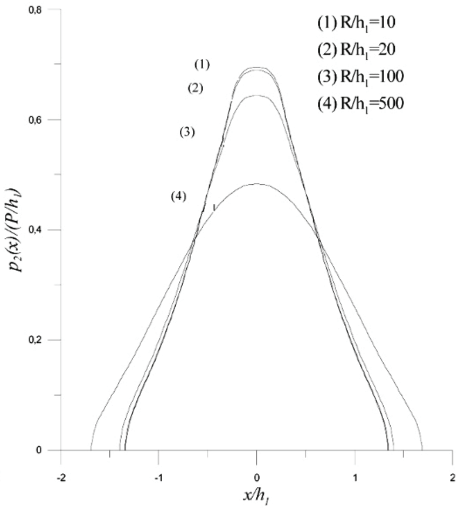


$(h_1/h_2 = 2, \beta h_1 = 0.01, \kappa_1 = \kappa_2 = 2, k_w = 1, k_p = 1)$

Fig. 4a - Contact stress distribution between circular rigid block and homogeneous layer for various values of μ_1/μ_0

Fig. 4b - Contact stress distribution between homogeneous layer and FGM layer for various values of μ_1/μ_0

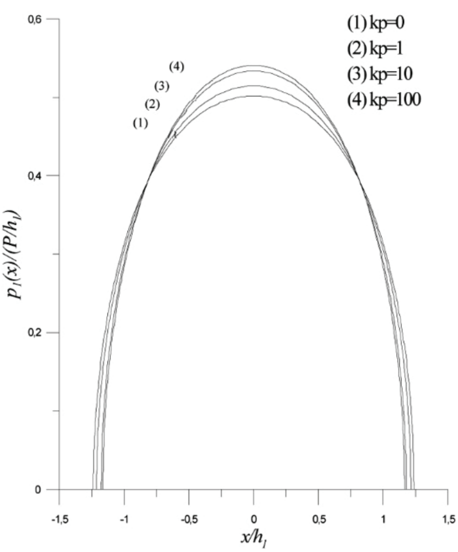


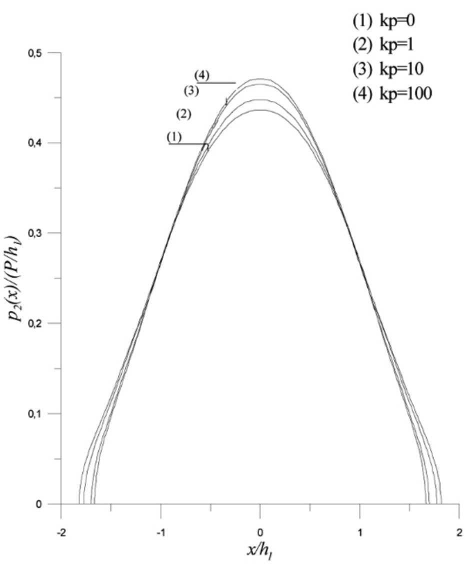


$(h_1/h_2 = 2\ , \beta h_1 = 0.01, \kappa_1 = \kappa_2 = 2, k_w = 1, k_p = 1)$

Fig. 5a - Contact stress distribution between circular rigid block and homogeneous layer for various values of R/h_1

Fig. 5b - Contact stress distribution between homogeneous layer and FGM layer for various values of R/h_1



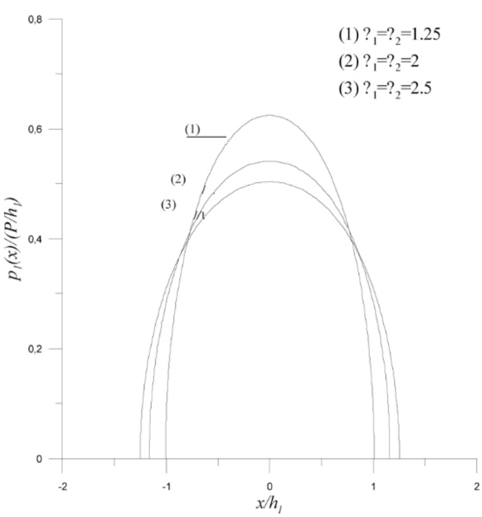


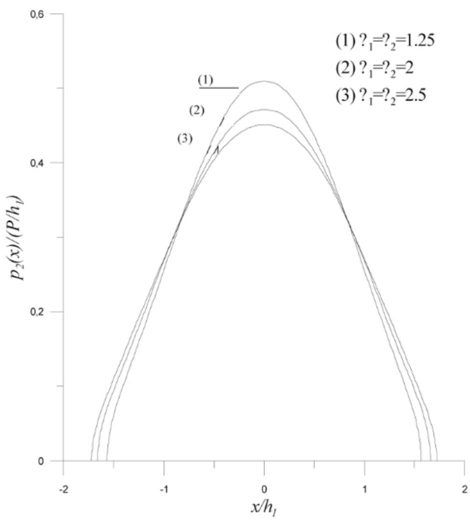
$(h_1/h_2 = 2, \mu_1/\mu_0 = 1, \kappa_1 = \kappa_2 = 2, \beta h_1 = 0.01, k_w = 1)$

Fig. 6a - Contact stress distribution between circular rigid block and homogeneous layer for various values of k_p

Fig.6b - Contact stress distribution between homogeneous layer and FGM layer for various values of k_p

Figure 6 illustrates the distribution of contact stresses along the contact distance between the rigid block and the homogeneous layer, as well as between the homogeneous layer and the functionally graded layer. These distributions are presented for varying values of the Pasternak foundation parameter k_p within a specified range from zero to infinity.





$(h_1/h_2 = 2, \mu_1/\mu_0 = 1, \kappa_1 = \kappa_2 = 2, \beta h_1 = 0.01, k_w = 1)$

Fig. 7a - Contact stress distribution between circular rigid block and homogeneous layer for various values of κ_1 *and* κ_2

Fig.7b - Contact stress distribution between homogeneous layer and FGM layer for various values of κ_1 *and* κ_2

Figure 7 presents the distribution of contact stresses along the contact distance between the rigid block and the homogeneous layer as well as the homogeneous layer and the functionally graded layer. The variations in these distributions are observed for different values of material properties.

The results of the study can be summarized as follows:

- As the ratio μ_0/μ_1 increases, minimal variations are observed in contact stresses along the contact distance between the circular rigid block and the homogeneous layer, consistent with expectations. However, between the homogeneous layer and the FGM, there is a reduction in the contact distance and a corresponding increase in contact stresses.
- With an increase in the radius of the circular rigid block, the contact distance between the block and the homogeneous layer, as well as between the homogeneous layer and the FGM, grows, leading to a decrease in contact stresses.
- As the value of Pasternak foundation moduli (k_p) increases, contact stresses increase and contact distances decrease. When compared with the study demonstrated in [23],

the results indicate that the Pasternak foundation behaves more rigidly. Consequently, contact distances decrease, leading to an increase in contact stresses.

- As the material properties (κ_1,κ_2) of the homogeneous and FD layer increase, the contact distance increase and the contact stresses decrease.
- In this study, unlike previous studies in the literature, the FG layer was brought into contact with the foundation and the results are presented in detail.
- The influence of the Pasternak foundation on contact problems has been demonstrated.

Symbols

a	: Circular rigid block half contact length
b	: Half contact length between layers.
P	: Singular load
$F(x)$	: A function defining the profile of a circular rigid block.
$p_1(x)$	: Contact stress beneath the circular rigid block.
$p_2(x)$	: Interlayer contact stress.
β	: Rigidity parameter.
μ_0	: Shear modulus on the bottom surface of the FG layer
μ_1	: Shear modulus of homogeneous layer
$\mu_2(y)$	: Shear modulus of FG layer
κ_1 , κ_2	: Material constant of homogeneous and FG layer
u, v, w	: Displacement components in x, y and z directions
$\sigma_x, \sigma_y, \sigma_z$	: Normal stress components in x, y and z directions
$\tau_{xy}, \tau_{xz}, \tau_{yz}$	: Shear stress components
h_1	: FD layer height
h_2	: Homogeneous layer height
h	: Total height of the layers
R	: Radius of the circular rigid block
k_w	: Non-dimensionless Winkler foundation moduli
k_p	: Non-dimensionless Pasternak foundation moduli

Note: Some symbols not given in this list are defined in the text where they are used.

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