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GENERALIZED TOPOLOGICAL OPERATOR ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-OPERATOR) THEORY IN GENERALIZED TOPOLOGICAL SPACES ($\mathscr{T}_{\mathfrak{g}}$-SPACES) PART IV. GENERALIZED DERIVED ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-DERIVED) AND GENERALIZED CODERIVED ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-CODERIVED) OPERATORS

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Abstract. In a recent paper (Cf. [1]), we have introduced the definitions and studied the essential properties of the generalized topological operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*derived* and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*coderived operators*) in a generalized topological space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ ($\mathscr{T}_{\mathfrak{g}}$-*space*). Mainly, we have shown that $(\mathfrak{g}\text{-Der}_{\mathfrak{g}}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}) : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ is a pair of both *dual and monotone* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*operators* that is $(\varnothing, \Omega)$, $(\cup, \cap)$-*preserving*, and $(\subseteq, \supseteq)$-*preserving* relative to $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-(open, closed) sets. We have also shown that $\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\right) : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ is a pair of *weaker* and *stronger* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*operators*. In this paper, we define by transfinite recursion on the class of successor ordinals the δ^{th}-iterates $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-*derived* and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-*coderived operators*) of $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, and study their basic properties in a $\mathscr{T}_{\mathfrak{g}}$-space. Moreover, we establish the necessary and sufficient conditions for $\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\right) : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ to be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators in $\mathfrak{T}_{\mathfrak{g}}$. Finally, we diagram various relationships amongst $\text{der}_{\mathfrak{g}}^{(\delta)}$, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}$, $\text{cod}_{\mathfrak{g}}^{(\delta)}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and present a nice application to support the overall study.

1. Introduction

Axiomatically, a generalized derived operator ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-*derived operator*) in an ordinary ($\mathfrak{a} = \mathfrak{o}$) or generalized ($\mathfrak{a} = \mathfrak{g}$) topological space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$ ($\mathscr{T}_{\mathfrak{a}}$-*space*) is a set-valued map $\begin{array}{rccl}\mathfrak{g}\text{-Der}_{\mathfrak{a}}: & \mathscr{P}(\Omega) & \longrightarrow & \mathscr{P}(\Omega) \\ & \mathscr{S}_{\mathfrak{a}} & \longmapsto & \mathfrak{g}\text{-Der}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}})\end{array}$ satisfying the following $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-*derived operator axioms*:

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– $\mathrm{Ax}_{\mathrm{DE},1}\left(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a}}\right) \stackrel{\mathrm{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a}}\left(\varnothing\right) = \varnothing$

– $\mathrm{Ax}_{\mathrm{DE},2}\left(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a}}\right) \stackrel{\mathrm{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a}}\left(\mathscr{R}_{\mathfrak{g}}\right) = \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a}}\big(\mathscr{R}_{\mathfrak{a}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{a}}\left(\{\xi\}\right)\big)$

– $\mathrm{Ax}_{\mathrm{DE},3}\left(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a}}\right) \stackrel{\mathrm{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a}} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a}}\left(\mathscr{R}_{\mathfrak{a}}\right) \subseteq \mathscr{R}_{\mathfrak{a}} \cup \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a}}\left(\mathscr{R}_{\mathfrak{a}}\right)$

– $\mathrm{Ax}_{\mathrm{DE},4}\left(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a}}\right) \stackrel{\mathrm{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a}}\left(\mathscr{R}_{\mathfrak{a}} \cup \mathscr{S}_{\mathfrak{a}}\right) = \bigcup_{\mathscr{U}_{\mathfrak{a}}=\mathscr{R}_{\mathfrak{a}},\mathscr{S}_{\mathfrak{a}}} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a}}\left(\mathscr{U}_{\mathfrak{a}}\right)$

for any $\left(\{\xi\}, \mathscr{R}_{\mathfrak{a}}, \mathscr{S}_{\mathfrak{a}}\right) \in \times_{\alpha\in I_3^*} \mathscr{P}\left(\Omega\right)$ such that $\{\xi\} \subset \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a}}\left(\mathscr{R}_{\mathfrak{a}}\right)$ [1, 2, 3, 4, 5, 6, 7, 8, 9]. A generalized coderived operator ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-*coderived operator*) in the $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}}$ is a set-valued map $\begin{array}{rccl}\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a}}: & \mathscr{P}\left(\Omega\right) & \longrightarrow & \mathscr{P}\left(\Omega\right) \\ & \mathscr{S}_{\mathfrak{a}} & \longmapsto & \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a}}\left(\mathscr{S}_{\mathfrak{a}}\right)\end{array}$ satisfying the following $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-*coderived operator axioms*:

– $\mathrm{Ax}_{\mathrm{CD},1}\left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a}}\right) \stackrel{\mathrm{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a}}\left(\Omega\right) = \Omega$

– $\mathrm{Ax}_{\mathrm{CD},2}\left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a}}\right) \stackrel{\mathrm{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a}}\big(\mathscr{U}_{\mathfrak{g}}\big) = \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a}}\big(\mathscr{U}_{\mathfrak{a}} \cup \{\zeta\}\big)$

– $\mathrm{Ax}_{\mathrm{CD},3}\left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a}}\right) \stackrel{\mathrm{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a}}\left(\mathscr{U}_{\mathfrak{a}}\right) \supseteq \mathscr{U}_{\mathfrak{a}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a}}\left(\mathscr{U}_{\mathfrak{a}}\right)$

– $\mathrm{Ax}_{\mathrm{CD},4}\left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a}}\right) \stackrel{\mathrm{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a}}\left(\mathscr{U}_{\mathfrak{a}} \cap \mathscr{V}_{\mathfrak{a}}\right) = \bigcap_{\mathscr{W}_{\mathfrak{a}}=\mathscr{U}_{\mathfrak{a}},\mathscr{V}_{\mathfrak{a}}} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a}}\left(\mathscr{W}_{\mathfrak{a}}\right)$

for any $\left(\{\zeta\}, \mathscr{U}_{\mathfrak{a}}, \mathscr{V}_{\mathfrak{a}}\right) \in \times_{\alpha\in I_3^*} \mathscr{P}\left(\Omega\right)$ [1, 2, 3, 4, 5, 6, 7, 8, 9]. Alternative axiomatic descriptions for $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{o}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{o}}$-coderived operators in $\mathscr{T}_{\mathfrak{o}}$-spaces can be found in the paper of Lei and Zhang [10].

If $\left(\mathscr{S}_{\mathfrak{a}}, \mathfrak{g}\text{-}\mathrm{Ope}_{\mathfrak{a}}\right) \in \mathscr{P}\left(\Omega\right) \times \{\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a}}, \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a}}\}$ be arbitrarily given, then β factors $\begin{array}{rccl}\mathfrak{g}\text{-}\mathrm{Ope}_{\mathfrak{a}}: & \mathscr{P}\left(\Omega\right) & \longrightarrow & \mathscr{P}\left(\Omega\right) \\ & \mathscr{S}_{\mathfrak{a}} & \longmapsto & \mathfrak{g}\text{-}\mathrm{Ope}_{\mathfrak{a}}\left(\mathscr{S}_{\mathfrak{a}}\right)\end{array}$ yields:

$$\mathbb{Z}_+^0 \ni \beta \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Ope}_{\mathfrak{a}}^{(\beta)}\left(\mathscr{S}_{\mathfrak{a}}\right) = \mathfrak{g}\text{-}\mathrm{Ope}_{\mathfrak{a}} \circ \cdots \circ \mathfrak{g}\text{-}\mathrm{Ope}_{\mathfrak{a}}\left(\mathscr{S}_{\mathfrak{a}}\right) \stackrel{\mathrm{def}}{=} \bigcirc_{\alpha\in I_\beta^0} \mathfrak{g}\text{-}\mathrm{Ope}_{\mathfrak{a}}\left(\mathscr{S}_{\mathfrak{a}}\right)$$

Thus, $\begin{array}{rccl}\left(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a}}^{(\beta)}, \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a}}^{(\beta)}\right): & \mathscr{P}\left(\Omega\right) & \longrightarrow & \mathscr{P}\left(\Omega\right) \\ & \mathscr{S}_{\mathfrak{a}} & \longmapsto & \left(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a}}^{(\beta)}, \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a}}^{(\beta)}\right)\left(\mathscr{S}_{\mathfrak{a}}\right)\end{array}$ is the β^{th} *order* of $\begin{array}{rccl}\left(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a}}, \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a}}\right): & \mathscr{P}\left(\Omega\right) & \longrightarrow & \mathscr{P}\left(\Omega\right) \\ & \mathscr{S}_{\mathfrak{a}} & \longmapsto & \left(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a}}, \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a}}\right)\left(\mathscr{S}_{\mathfrak{a}}\right)\end{array}$ and, for any pair $\left(\mathscr{S}_{\mathfrak{a}}, \mathfrak{g}\text{-}\mathrm{Ope}_{\mathfrak{a}}\right) \in \mathscr{P}\left(\Omega\right) \times \{\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a}}, \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a}}\}$, it holds that:

$$\Big[\big(\exists\beta \in \mathbb{Z}_+^0\big)\big(\mathfrak{g}\text{-}\mathrm{Ope}_{\mathfrak{a}}^{(\beta)}\left(\mathscr{S}_{\mathfrak{a}}\right) = \varnothing\big)\Big] \veebar \Big[\big(\forall\beta \in \mathbb{Z}_+^0\big)\big(\mathfrak{g}\text{-}\mathrm{Ope}_{\mathfrak{a}}^{(\beta)}\left(\mathscr{S}_{\mathfrak{a}}\right) \neq \varnothing\big)\Big]$$

If $\mathfrak{g}\text{-}\mathrm{Ope}_{\mathfrak{a}}^{(\beta)}\left(\mathscr{S}_{\mathfrak{a}}\right) = \varnothing$ for some $\beta \in \mathbb{Z}_+^0$, then β is a type of *density measure* of $\mathscr{S}_{\mathfrak{a}}$ to achieve *emptiness* (if this is ever achieved). But if $\mathscr{S}_{\mathfrak{a}}^{(\lambda)} \stackrel{\mathrm{def}}{=} \bigcap_{\beta\in\mathbb{Z}_+^*} \mathfrak{g}\text{-}\mathrm{Ope}_{\mathfrak{a}}^{(\beta)}\left(\mathscr{S}_{\mathfrak{a}}\right) \neq \varnothing$, then λ is a type of *limit order* of $\mathscr{S}_{\mathfrak{a}}$, in which case the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-operators $\mathfrak{g}\text{-}\mathrm{Ope}_{\mathfrak{a}}^{(1)}$, $\mathfrak{g}\text{-}\mathrm{Ope}_{\mathfrak{a}}^{(2)}, \ldots : \mathscr{P}\left(\Omega\right) \longrightarrow \mathscr{P}\left(\Omega\right)$ can again be applied on $\mathscr{S}_{\mathfrak{a}}^{(\omega)} \in \mathscr{P}\left(\Omega\right)$, yielding $\mathfrak{g}\text{-}\mathrm{Ope}_{\mathfrak{a}}^{(\lambda+1)}\left(\mathscr{S}_{\mathfrak{a}}\right)$, $\mathfrak{g}\text{-}\mathrm{Ope}_{\mathfrak{a}}^{(\lambda+2)}\left(\mathscr{S}_{\mathfrak{a}}\right)$, …. Viewing $\delta = 0, 1, 2, \ldots$ as *successor ordinals* while $\delta = \lambda$ as *limit ordinal*, the foregoing descriptions surprisingly introduce by transfinite recursion on the class of successor ordinals the definitions of the δ^{th}-iterates $\begin{array}{rccl}\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a}}^{(\delta)},\ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a}}^{(\delta)}: & \mathscr{P}\left(\Omega\right) & \longrightarrow & \mathscr{P}\left(\Omega\right) \\ & \mathscr{S}_{\mathfrak{a}} & \longmapsto & \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a}}^{(\delta)}\left(\mathscr{S}_{\mathfrak{a}}\right),\ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a}}^{(\delta)}\left(\mathscr{S}_{\mathfrak{a}}\right)\end{array}$

($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}^{(\delta)}$*-derived* and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}^{(\delta)}$*-coderived operators*) of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-coderived operators $\begin{array}{rll}\mathfrak{g}\text{-Der}_{\mathfrak{a}},\ \mathfrak{g}\text{-Cod}_{\mathfrak{a}}: & \mathscr{P}(\Omega) & \longrightarrow \mathscr{P}(\Omega)\\ & \mathscr{S}_{\mathfrak{a}} & \longmapsto \mathfrak{g}\text{-Der}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}),\ \mathfrak{g}\text{-Cod}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}),\end{array}$ respectively, in a $\mathscr{T}_{\mathfrak{a}}$-space.

In a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, by virtue of $\mathrm{Ax}_{\mathrm{DE},1}(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}), \ldots, \mathrm{Ax}_{\mathrm{DE},4}(\mathfrak{g}\text{-Cod}_{\mathfrak{g}})$ and $\mathrm{Ax}_{\mathrm{CD},1}(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}), \ldots, \mathrm{Ax}_{\mathrm{CD},4}(\mathfrak{g}\text{-Cod}_{\mathfrak{g}})$, generalized characterizations of $\mathscr{T}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ can be realized by specifying either the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator $\begin{array}{rll}\mathfrak{g}\text{-Der}_{\mathfrak{g}}: & \mathscr{P}(\Omega) & \longrightarrow \mathscr{P}(\Omega)\\ & \mathscr{S}_{\mathfrak{g}} & \longmapsto \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\end{array}$ or the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator $\begin{array}{rll}\mathfrak{g}\text{-Cod}_{\mathfrak{g}}: & \mathscr{P}(\Omega) & \longrightarrow \mathscr{P}(\Omega)\\ & \mathscr{S}_{\mathfrak{g}} & \longmapsto \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}),\end{array}$ respectively [1]. Moreover, if the δ^{th}-iterates $\begin{array}{rll}\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)},\ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}: & \mathscr{P}(\Omega) & \longrightarrow \mathscr{P}(\Omega)\\ & \mathscr{S}_{\mathfrak{g}} & \longmapsto \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}),\ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\end{array}$ are also themselves $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, then similar roles can be played, thereby realizing other generalized characterizations of $\mathscr{T}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in $\mathfrak{T}_{\mathfrak{g}}$.

Although the literature of $\mathscr{T}_{\mathfrak{a}}$-spaces contains a wealth of information on the study of different types of $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators in $\mathscr{T}_{\mathfrak{g}}$-spaces [2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22], including the study of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{o}}^{(\delta)}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{o}}^{(\delta)}$-coderived operators in $\mathscr{T}_{\mathfrak{o}}$-spaces [23, 24, 25, 26, 27], it does, unfortunately, not contain a study of any $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-coderived operators in $\mathscr{T}_{\mathfrak{g}}$-spaces.

In investigating the convergence of Fourier series, Cantor [23, 24] has introduced and considered $\begin{array}{rll}\mathrm{der}_{\mathfrak{o}|\mathbb{R}}: & \mathscr{P}(\mathbb{R}) & \longrightarrow \mathscr{P}(\mathbb{R})\\ & \mathscr{S}_{\mathfrak{o}} & \longmapsto \mathrm{der}_{\mathfrak{o}|\mathbb{R}}(\mathscr{S}_{\mathfrak{o}})\end{array}$ in $\mathbb{R}$. He has also considered its iteration, thereby introducing the notion of ordinal and then the definition of $\begin{array}{rll}\mathrm{der}_{\mathfrak{o}|\mathbb{R}}^{(\delta)}: & \mathscr{P}(\mathbb{R}) & \longrightarrow \mathscr{P}(\mathbb{R})\\ & \mathscr{S}_{\mathfrak{o}} & \longmapsto \mathrm{der}_{\mathfrak{o}|\mathbb{R}}^{(\delta)}(\mathscr{S}_{\mathfrak{o}})\end{array}$ in $\mathbb{R}$ for some ordinal δ. Later on, Rutt [25] has introduced a weaker form of $\begin{array}{rll}\mathrm{der}_{\mathfrak{o}}: & \mathscr{P}(\Omega) & \longrightarrow \mathscr{P}(\Omega)\\ & \mathscr{S}_{\mathfrak{o}} & \longmapsto \mathrm{der}_{\mathfrak{o}}(\mathscr{S}_{\mathfrak{o}})\end{array}$ and investigated some of its properties as well as the properties of its δ^{th}-order iterate $\begin{array}{rll}\mathrm{der}_{\mathfrak{o}}^{(\delta)}: & \mathscr{P}(\Omega) & \longrightarrow \mathscr{P}(\Omega)\\ & \mathscr{S}_{\mathfrak{o}} & \longmapsto \mathrm{der}_{\mathfrak{o}}^{(\delta)}(\mathscr{S}_{\mathfrak{o}})\end{array}$ from a sequential point of view. Adopting a point of view similar to Rutt [25], Tucker [26] has presented a theorem concerning the period of periodic sequences of $\mathfrak{T}_{\mathfrak{o}}$-derived sets with respect to the $\mathfrak{T}_{\mathfrak{o}}^{(\delta)}$-derived operator $\begin{array}{rll}\mathrm{der}_{\mathfrak{o}}^{(\delta)}: & \mathscr{P}(\Omega) & \longrightarrow \mathscr{P}(\Omega)\\ & \mathscr{S}_{\mathfrak{o}} & \longmapsto \mathrm{der}_{\mathfrak{o}}^{(\delta)}(\mathscr{S}_{\mathfrak{o}})\end{array}$ and has studied other properties in a $\mathscr{T}_{\mathfrak{o}}$-space. Noticing that, for a large class of real $\mathscr{T}_{\mathfrak{o}|\mathbb{R}}$-spaces of the type $\mathfrak{T}_{\mathfrak{o}|\mathbb{R}} = (\mathbb{R}, \mathscr{T}_{\mathfrak{o}|\mathbb{R}})$, the $\mathfrak{T}_{\mathfrak{o}|\mathbb{R}}$-derived operator $\begin{array}{rll}\mathrm{der}_{\mathfrak{o}|\mathbb{R}}^{(\delta)}: & \mathscr{P}(\mathbb{R}) & \longrightarrow \mathscr{P}(\mathbb{R})\\ & \mathscr{S}_{\mathfrak{o}} & \longmapsto \mathrm{der}_{\mathfrak{o}|\mathbb{R}}^{(\delta)}(\mathscr{S}_{\mathfrak{o}})\end{array}$ itself realizes an ordinary characterization of $\mathscr{T}_{\mathfrak{o}|\mathbb{R}} : \mathscr{P}(\mathbb{R}) \longrightarrow \mathscr{P}(\mathbb{R})$ in the $\mathscr{T}_{\mathfrak{o}|\mathbb{R}}$-space $\mathfrak{T}_{\mathfrak{o}|\mathbb{R}}$, Higgs [27] has given characterizations of $\mathscr{T}_{\mathfrak{o}|\mathbb{R}}$-spaces for which the δ^{th}-iterate $\begin{array}{rll}\mathrm{der}_{\mathfrak{o}|\mathbb{R}}^{(\delta)}: & \mathscr{P}(\mathbb{R}) & \longrightarrow \mathscr{P}(\mathbb{R})\\ & \mathscr{S}_{\mathfrak{g}} & \longmapsto \mathrm{der}_{\mathfrak{o}|\mathbb{R}}^{(\delta)}(\mathscr{S}_{\mathfrak{o}})\end{array}$ is a $\mathfrak{T}_{\mathfrak{o}|\mathbb{R}}$-derived operator. He has also considered the unfortunate extent to which δ^{th}-iteration fails to relate well to several $\mathscr{T}_{\mathfrak{o}|\mathbb{R}}$-concepts and defined the limit δ^{th}-iterate of the $\mathfrak{T}_{\mathfrak{o}|\mathbb{R}}^{(\delta)}$-derived operator in $\mathfrak{T}_{\mathfrak{o}|\mathbb{R}}$.

Having introduced the definitions and then investigated the properties of a new type of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators in $\mathscr{T}_{\mathfrak{g}}$-spaces [1], it may be another good research investigation to introduce the definitions and then investigate the properties of the δ^{th}-order derivative $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators defined by transfinite recursion on the class of successor ordinals in $\mathscr{T}_{\mathfrak{g}}$-spaces. Such inquiry is what we endeavor to undertake in the present paper.

Hereafter, the paper is structured as thus: In § 2, the preliminary and main concepts are described in §§ 2.1 and §§ 2.2, respectively. The main results are reported in § 3. In § 4, the various relationships amongst the $\mathfrak{T}_{\mathfrak{a}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-derived and $\mathfrak{T}_{\mathfrak{a}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-coderived operators in a $\mathscr{T}_{\mathfrak{a}}$-space are diagrammed in §§ 4.1, and a nice application supporting the overall study is presented in §§ 4.2. Finally, the work is concluded in § 5.

2. Theory

2.1. **Preliminary Concepts.** The standard reference for $\mathscr{T}_{\mathfrak{a}}$-space notations and notions is the Ph.D. Thesis of Khodabocus, M. I. [9], whereas that for $\mathfrak{T}_{\mathfrak{a}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-derived and $\mathfrak{T}_{\mathfrak{a}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-coderived operators notations and preliminary concepts in $\mathscr{T}_{\mathfrak{a}}$-spaces is our recent paper on the subject matter [1] (Cf. [2, 3, 4, 5, 6, 7, 8]).

The notation $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$ designates a topological structure called $\mathscr{T}_{\mathfrak{a}}$*-space* on which no separation axioms are assumed unless otherwise mentioned [7, 8, 9]. The relation $(\alpha_1, \alpha_2, \ldots)\,\mathrm{R}\,\mathscr{A}_1 \times \mathscr{A}_2 \times \cdots$ is made a rule to mean $\alpha_1\,\mathrm{R}\,\mathscr{A}_1$, $\alpha_2\,\mathrm{R}\,\mathscr{A}_2$, … where $\mathrm{R} = \in, \subset, \supset, \ldots$. Accordingly, $\left(I_n^0, I_n^*\right) = \left(\llbracket 0, n\rrbracket, \llbracket 1, n\rrbracket\right) \subset \mathbb{Z}_+^0 \times \mathbb{Z}_+^*$ and $\left(I_\infty^0, I_\infty^*\right) = \left(\llbracket 0, \infty\rrbracket, \llbracket 1, \infty\rrbracket\right) \sim \mathbb{Z}_+^0 \times \mathbb{Z}_+^*$ are pairs of *finite* and *infinite index sets*, respectively, [8, 9]. For any $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$, the relations $\Gamma \subset \Omega$, $\mathscr{O}_{\mathfrak{a}} \in \mathscr{T}_{\mathfrak{a}}$, $\mathscr{K}_{\mathfrak{a}} \in \neg\mathscr{T}_{\mathfrak{a}} \stackrel{\text{def}}{=} \left\{\mathscr{K}_{\mathfrak{a}} :\ \complement_{\Omega}\left(\mathscr{K}_{\mathfrak{a}}\right) \in \mathscr{T}_{\mathfrak{a}}\right\}$ and $\mathscr{S}_{\mathfrak{a}} \subset \mathfrak{T}_{\mathfrak{a}}$ state that Γ, $\mathscr{O}_{\mathfrak{a}}$, $\mathscr{K}_{\mathfrak{a}}$ and $\mathscr{S}_{\mathfrak{a}}$ are a Ω*-subset*, $\mathscr{T}_{\mathfrak{a}}$*-open set*, $\mathscr{T}_{\mathfrak{a}}$*-closed set* and $\mathfrak{T}_{\mathfrak{a}}$*-set*, respectively [8, 9]. The $\mathfrak{T}_{\mathfrak{a}}$-operators $\begin{array}{rl} \mathrm{int}_{\mathfrak{a}},\ \mathrm{cl}_{\mathfrak{a}} : & \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega) \\ & \mathscr{S}_{\mathfrak{a}} \longmapsto \mathrm{int}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}),\ \mathrm{cl}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}) \end{array}$ are the $\mathfrak{T}_{\mathfrak{a}}$*-interior* and $\mathfrak{T}_{\mathfrak{a}}$*-closure operators*, respectively [8, 9]. Let the class of all possible pairs of compositions of these $\mathfrak{T}_{\mathfrak{a}}$-operators in $\mathfrak{T}_{\mathfrak{a}}$ be $\mathscr{L}_{\mathfrak{a}}[\Omega] \stackrel{\text{def}}{=} \left\{\mathfrak{op}_{\mathfrak{a},\nu} = \left(\mathrm{op}_{\mathfrak{a},\nu}, \neg\,\mathrm{op}_{\mathfrak{a},\nu}\right) :\ \nu \in I_3^0\right\}$, where

$$\begin{aligned} \left\langle \mathrm{op}_{\mathfrak{a},\nu} :\ \nu \in I_3^0\right\rangle &= \left\langle \mathrm{int}_{\mathfrak{a}},\ \mathrm{cl}_{\mathfrak{a}}\circ\mathrm{int}_{\mathfrak{a}},\ \mathrm{int}_{\mathfrak{a}}\circ\mathrm{cl}_{\mathfrak{a}},\ \mathrm{cl}_{\mathfrak{a}}\circ\mathrm{int}_{\mathfrak{a}}\circ\mathrm{cl}_{\mathfrak{a}}\right\rangle \\ \left\langle \neg\,\mathrm{op}_{\mathfrak{a},\nu} :\ \nu \in I_3^0\right\rangle &= \left\langle \mathrm{cl}_{\mathfrak{a}},\ \mathrm{int}_{\mathfrak{a}}\circ\mathrm{cl}_{\mathfrak{a}},\ \mathrm{cl}_{\mathfrak{a}}\circ\mathrm{int}_{\mathfrak{a}},\ \mathrm{int}_{\mathfrak{a}}\circ\mathrm{cl}_{\mathfrak{a}}\circ\mathrm{int}_{\mathfrak{a}}\right\rangle \end{aligned}$$

Then, $\mathscr{S}_{\mathfrak{a}} \subset \mathfrak{T}_{\mathfrak{a}}$ is called a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-set if and only if it holds that

$$(\exists\xi)\left[\left(\xi \in \mathscr{S}_{\mathfrak{a}}\right) \wedge \left(\left(\mathscr{S}_{\mathfrak{a}} \subseteq \mathrm{op}_{\mathfrak{a}}\left(\mathscr{O}_{\mathfrak{a}}\right)\right) \vee \left(\mathscr{S}_{\mathfrak{a}} \supseteq \neg\,\mathrm{op}_{\mathfrak{a}}\left(\mathscr{K}_{\mathfrak{a}}\right)\right)\right)\right] \tag{2.1}$$

for some $\left(\mathscr{O}_{\mathfrak{a}}, \mathscr{K}_{\mathfrak{a}}, \mathrm{op}_{\mathfrak{a}}\right) \in \mathscr{T}_{\mathfrak{a}} \times \neg\mathscr{T}_{\mathfrak{a}} \times \mathscr{L}_{\mathfrak{a}}[\Omega]$. In this way, the derived class $\mathfrak{g}$-ν-$\mathrm{S}\left[\mathfrak{T}_{\mathfrak{a}}\right] = \bigcup_{\mathrm{E}\in\{\mathrm{O},\mathrm{K}\}} \mathfrak{g}$-$\nu$-$\mathrm{E}\left[\mathfrak{T}_{\mathfrak{a}}\right]$ collects all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-sets of category $\nu \in I_3^0$ ($\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$*-sets*), whereas

$$\mathfrak{g}\text{-}\mathrm{S}\left[\mathfrak{T}_{\mathfrak{a}}\right] = \bigcup_{\nu\in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{S}\left[\mathfrak{T}_{\mathfrak{a}}\right] = \bigcup_{(\nu,\mathrm{E})\in I_3^0\times\{\mathrm{O},\mathrm{K}\}} \mathfrak{g}\text{-}\nu\text{-}\mathrm{E}\left[\mathfrak{T}_{\mathfrak{a}}\right] = \bigcup_{\mathrm{E}\in\{\mathrm{O},\mathrm{K}\}} \mathfrak{g}\text{-}\mathrm{E}\left[\mathfrak{T}_{\mathfrak{a}}\right] \tag{2.2}$$

collects all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-sets irrespective of their categories in $\mathfrak{T}_{\mathfrak{a}}$ [8, 9]. In particular, $\mathrm{S}[\mathfrak{T}_{\mathfrak{a}}] = \bigcup_{(\nu,\mathrm{E})\in\{0\}\times\{\mathrm{O},\mathrm{K}\}} \mathfrak{g}\text{-}\nu\text{-}\mathrm{E}[\mathfrak{T}_{\mathfrak{a}}] = \bigcup_{\mathrm{E}\in\{\mathrm{O},\mathrm{K}\}} \mathrm{E}[\mathfrak{T}_{\mathfrak{a}}]$ collects all $\mathfrak{T}_{\mathfrak{a}}$-sets in $\mathfrak{T}_{\mathfrak{a}}$ [8, 9].

Definition 2.1 ($\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-*Interior,* $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-*Closure Operators* [2, 3])**.** In a $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$, the one-valued maps

$$\begin{aligned} \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{a},\nu} : \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \\ \mathscr{S}_{\mathfrak{a}} &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{a}}\in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{a}}]}[\mathscr{S}_{\mathfrak{a}}]} \mathscr{O}_{\mathfrak{a}} \end{aligned} \tag{2.3}$$

$$\begin{aligned} \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{a},\nu} : \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \\ \mathscr{S}_{\mathfrak{a}} &\longmapsto \bigcap_{\mathscr{K}_{\mathfrak{a}}\in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{a}}]}[\mathscr{S}_{\mathfrak{a}}]} \mathscr{K}_{\mathfrak{a}} \end{aligned} \tag{2.4}$$

where $\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{a}}]}[\mathscr{S}_{\mathfrak{a}}] \stackrel{\mathrm{def}}{=} \{\mathscr{O}_{\mathfrak{a}} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{a}}] : \mathscr{O}_{\mathfrak{a}} \subseteq \mathscr{S}_{\mathfrak{a}}\}$ and $\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{a}}]}[\mathscr{S}_{\mathfrak{a}}] \stackrel{\mathrm{def}}{=} \{\mathscr{K}_{\mathfrak{a}} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{a}}] : \mathscr{K}_{\mathfrak{a}} \supseteq \mathscr{S}_{\mathfrak{a}}\}$ are called $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-interior and $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-closure operators, respectively. Then, $\mathfrak{g}\text{-}\mathrm{I}[\mathfrak{T}_{\mathfrak{a}}] \stackrel{\mathrm{def}}{=} \{\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{a},\nu} : \nu \in I^0_3\}$ and $\mathfrak{g}\text{-}\mathrm{C}[\mathfrak{T}_{\mathfrak{a}}] \stackrel{\mathrm{def}}{=} \{\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{a},\nu} : \nu \in I^0_3\}$ are the classes of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-closure operators, respectively.

Definition 2.2 ($\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-*Vector Operator* [2, 3])**.** In a $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$, the two-valued map

$$\begin{aligned} \mathfrak{g}\text{-}\mathrm{Ic}_{\mathfrak{a},\nu} : \mathscr{P}(\Omega)\times\mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega)\times\mathscr{P}(\Omega) \\ (\mathscr{R}_{\mathfrak{a}}, \mathscr{S}_{\mathfrak{a}}) &\longmapsto \left(\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{a},\nu}(\mathscr{R}_{\mathfrak{a}}), \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{a},\nu}(\mathscr{S}_{\mathfrak{a}})\right) \end{aligned} \tag{2.5}$$

is called a $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-vector operator. Then, $\mathfrak{g}\text{-}\mathrm{IC}[\mathfrak{T}_{\mathfrak{a}}] \stackrel{\mathrm{def}}{=} \{\mathfrak{g}\text{-}\mathrm{Ic}_{\mathfrak{a},\nu} = (\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{a},\nu}, \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{a},\nu}) : \nu \in I^0_3\}$ is the class of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-vector operators.

Remark 2.3 ($\mathfrak{T}_{\mathfrak{a}}$-*Vector Operator* [1]). For each $\nu \in I^0_3$, $\mathfrak{g}\text{-}\mathrm{Ic}_{\mathfrak{a},\nu} = \mathrm{ic}_{\mathfrak{a}} \stackrel{\mathrm{def}}{=} (\mathrm{int}_{\mathfrak{a}}, \mathrm{cl}_{\mathfrak{a}})$ if based on $\mathrm{O}[\mathfrak{T}_{\mathfrak{a}}]\times\mathrm{K}[\mathfrak{T}_{\mathfrak{a}}]$. Then, $\begin{aligned} \mathbf{ic}_{\mathfrak{a}} : \mathscr{P}(\Omega)\times\mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega)\times\mathscr{P}(\Omega) \\ (\mathscr{R}_{\mathfrak{a}}, \mathscr{S}_{\mathfrak{a}}) &\longmapsto (\mathrm{int}_{\mathfrak{a}}(\mathscr{R}_{\mathfrak{a}}), \mathrm{cl}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}})) \end{aligned}$ is a $\mathfrak{T}_{\mathfrak{a}}$-vector operator in a $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$.

Definition 2.4 (*Complement* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-*Operator* [2, 3])**.** Let $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$ be a $\mathscr{T}_{\mathfrak{a}}$-space. Then, the one-valued map

$$\begin{aligned} \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{a},\mathscr{R}_{\mathfrak{a}}} : \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \\ \mathscr{S}_{\mathfrak{a}} &\longmapsto \complement_{\mathscr{R}_{\mathfrak{a}}}(\mathscr{S}_{\mathfrak{a}}) \end{aligned} \tag{2.6}$$

where $\complement_{\mathscr{R}_{\mathfrak{a}}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ denotes the relative complement of its operand with respect to $\mathscr{R}_{\mathfrak{a}} \in \mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{a}}]$, is called a natural complement $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-operator on $\mathscr{P}(\Omega)$.

For the sake of clarity, $\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{a},\mathscr{R}_{\mathfrak{a}}} = \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{a}}$ whenever $\mathscr{R}_{\mathfrak{a}} = \Omega$, and $\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{a},\mathscr{R}_{\mathfrak{a}}} = \mathrm{Op}_{\mathfrak{a},\mathscr{R}_{\mathfrak{a}}}$ whenever $\mathscr{R}_{\mathfrak{a}} \in \mathrm{S}[\mathfrak{T}_{\mathfrak{a}}]$ in which case, the term natural complement $\mathfrak{T}_{\mathfrak{g}}$-operator is employed and it stand for $\mathrm{Op}_{\mathfrak{a},\mathscr{R}_{\mathfrak{a}}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$.

Definition 2.5 ($\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-*Derived,* $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-*Coderived Operators* [1])**.** Let $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{a},\nu}$, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{a},\nu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, denote the $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-interior and $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-closure operators and, $\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{a}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ denote the absolute complement

$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-operator in a $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$. Then, the one-valued maps

$$\begin{array}{rrcl} (2.7) & \mathfrak{g}\text{-Der}_{\mathfrak{a},\nu} : \mathscr{P}(\Omega) & \longrightarrow & \mathscr{P}(\Omega) \\ & \mathscr{S}_{\mathfrak{a}} & \longmapsto & \left\{\xi \in \mathfrak{T}_{\mathfrak{a}} :\ \xi \in \mathfrak{g}\text{-Cl}_{\mathfrak{a},\nu}\big(\mathscr{S}_{\mathfrak{a}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{a}}(\{\xi\})\big)\right\} \\ (2.8) & \mathfrak{g}\text{-Cod}_{\mathfrak{a},\nu} : \mathscr{P}(\Omega) & \longrightarrow & \mathscr{P}(\Omega) \\ & \mathscr{S}_{\mathfrak{a}} & \longmapsto & \left\{\zeta \in \mathfrak{T}_{\mathfrak{a}} :\ \zeta \in \mathfrak{g}\text{-Int}_{\mathfrak{a},\nu}\big(\mathscr{S}_{\mathfrak{a}} \cup \{\zeta\}\big)\right\} \end{array}$$

on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ are called, respectively, a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-derived operator of category ν and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-coderived operator of category ν. The classes $\mathfrak{g}\text{-DE}[\mathfrak{T}_{\mathfrak{a}}] \overset{\text{def}}{=} \left\{\mathfrak{g}\text{-Der}_{\mathfrak{a},\nu} :\ \nu \in I_3^0\right\}$ and $\mathfrak{g}\text{-CD}[\mathfrak{T}_{\mathfrak{a}}] \overset{\text{def}}{=} \left\{\mathfrak{g}\text{-Cod}_{\mathfrak{a},\nu} :\ \nu \in I_3^0\right\}$ are called, respectively, the class of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-derived operators and the class of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-coderived operators.

Remark 2.6 ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-*Derived*, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-*Coderived Sets* [1]). In a $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}}$, suppose $\big(\mathfrak{g}\text{-Der}_{\mathfrak{a}}(\xi;\mathscr{S}_{\mathfrak{a}}), \mathfrak{g}\text{-Cod}_{\mathfrak{a}}(\zeta;\mathscr{S}_{\mathfrak{a}})\big)$ denotes a pair $(\xi,\zeta) \in \mathfrak{T}_{\mathfrak{a}} \times \mathfrak{T}_{\mathfrak{a}}$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-*derived* and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-*coderived points* of $\mathscr{S}_{\mathfrak{a}} \in \mathscr{P}(\Omega)$, then $\big(\mathfrak{g}\text{-Der}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}), \mathfrak{g}\text{-Cod}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}})\big)$ denotes the pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-*derived* and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-*coderived sets* of $\mathscr{S}_{\mathfrak{a}}$ in $\mathfrak{T}_{\mathfrak{a}}$, where

$$(2.9) \qquad \begin{cases} \mathfrak{g}\text{-Der}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}) \overset{\text{def}}{=} \left\{\mathfrak{g}\text{-Der}_{\mathfrak{a}}(\xi;\mathscr{S}_{\mathfrak{a}}) :\ \xi \in \mathfrak{T}_{\mathfrak{a}}\right\} \\ \mathfrak{g}\text{-Cod}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}) \overset{\text{def}}{=} \left\{\mathfrak{g}\text{-Cod}_{\mathfrak{a}}(\zeta;\mathscr{S}_{\mathfrak{a}}) :\ \xi \in \mathfrak{T}_{\mathfrak{a}}\right\} \end{cases}$$

denote the pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-*derived* and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-*coderived sets* of $\mathscr{S}_{\mathfrak{a}}$ in $\mathfrak{T}_{\mathfrak{a}}$.

Definition 2.7 ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-*Vector Operator* [1])**.** Let $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$ be a $\mathscr{T}_{\mathfrak{a}}$-space. Then, an operator of the type

$$\begin{array}{rrcl} (2.10) & \mathfrak{g}\text{-Dc}_{\mathfrak{a},\nu} : \times_{\alpha \in I_2^*} \mathscr{P}(\Omega) & \longrightarrow & \displaystyle\bigtimes_{\alpha \in I_2^*} \mathscr{P}(\Omega) \\ & (\mathscr{R}_{\mathfrak{a}}, \mathscr{S}_{\mathfrak{a}}) & \longmapsto & \big(\mathfrak{g}\text{-Der}_{\mathfrak{a},\nu}(\mathscr{R}_{\mathfrak{a}}), \mathfrak{g}\text{-Cod}_{\mathfrak{a},\nu}(\mathscr{S}_{\mathfrak{a}})\big) \end{array}$$

on $\times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$ ranging in $\times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$ is called a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-vector operator of category ν and, $\mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{a}}] \overset{\text{def}}{=} \left\{\mathfrak{g}\text{-Dc}_{\mathfrak{a},\nu} = \big(\mathfrak{g}\text{-Der}_{\mathfrak{a},\nu}, \mathfrak{g}\text{-Cod}_{\mathfrak{a},\nu}\big) :\ \nu \in I_3^0\right\}$ is called the class of all such $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-vector operators.

Remark 2.8 ($\mathfrak{T}_{\mathfrak{a}}$-*Vector Operator* [1]). For any $\nu \in I_3^0$, $\mathfrak{g}\text{-Dc}_{\mathfrak{a},\nu} = \text{dc}_{\mathfrak{a}} \overset{\text{def}}{=} \big(\text{der}_{\mathfrak{a}}, \text{cod}_{\mathfrak{a}}\big)$ if based on $\big(\text{cl}_{\mathfrak{g}}, \text{int}_{\mathfrak{g}}\big)$. Then, $\begin{array}{rrl} \mathbf{dc}_{a} : & \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) & \longrightarrow \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \\ & (\mathscr{R}_{\mathfrak{a}}, \mathscr{S}_{\mathfrak{a}}) & \longmapsto \big(\text{der}_{\mathfrak{a}}(\mathscr{R}_{\mathfrak{a}}), \text{cod}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}})\big) \end{array}$ is a $\mathfrak{T}_{\mathfrak{a}}$-vector operator in a $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$.

Accordingly,

$$\begin{array}{rcl} \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{a}}] & \overset{\text{def}}{=} & \left\{\mathfrak{g}\text{-Dc}_{\mathfrak{a},\nu} = \big(\mathfrak{g}\text{-Der}_{\nu}, \mathfrak{g}\text{-Cod}_{\mathfrak{a},\nu}\big) : \nu \in I_3^0\right\} \\ (2.11) \qquad & \subseteq & \left\{\mathfrak{g}\text{-Der}_{\mathfrak{a},\nu} :\ \nu \in I_3^0\right\} \times \left\{\mathfrak{g}\text{-Cod}_{\mathfrak{a},\nu} : \nu \in I_3^0\right\} \overset{\text{def}}{=} \mathfrak{g}\text{-DE}[\mathfrak{T}_{\mathfrak{a}}] \times \mathfrak{g}\text{-CD}[\mathfrak{T}_{\mathfrak{a}}] \end{array}$$

Then, $\mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{a}}]$ denotes the class of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-vector operators in the $\mathscr{T}$-space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$; $\mathfrak{g}\text{-DE}[\mathfrak{T}_{\mathfrak{a}}]$ denotes the class of all $\mathfrak{g}$-$\mathfrak{T}$-derived operators while $\mathfrak{g}\text{-CD}[\mathfrak{T}_{\mathfrak{a}}]$ denotes the class of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-coderived operators in the $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$.

2.2. **Main Concepts.** The main concepts underlying the δ^{th}-order derivative $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators defined by transfinite recursion on the class of successor ordinals in $\mathscr{T}_{\mathfrak{g}}$-spaces, $\mathfrak{a} \in \{\mathfrak{o}, \mathfrak{g}\}$, are now presented.

For any $(\mathscr{S}_{\mathfrak{g}}, \mathfrak{g}\text{-Ope}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \{\mathfrak{g}\text{-Der}_{\mathfrak{g}}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\}$, consider the description:

$$\begin{array}{ccl} 0 & \longleftrightarrow & \mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(0)}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \bigcirc_{\alpha \in I_0^0} \mathfrak{g}\text{-Ope}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ 1 & \longleftrightarrow & \mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(1)}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \bigcirc_{\alpha \in I_1^0} \mathfrak{g}\text{-Ope}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ 2 & \longleftrightarrow & \mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(2)}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \bigcirc_{\alpha \in I_2^0} \mathfrak{g}\text{-Ope}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ & \vdots & \\ \beta - 1 & \longleftrightarrow & \mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(\beta-1)}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \bigcirc_{\alpha \in I_{\beta-1}^0} \mathfrak{g}\text{-Ope}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ \beta & \longleftrightarrow & \mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(\beta)}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \bigcirc_{\alpha \in I_{\beta}^0} \mathfrak{g}\text{-Ope}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \end{array} \tag{2.12}$$

where $\bigcirc_{\alpha \in I_0^0} \mathfrak{g}\text{-Ope}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathscr{S}_{\mathfrak{g}}$; next, $\bigcirc_{\alpha \in I_1^0} \mathfrak{g}\text{-Ope}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Ope}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and $\bigcirc_{\alpha \in I_2^0} \mathfrak{g}\text{-Ope}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Ope}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ope}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$; more generally,

$$\bigcirc_{\alpha \in I_{\beta}^0} \mathfrak{g}\text{-Ope}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Ope}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ope}_{\mathfrak{g}} \circ \cdots \circ \mathfrak{g}\text{-Ope}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$$

β factors $\mathfrak{g}\text{-Ope}_{\mathfrak{g}}$. Thus, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(0)}$, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(1)}$, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(2)}$, …, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\beta)}$, … : $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are the 0^{th}, 1^{st}, 2^{nd}, …, β^{th}, … *order derivative* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived operators* of $\mathfrak{g}\text{-Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$; $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(0)}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(1)}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(2)}$, …, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\beta)}$, … : $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are the 0^{th}, 1^{st}, 2^{nd}, …, β^{th}, … *order derivative* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operators* of $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$. Then, for any pair $(\mathscr{S}_{\mathfrak{g}}, \mathfrak{g}\text{-Ope}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \{\mathfrak{g}\text{-Der}_{\mathfrak{g}}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\}$, it holds that:

$$\left[(\exists \beta \in I_{\infty}^0)\left(\mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(\beta)}(\mathscr{S}_{\mathfrak{g}}) = \varnothing\right)\right] \veebar \left[(\forall \beta \in I_{\infty}^0)\left(\mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(\beta)}(\mathscr{S}_{\mathfrak{g}}) \neq \varnothing\right)\right]$$

Suppose the statement preceding $\veebar$ hold, then the number of iterations of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operator $\mathfrak{g}\text{-Ope}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ required to achieve *emptiness* (if this is ever achieved) is a type of *density measure* of $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$. But if the statement following $\veebar$ holds, then $\mathscr{S}_{\mathfrak{g}}^{(\lambda)} \stackrel{\text{def}}{=} \bigcap_{\beta \in I_{\infty}^*} \mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(\beta)}(\mathscr{S}_{\mathfrak{g}}) \neq \varnothing$. Therefore, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(1)}$, $\mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(2)}$, …, $\mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(\beta)}$, … : $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ can again be applied on $\mathscr{S}_{\mathfrak{g}}^{(\omega)} \in \mathscr{P}(\Omega)$, yielding $\mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(\lambda+1)}(\mathscr{S}_{\mathfrak{g}})$, $\mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(\lambda+2)}(\mathscr{S}_{\mathfrak{g}})$, …, $\mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(\lambda+\beta)}(\mathscr{S}_{\mathfrak{g}})$, …, with $\mathfrak{g}\text{-Ope}_{\mathfrak{g}} \in \{\mathfrak{g}\text{-Der}_{\mathfrak{g}}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\}$.

In view of the above descriptions, 1, 2, …, β, … may be viewed as *successor ordinals* while λ as *limit ordinal* and, despite the absence of a *predecessor ordinal*, 0 may, for conveniency, be included in the class of successor ordinals. To define the notion of *ordinal*, the concepts of *everywhere-ordered set*, *similarity* and *order-type* in chronological order have first to be defined. The definition of the first concept (*everywhere-ordered set*) follows.

Definition 2.9 (*Everywhere-Ordered Set*)**.** An "everywhere-ordered set" is an ordered structure of the type

$$\mathfrak{W} \stackrel{\text{def}}{=} (\mathscr{W}, \preccurlyeq) \stackrel{\text{def}}{\longleftrightarrow} \langle \alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{\nu}, \ldots \rangle \tag{2.13}$$

in which $\mathscr{W} \subset \mathfrak{U}$ is an "underlying set" and,

$$\begin{array}{rcll} \preccurlyeq: & \mathscr{W} \times \mathscr{W} & \longrightarrow & \mathbb{W} \overset{\text{def}}{=} \{\alpha \preccurlyeq \beta : (\alpha,\beta) \in \mathscr{W} \times \mathscr{W}\} \\ & (\alpha,\beta) & \longmapsto & \alpha \preccurlyeq \beta \end{array} \tag{2.14}$$

is a "2-ary rule" satisfying these "everywhere-ordering relation axioms:"

– $\mathrm{Ax}_1(\preccurlyeq) \overset{\text{def}}{\longleftrightarrow} (\forall \alpha \in \mathfrak{W})[\alpha \preccurlyeq \alpha \longleftrightarrow \alpha = \alpha]$

– $\mathrm{Ax}_2(\preccurlyeq) \overset{\text{def}}{\longleftrightarrow} (\forall (\alpha,\beta) \in \mathfrak{W}^2)[(\alpha \preccurlyeq \beta) \wedge (\beta \preccurlyeq \alpha) \longrightarrow \alpha = \beta]$

– $\mathrm{Ax}_3(\preccurlyeq) \overset{\text{def}}{\longleftrightarrow} (\forall (\alpha,\beta,\gamma) \in \mathfrak{W}^3)[(\alpha \preccurlyeq \beta) \wedge (\beta \preccurlyeq \gamma) \longrightarrow \alpha \preccurlyeq \gamma]$

– $\mathrm{Ax}_4(\preccurlyeq) \overset{\text{def}}{\longleftrightarrow} (\forall \mathfrak{V} \subseteq \mathfrak{W})[\mathfrak{V} \overset{\text{def}}{\longleftrightarrow} \langle \beta_0, \beta_1, \beta_2, \ldots \rangle \longrightarrow \beta_0 \preccurlyeq \beta_1 \preccurlyeq \beta_2 \preccurlyeq \cdots]$

The above definition requires some few explanations. By $\mathrm{Ax}_1(\preccurlyeq)$, $\mathrm{Ax}_2(\preccurlyeq)$ and $\mathrm{Ax}_3(\preccurlyeq)$ are meant that $\begin{array}{rcl} \preccurlyeq: \mathscr{W} \times \mathscr{W} & \longrightarrow \mathbb{W} \\ (\alpha,\beta) & \longmapsto \alpha \preccurlyeq \beta \end{array}$ is *reflexive*, *antisymmetric* and *transitive*, respectively; by $\mathrm{Ax}_4(\preccurlyeq)$ is meant that any ordered structure $\mathfrak{V} = (\mathscr{V}, \preccurlyeq)$ derived from $\mathfrak{W} = (\mathscr{W}, \preccurlyeq)$ has a *first* element (*i.e.*, $\beta_0 \in \mathfrak{V} \subseteq \mathfrak{W}$). Moreover, the following statement holds true:

$$(\forall (\alpha,\beta) \in \mathfrak{W} \times \mathfrak{W})[(\alpha \preccurlyeq \beta) \veebar (\beta = \alpha) \veebar (\beta \preccurlyeq \alpha)] \tag{2.15}$$

Thus, given $(\alpha,\beta) \in \mathfrak{W} \times \mathfrak{W}$ then, either α *preceeds* β (*i.e.*, $\alpha \preccurlyeq \beta$), α *is of the same order as* β (*i.e.*, $\beta = \alpha$) or α *succeeds* β (*i.e.*, $\beta \preccurlyeq \alpha$). The remark below is presented in order to avoid any danger of confusing the notations of *underlying* (not ordered) and *everywhere-ordered* sets.

Remark 2.10 (*Everywhere-Ordered Set*). Instead of such *plain sets* notations as $\alpha \in \mathscr{W}$, $(\alpha,\beta) \in \mathscr{W} \times \mathscr{W}$, ... which, in actual fact, are improper, the *ordered sets* notations $\alpha \in \mathfrak{W}$, $(\alpha,\beta) \in \mathfrak{W} \times \mathfrak{W}$, ... are employed solely to stress that α, β, ... are elements of their *ordered set* $\mathfrak{W}$, not of the *underlying set* $\mathscr{W}$ of the ordered set $\mathfrak{W}$. Indeed, in the present context, it does not hold that $\langle \alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_\nu, \ldots \rangle \neq \{\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_\nu, \ldots\}$, though it does hold that $\{\alpha : \alpha \in \mathfrak{W}\} = \{\alpha : \alpha \in \mathscr{W}\}$.

For each $\mathscr{U} \in \{\mathscr{V}, \mathscr{W}\}$, set $\mathbb{W}_{\mathscr{U}} = \{\alpha \preccurlyeq_{\mathscr{U}} \beta : (\alpha,\beta) \in \mathscr{W}_{\mathscr{U}} \times \mathscr{W}_{\mathscr{U}}\}$. Then, the second concept (*similarity*) may be defined as thus.

Definition 2.11 (*Similarity*)**.** The everywhere-ordered sets $\mathfrak{V} = (\mathscr{V}, \preccurlyeq_{\mathscr{V}})$ and $\mathfrak{W} = (\mathscr{W}, \preccurlyeq_{\mathscr{W}})$, where $\begin{array}{rcl} \preccurlyeq_{\mathscr{V}}: \mathscr{V} \times \mathscr{V} & \longrightarrow \mathbb{W}_{\mathscr{V}} \\ (\alpha,\beta) & \longmapsto \alpha \preccurlyeq_{\mathscr{V}} \beta \end{array}$ and $\begin{array}{rcl} \preccurlyeq_{\mathscr{W}}: \mathscr{W} \times \mathscr{W} & \longrightarrow \mathbb{W}_{\mathscr{W}} \\ (\alpha,\beta) & \longmapsto \alpha \preccurlyeq_{\mathscr{W}} \beta, \end{array}$ respectively, are said to be "similar," written $\mathfrak{V} \approx \mathfrak{W}$, if and only if there is an "order isomorphism" $\varphi : \mathfrak{V} \cong \mathfrak{W}$ relating the elements α_0, α_1, α_2, ... of $\mathfrak{V}$ to the elements β_0, β_1, β_2, ... of $\mathfrak{W}$ as:

$$\begin{array}{rcl} \mathfrak{V} = (\mathscr{V}, \preccurlyeq_{\mathscr{V}}) & \overset{\text{def}}{\longleftrightarrow} & \langle \alpha_0,\ \alpha_1,\ \alpha_2,\ \ldots \rangle \\ \approx & & \updownarrow \varphi \\ \mathfrak{W} = (\mathscr{W}, \preccurlyeq_{\mathscr{W}}) & \overset{\text{def}}{\longleftrightarrow} & \langle \beta_0,\ \beta_1,\ \beta_2,\ \ldots \rangle \end{array} \tag{2.16}$$

From this definition, given $\mathfrak{Y} = (\mathscr{Y}, \preccurlyeq_{\mathscr{Y}}) \longleftrightarrow \langle \gamma_0,\ \gamma_1,\ \gamma_2,\ \ldots,\ \gamma_\nu,\ \ldots \rangle$ with $(\mathfrak{Y}, \mathscr{Y}, \gamma) \in \{(\mathfrak{V}, \mathscr{V}, \alpha), (\mathfrak{W}, \mathscr{W}, \beta)\}$ and $\varphi : \mathfrak{V} \cong \mathfrak{W}$, then $\alpha_0 \preccurlyeq_{\mathscr{V}} \alpha_1 \overset{\varphi}{\longrightarrow} \beta_0 \preccurlyeq_{\mathscr{W}} \beta_1$, $\alpha_1 \preccurlyeq_{\mathscr{V}} \alpha_2 \overset{\varphi}{\longrightarrow} \beta_1 \preccurlyeq_{\mathscr{W}} \beta_2$, ..., $\alpha_{\nu-1} \preccurlyeq_{\mathscr{V}} \alpha_\nu \overset{\varphi}{\longrightarrow} \beta_{\nu-1} \preccurlyeq_{\mathscr{W}} \alpha_\nu$, ... For any

$(\mathfrak{V},\mathfrak{W},\mathfrak{Y})\in\times_{\mu\in I_3^*}\{\mathfrak{W}_\nu=(\mathscr{W}_\nu,\preccurlyeq_\nu):\ \nu\in I_\infty^*\}$, the relations $\mathfrak{V}\approx\mathfrak{V}$, $\mathfrak{V}\approx\mathfrak{W}\longleftrightarrow\mathfrak{W}\approx\mathfrak{V}$ and $(\mathfrak{V}\approx\mathfrak{W})\wedge(\mathfrak{W}\approx\mathfrak{Y})\longrightarrow(\mathfrak{V}\approx\mathfrak{Y})$ hold. Therefore, the relation of similarity $\approx:\ (\mathfrak{V},\mathfrak{W})\longmapsto\mathfrak{V}\approx\mathfrak{W}$ is *reflexive*, *symmetrical* and *transitive*.

The definition of the third concept (*order-type*) may be stated as thus.

Definition 2.12 (*Order-Type*)**.** An operator of the type

$$\mathrm{OTyp}\ :\ \mathfrak{W}\longmapsto\mathrm{OTyp}\,(\mathfrak{W})\stackrel{\mathrm{def}}{=}\tau_{\mathscr{W}} \tag{2.17}$$

assigning to any everywhere-ordered set $\mathfrak{W}=(\mathscr{W},\preccurlyeq_{\mathscr{W}})$ a uniquely determined symbol $\tau_{\mathscr{W}}$ is called the "order-type" of $\mathfrak{W}$, provided that if $\mathfrak{V}=(\mathscr{V},\preccurlyeq_{\mathscr{V}})$ be any other everywhere-ordered set together with its uniquely determined order-type $\mathrm{OTyp}\,(\mathfrak{V})\stackrel{\mathrm{def}}{=}\tau_{\mathscr{V}}$, the following statement holds:

$$\mathfrak{V}\approx\mathfrak{W}\ \longleftrightarrow\ \tau_{\mathscr{V}}=\tau_{\mathscr{W}}. \tag{2.18}$$

Clearly, the manner of proceeding from the relation of similarity to the concept of order-type is exactly the same as that from the relation of equivalence to the concept of cardinal number. For, given any $\mathfrak{V}=(\mathscr{V},\preccurlyeq_{\mathscr{V}})$ and $\mathfrak{W}=(\mathscr{W},\preccurlyeq_{\mathscr{W}})$, then $\mathfrak{V}\approx\mathfrak{W}\ \longleftrightarrow\ \mathrm{OTyp}\,(\mathfrak{V})=\mathrm{OTyp}\,(\mathfrak{W})$ is analogous to $\mathscr{V}\sim\mathscr{W}\ \longleftrightarrow\ \mathrm{card}\,(\mathscr{V})=\mathrm{card}\,(\mathscr{W})$.

Remark 2.13. By $\mathfrak{V}\approx\mathfrak{W}\ \longleftrightarrow\ \tau_{\mathscr{V}}=\tau_{\mathscr{W}}$ is meant that a uniquely determined symbol actually is assigned not to a single set but to a class of everywhere-ordered sets which are similar to each other.

Granted the definitions of the concepts of *everywhere-ordered set*, *similarity* and *order-type*, the definition of the concept of *ordinal* may be stated as thus.

Definition 2.14 (*Ordinal*)**.** The order-type $\mathrm{OTyp}\,(\mathfrak{W})=\tau_{\mathscr{W}}$ of an everywhere-ordered set $\mathfrak{W}=(\mathscr{W},\preccurlyeq_{\mathscr{W}})$ is called "ordinal," written $\mathrm{ord}\,(\mathfrak{W})\stackrel{\mathrm{def}}{=}\delta_{\mathscr{W}}$. Moreover:

– I. $\delta_{\mathscr{W}}$ is called a "predecessor ordinal" if and only if there exists no ordinal $\mathrm{ord}\,(\mathfrak{W})$ such that $\delta_{\mathscr{W}}=\mathrm{ord}\,(\mathfrak{W})+1$.

– II. $\delta_{\mathscr{W}}$ is called a "successor ordinal" if and only if there exists an ordinal $\mathrm{ord}\,(\mathfrak{W})$ such that $\delta_{\mathscr{W}}=\mathrm{ord}\,(\mathfrak{W})+1$.

– III. $\delta_{\mathscr{W}}$ is called a "limit ordinal," denoted as $\delta_{\mathscr{W}}\stackrel{\mathrm{def}}{=}\lambda_{\mathscr{W}}$, if and only if it has no immediate predecessor.

Let the symbols 0, δ, and λ (instead of the symbols $0_{\mathscr{W}}$, $\delta_{\mathscr{W}}$, and $\lambda_{\mathscr{W}}$) stand for *predecessor ordinal*, *successor ordinal* and *limit ordinal*, respectively. Then, the definitions of the notions of *ordered derivative* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived and* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operators* of $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}:\mathscr{P}\,(\Omega)\longrightarrow\mathscr{P}\,(\Omega)$, respectively, may well be stated as thus.

Definition 2.15 (δ^{th}*-Iterations:* $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$*-Derived,* $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$*-Coderived Operators*)**.** Let $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\nu}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},\nu}:\mathscr{P}\,(\Omega)\longrightarrow\mathscr{P}\,(\Omega)$, respectively, be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators of category ν in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$. Then:

— I. The "δ^{th}-iterate of $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\nu}:\mathscr{P}\,(\Omega)\longrightarrow\mathscr{P}\,(\Omega)$" is a set-valued map $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\nu}^{(\delta)}:\mathscr{S}_{\mathfrak{g}}\in\mathscr{P}\,(\Omega)\longmapsto\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\nu}^{(\delta)}\,(\mathscr{S}_{\mathfrak{g}})$ defined by transfinite recursion on the class of successor ordinals as,

– I. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\nu}^{(0)}\,(\mathscr{S}_{\mathfrak{g}})\ \stackrel{\mathrm{def}}{\longleftrightarrow}\ \mathscr{S}_{\mathfrak{g}}$

– II. $\mathfrak{g}\text{-Der}^{(1)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}}) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})$
– III. $\mathfrak{g}\text{-Der}^{(\delta+1)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}}) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Der}_{\mathfrak{g},\nu} \circ \mathfrak{g}\text{-Der}^{(\delta)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})$
– IV. $\mathfrak{g}\text{-Der}^{(\lambda)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}}) \overset{\text{def}}{\longleftrightarrow} \bigcap_{\delta<\lambda} \mathfrak{g}\text{-Der}^{(\delta)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})$

— II. The "δ^{th}-iterate of $\mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$" is a set-valued map $\mathfrak{g}\text{-Cod}^{(\delta)}_{\mathfrak{g},\nu} : \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega) \longmapsto \mathfrak{g}\text{-Cod}^{(\delta)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})$ defined by transfinite recursion on the class of successor ordinals as,

– I. $\mathfrak{g}\text{-Cod}^{(0)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}}) \overset{\text{def}}{\longleftrightarrow} \mathscr{S}_{\mathfrak{g}}$
– II. $\mathfrak{g}\text{-Cod}^{(1)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}}) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})$
– III. $\mathfrak{g}\text{-Cod}^{(\delta+1)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}}) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu} \circ \mathfrak{g}\text{-Cod}^{(\delta)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})$
– IV. $\mathfrak{g}\text{-Cod}^{(\lambda)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}}) \overset{\text{def}}{\longleftrightarrow} \bigcap_{\delta<\lambda} \mathfrak{g}\text{-Cod}^{(\delta)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})$

In the following remark, the concepts of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived sets of category ν and order δ ($\mathfrak{g}$-ν-$\mathfrak{T}^{(\delta)}_{\mathfrak{g}}$*-derived,* $\mathfrak{g}$-ν-$\mathfrak{T}^{(\delta)}_{\mathfrak{g}}$*-coderived sets*) are presented.

Remark 2.16 ($\mathfrak{g}$-ν-$\mathfrak{T}^{(\delta)}_{\mathfrak{g}}$*-Derived,* $\mathfrak{g}$-ν-$\mathfrak{T}^{(\delta)}_{\mathfrak{g}}$*-Coderived sets*). Suppose $\left(\mathscr{R}^{(\delta)}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\right) \in \times_{\alpha\in I^*_2} \mathscr{P}(\Omega)$ such that $\mathscr{R}^{(\delta)}_{\mathfrak{g}} = \mathfrak{g}\text{-Der}^{(\delta)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})$ for some ordinal δ, then $\mathscr{R}^{(\delta)}_{\mathfrak{g}}$ may be called a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived set of* $\mathscr{S}_{\mathfrak{g}}$ *of category* ν *and order* δ. Likewise, given $\left(\mathscr{U}^{(\delta)}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\right) \in \times_{\alpha\in I^*_2} \mathscr{P}(\Omega)$ such that $\mathscr{U}^{(\delta)}_{\mathfrak{g}} = \mathfrak{g}\text{-Cod}^{(\delta)}_{\mathfrak{g},\nu}(\mathscr{V}_{\mathfrak{g}})$ for some ordinal δ, then $\mathscr{U}^{(\delta)}_{\mathfrak{g}}$ may be called a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived set of* $\mathscr{V}_{\mathfrak{g}}$ *of category* ν *and order* δ. Hence, any $\{\xi\} \in \mathscr{P}(\Omega)$ such that $\left(\xi \in \mathscr{R}^{(\delta)}_{\mathfrak{g}} \in \mathscr{P}(\Omega)\right) \wedge \left(\xi \notin \mathscr{R}^{(\delta+1)}_{\mathfrak{g}} \in \mathscr{P}(\Omega)\right)$ may be called a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived unit set of* $\mathscr{R}_{\mathfrak{g}}$ *of category* ν *and order* δ, and any $\{\zeta\} \in \mathscr{P}(\Omega)$ such that $\left(\zeta \in \mathscr{U}^{(\delta)}_{\mathfrak{g}} \in \mathscr{P}(\Omega)\right) \wedge \left(\zeta \notin \mathscr{U}^{(\delta+1)}_{\mathfrak{g}} \in \mathscr{P}(\Omega)\right)$ may be called a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived unit set of* $\mathscr{U}_{\mathfrak{g}}$ *of category* ν *and order* δ.

Evidently, the use of $\text{der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Der}_{\nu}$, $\text{der} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ instead of $\mathfrak{g}\text{-Der}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ introduce the notions of $\mathfrak{T}_{\mathfrak{g}}$*-derived set of* $\mathscr{S}_{\mathfrak{g}}$ *of order* δ, $\mathfrak{g}$-$\mathfrak{T}$*-derived set of* $\mathscr{S}_{\mathfrak{g}}$ *of category* ν *and order* δ, and $\mathfrak{T}$*-derived set of* $\mathscr{S}_{\mathfrak{g}}$ *of order* δ, respectively; the use of $\text{doc}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\nu}$, $\text{cod} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ instead of $\mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ introduce the notions of $\mathfrak{T}_{\mathfrak{g}}$*-coderived set of* $\mathscr{S}_{\mathfrak{g}}$ *of order* δ, $\mathfrak{g}$-$\mathfrak{T}$*-coderived set of* $\mathscr{S}_{\mathfrak{g}}$ *of category* ν *and order* δ, and $\mathfrak{T}$*-coderived set of* $\mathscr{S}_{\mathfrak{g}}$ *of order* δ, respectively.

Of the notations $\mathfrak{T}_{\mathfrak{o}} = (\Omega, \mathscr{T}_{\mathfrak{o}})$ and $\mathfrak{T} = (\Omega, \mathscr{T})$, either the first will be used instead of the second, or both will be used interchangeably.

3. Main Results

In this section, the basic properties of the $\mathfrak{g}$-$\mathfrak{T}^{(\delta)}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}^{(\delta)}_{\mathfrak{g}}$-coderived operators $\mathfrak{g}\text{-Der}^{(\delta)}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}^{(\delta)}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, are studied in $\mathscr{T}_{\mathfrak{g}}$-spaces.

In a $\mathscr{T}_{\mathfrak{g}}$-space, every $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived set is contained in all the preceding $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived sets and, every $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived set contains all the preceding $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived sets. The theorem follows.

Theorem 3.1. *Let* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived and a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operators, respectively, and let* $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*. Then:*

– I. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall\delta:\ 1 \leqslant \delta < \lambda)$

– II. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall\delta:\ 1 \leqslant \delta < \lambda)$

Proof. Let $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-coderived operators, respectively, and let $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

— I. Introduce $\mathbb{B} = \{0,1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{P}(\delta) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall\delta:\ 1 \leqslant \delta < \lambda)$$

Then, to prove Item I., it only suffices to prove that,

$$\big(\forall\delta:\ 1 \leqslant \delta < \lambda\big)\Big[\big(\mathrm{P}(1) = 1\big) \wedge \big(\mathrm{P}(\delta) = 1 \longrightarrow \mathrm{P}(\delta+1) = 1\big)\Big]$$

– Case I. Let $1 = \delta$. Since $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, it follows that

$$\begin{aligned}
\mathfrak{g}\text{-Der}_{\mathfrak{g}}:\ \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longmapsto \big\{\xi \in \mathfrak{T}_{\mathfrak{g}}:\ \xi \in \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\big\} \\
&\subseteq \big\{\xi \in \mathfrak{T}_{\mathfrak{g}}:\ \xi \in \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big\} \\
&\longleftrightarrow \big\{\xi \in \mathfrak{T}_{\mathfrak{g}}:\ \xi \in \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big\} \\
&\longleftrightarrow \big\{\xi \in \mathfrak{T}_{\mathfrak{g}}:\ \xi \in \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\big\} \\
&\longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})
\end{aligned}$$

Thus, $\mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. But, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(1)}(\mathscr{S}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(2)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Thus, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(2)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(1)}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{P}(1) = 1$. The base case therefore holds.

– Case II. Let $1 < \delta < \lambda$ and assume that the inductive hypothesis $\mathrm{P}(\delta) = 1$ holds true. Then, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$ and consequently, it results that $\mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$. But, for each $\eta \in \{\delta, \delta+1\}$,

$$\mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(1)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\eta+1)}(\mathscr{S}_{\mathfrak{g}})$$

Hence, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{((\delta+1)+1)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{P}(\delta+1) = 1$. The inductive case therefore holds.

Since $\mathrm{P}(\delta) = 1$ for all δ such that $1 < \delta < \lambda$, it follows that

$$\begin{aligned}
\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\lambda+1)}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\lambda)}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}\Bigg(\bigcap_{\delta<\lambda} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\Bigg) \\
&\subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \\
&\qquad \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})
\end{aligned}$$

for all δ such that $1 < \delta < \lambda$, from which $\mathrm{P}(\lambda) = 1$ follows.

— II. Introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{Q}(\delta) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall\delta:\ 1 \leqslant \delta < \lambda)$$

Then, to prove ITEM II., it only suffices to prove that,

$$\left(\forall\delta:\ 1\leqslant\delta<\lambda\right)\left[\left(\mathrm{Q}\left(0\right)=1\right)\wedge\left(\mathrm{Q}\left(\delta\right)=1\ \longrightarrow\ \mathrm{Q}\left(\delta+1\right)=1\right)\right]$$

– CASE I. Let $1=\delta$. Since $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\subseteq\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)$, it results that

$$\begin{aligned}\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}:\ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\ &\longmapsto\ \left\{\zeta\in\mathfrak{T}_{\mathfrak{g}}:\ \zeta\in\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\big(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\cup\{\zeta\}\big)\right\}\\ &\supseteq\ \left\{\zeta\in\mathfrak{T}_{\mathfrak{g}}:\ \zeta\in\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\right\}\\ &\longleftrightarrow\ \left\{\zeta\in\mathfrak{T}_{\mathfrak{g}}:\ \zeta\in\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\right\}\\ &\longleftrightarrow\ \left\{\zeta\in\mathfrak{T}_{\mathfrak{g}}:\ \xi\in\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}\cup\{\zeta\}\big)\right\}\\ &\longleftrightarrow\ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\end{aligned}$$

Thus, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\supseteq\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)$. But, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\longleftrightarrow\ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(1)}\left(\mathscr{S}_{\mathfrak{g}}\right)$ and $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(2)}\left(\mathscr{S}_{\mathfrak{g}}\right)\ \longleftrightarrow\ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)$. Thus, the relation $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(2)}\left(\mathscr{S}_{\mathfrak{g}}\right)\supseteq\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(1)}\left(\mathscr{S}_{\mathfrak{g}}\right)$ holds true, implying $\mathrm{Q}\left(1\right)=1$. The base case therefore holds.

– CASE II. Let $1<\delta<\lambda$ and assume that the inductive hypothesis $\mathrm{Q}\left(\delta\right)=1$ holds true. Then, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+1)}\left(\mathscr{S}_{\mathfrak{g}}\right)\supseteq\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\left(\mathscr{S}_{\mathfrak{g}}\right)$ and consequently, it results that $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+1)}\left(\mathscr{S}_{\mathfrak{g}}\right)\supseteq\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\left(\mathscr{S}_{\mathfrak{g}}\right)$. But, for each $\eta\in\{\delta,\delta+1\}$,

$$\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\eta)}\left(\mathscr{S}_{\mathfrak{g}}\right)\longleftrightarrow\ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(1)}\circ\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\eta)}\left(\mathscr{S}_{\mathfrak{g}}\right)\ \longleftrightarrow\ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\eta+1)}\left(\mathscr{S}_{\mathfrak{g}}\right)$$

Hence, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{((\delta+1)+1)}\left(\mathscr{S}_{\mathfrak{g}}\right)\supseteq\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+1)}\left(\mathscr{S}_{\mathfrak{g}}\right)$, implying $\mathrm{Q}\left(\delta+1\right)=1$. The inductive case therefore holds.

Since $\mathrm{Q}\left(\delta\right)=1$ for all δ such that $1<\delta<\lambda$, it follows that

$$\begin{aligned}\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\lambda+1)}\left(\mathscr{S}_{\mathfrak{g}}\right)\ &\longleftrightarrow\ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\lambda)}\left(\mathscr{S}_{\mathfrak{g}}\right)\\ &\longleftrightarrow\ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\left(\bigcap_{\delta<\lambda}\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\left(\mathscr{S}_{\mathfrak{g}}\right)\right)\\ &\supseteq\ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\left(\mathscr{S}_{\mathfrak{g}}\right)\longleftrightarrow\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+1)}\left(\mathscr{S}_{\mathfrak{g}}\right)\\ &\qquad\supseteq\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\left(\mathscr{S}_{\mathfrak{g}}\right)\end{aligned}$$

for all δ such that $1<\delta<\lambda$, from which $\mathrm{Q}\left(\lambda\right)=1$ follows. The proof of the theorem is complete. □

The corollary stated below is an immediate consequence of the above theorem.

Corollary 3.2. *Let* $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}:\mathscr{P}\left(\Omega\right)\longrightarrow\mathscr{P}\left(\Omega\right)$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived and a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operators, respectively, and let* $\mathscr{S}_{\mathfrak{g}}\in\mathscr{P}\left(\Omega\right)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=\left(\Omega,\mathscr{T}_{\mathfrak{g}}\right)$*. Then:*

– I. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}\left(\mathscr{S}_{\mathfrak{g}}\right)\subseteq\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\quad\left(\forall\delta:\ 1\leqslant\delta<\lambda\right)$

– II. $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\left(\mathscr{S}_{\mathfrak{g}}\right)\supseteq\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\quad\left(\forall\delta:\ 1\leqslant\delta<\lambda\right)$

In a $\mathscr{T}_{\mathfrak{g}}$-space, just as $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}:\mathscr{P}\left(\Omega\right)\longrightarrow\mathscr{P}\left(\Omega\right)$ is *coarser* (or, *smaller*, *weaker*) than $\mathrm{der}_{\mathfrak{g}}:\mathscr{P}\left(\Omega\right)\longrightarrow\mathscr{P}\left(\Omega\right)$ (or, $\mathrm{der}_{\mathfrak{g}}:\mathscr{P}\left(\Omega\right)\longrightarrow\mathscr{P}\left(\Omega\right)$ is *finer* (or, *larger*, *stronger*) than $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}:\mathscr{P}\left(\Omega\right)\longrightarrow\mathscr{P}\left(\Omega\right)$) [1], so is $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}:\mathscr{P}\left(\Omega\right)\longrightarrow\mathscr{P}\left(\Omega\right)$ *coarser* (or, *smaller*, *weaker*) than $\mathrm{der}_{\mathfrak{g}}^{(\delta)}:\mathscr{P}\left(\Omega\right)\longrightarrow\mathscr{P}\left(\Omega\right)$ (or, $\mathrm{der}_{\mathfrak{g}}^{(\delta)}:\mathscr{P}\left(\Omega\right)\longrightarrow\mathscr{P}\left(\Omega\right)$ *finer* (or, *larger*, *stronger*) than $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}:\mathscr{P}\left(\Omega\right)\longrightarrow\mathscr{P}\left(\Omega\right)$); likewise, just as $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\text{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\text{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ [1], so is $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *finer* (or, *larger*, *stronger*) than $\text{cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ (or, $\text{cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *coarser* (or, *smaller*, *weaker*) than $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$). Accordingly, the proposition follows.

Proposition 1. If $\mathfrak{g}\text{-Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a given pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and $\mathbf{dc}_g \in \text{DC}[T_g]$ be a given pair of $\mathfrak{T}_{\mathfrak{g}}$-operators $\text{der}_{\mathfrak{g}}$, $\text{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, and let $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then:

– I. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \text{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \delta :\ 1 \preccurlyeq \delta \prec \lambda)$

– II. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \text{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \delta :\ 1 \preccurlyeq \delta \prec \lambda)$

Proof. Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. Suppose $\mathfrak{g}\text{-Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathbf{dc}_g \in \text{DC}[T_g]$ be given and $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be arbitrary. Then:

— I. Introduce $\mathbb{B} = \{0, 1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \text{P}(\delta) \quad \overset{\text{def}}{\longleftrightarrow} \quad \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \text{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \delta :\ 1 \preccurlyeq \delta \prec \lambda)$$

Then, to prove Item i., it only suffices to prove that,

$$\big(\forall \delta :\ 1 \preccurlyeq \delta \prec \lambda\big)\big[\big(\text{P}(1) = 1\big) \wedge \big(\text{P}(\delta) = 1 \ \longrightarrow\ \text{P}(\delta + 1) = 1\big)\big]$$

– Case i. Let $1 = \delta$. Then, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \text{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ holds true, implying $\text{P}(1) = 1$. The base case therefore holds.

– Case ii. Let $1 \prec \delta \prec \lambda$ and assume that the inductive hypothesis $\text{P}(\delta) = 1$ holds true. Then, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \text{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$ and consequently, it follows that

$$\begin{aligned} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \quad &\longleftrightarrow \quad \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \\ &\subseteq \quad \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \text{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \\ &\subseteq \quad \text{der}_{\mathfrak{g}} \circ \text{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \ \longleftrightarrow\ \text{der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \end{aligned}$$

Hence, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \text{der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})$, implying $\text{P}(\delta + 1) = 1$. The inductive case therefore holds.

Since $\text{P}(\delta) = 1$ for all δ such that $1 \prec \delta \prec \lambda$, it follows that

$$\begin{aligned} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\lambda+1)}(\mathscr{S}_{\mathfrak{g}}) \quad &\longleftrightarrow \quad \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\lambda)}(\mathscr{S}_{\mathfrak{g}}) \\ &\subseteq \quad \text{der}_{\mathfrak{g}} \circ \text{der}_{\mathfrak{g}}^{(\lambda)}(\mathscr{S}_{\mathfrak{g}}) \\ &\longleftrightarrow \quad \text{der}_{\mathfrak{g}}\Bigg(\bigcap_{\delta \prec \lambda} \text{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\Bigg) \\ &\subseteq \quad \text{der}_{\mathfrak{g}} \circ \text{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \text{der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \\ &\qquad\qquad \subseteq \text{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \end{aligned}$$

for all δ such that $1 \prec \delta \prec \lambda$, from which $\text{P}(\lambda) = 1$ follows.

— II. Introduce $\mathbb{B} = \{0,1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{Q}(\delta) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \delta:\ 1 \leqslant \delta < \lambda)$$

Then, to prove Item ii., it only suffices to prove that,

$$\big(\forall \delta:\ 1 \leqslant \delta < \lambda\big)\Big[\big(\mathrm{Q}(1) = 1\big) \wedge \big(\mathrm{Q}(\delta) = 1 \longrightarrow \mathrm{P}(\delta+1) = 1\big)\Big]$$

– Case i. Let $1 = \delta$. Then, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ holds true, implying $\mathrm{Q}(1) = 1$. The base case therefore holds.

– Case ii. Let $1 < \delta < \lambda$ and assume that the inductive hypothesis $\mathrm{Q}(\delta) = 1$ holds true. Then, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$ and consequently, it follows that

$$\begin{aligned}
\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \\
&\supseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} \circ \mathrm{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \\
&\supseteq \mathrm{cod}_{\mathfrak{g}} \circ \mathrm{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathrm{cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})
\end{aligned}$$

Hence, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{Q}(\delta+1) = 1$. The inductive case therefore holds.

Since $\mathrm{Q}(\delta) = 1$ for all δ such that $1 < \delta < \lambda$, it follows that

$$\begin{aligned}
\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\lambda+1)}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\lambda)}(\mathscr{S}_{\mathfrak{g}}) \\
&\supseteq \mathrm{cod}_{\mathfrak{g}} \circ \mathrm{cod}_{\mathfrak{g}}^{(\lambda)}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathrm{cod}_{\mathfrak{g}}\Bigg(\bigcap_{\delta<\lambda} \mathrm{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\Bigg) \\
&\supseteq \mathrm{cod}_{\mathfrak{g}} \circ \mathrm{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathrm{cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \\
&\qquad \supseteq \mathrm{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})
\end{aligned}$$

for all δ such that $1 < \delta < \lambda$, from which $\mathrm{Q}(\lambda) = 1$ follows. The proof of the proposition is complete. □

For any δ such that $1 \leqslant \delta < \lambda$, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathrm{der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\mathrm{der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$; $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathrm{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\mathrm{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$. Accordingly, the following corollary is an immediate consequence of the above proposition.

Corollary 3.3. *If* $\mathfrak{g}\text{-}\mathrm{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *and* $\mathbf{dc}_g \in \mathrm{DC}[T_g]$ *be a given pair of* $\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathrm{der}_{\mathfrak{g}}$, $\mathrm{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, *and let* $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

– i. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \delta:\ 1 \leqslant \delta < \lambda)$

– ii. $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \delta:\ 1 \leqslant \delta < \lambda)$

For any δ such that $1 \preccurlyeq \delta \prec \lambda$, the notions of δ^{th}-order $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived set operators can be interrelated among themselves and presented δ^{th}*-order* $\mathfrak{T}_{\mathfrak{g}}$*,* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived set operators finness-coarseness diagrams*; similarly, the notions of δ^{th}-order $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived set operators can be interrelated among themselves and presented δ^{th}*-order* $\mathfrak{T}_{\mathfrak{g}}$*,* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived set operators finness-coarseness diagrams.* A further corollary follows.

Corollary 3.4. *If* $\mathfrak{g}\text{-Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *and* $\mathbf{dc}_{g} \in \text{DC}[T_{g}]$ *be a given pair of* $\mathfrak{T}_{\mathfrak{g}}$*-operators* $\text{der}_{\mathfrak{g}}$, $\text{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

– I. *For any* $\mathscr{R}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$*,*

$$\begin{array}{ccc} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) & \longrightarrow & \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}) \subseteq \text{der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \\ & \nwarrow & \uparrow \\ & & \text{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}) \subseteq \text{der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda) \end{array} \tag{3.1}$$

– II. *For any* $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$*,*

$$\begin{array}{ccc} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) & \longleftarrow & \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \text{cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ & \searrow & \downarrow \\ & & \text{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \text{cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda) \end{array} \tag{3.2}$$

For any δ such that $1 \preccurlyeq \delta \prec \lambda$, the δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived set operator is $\varnothing$-*grounded* (alternatively, $\varnothing$*-preserving*); the δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived set operator is Ω-*grounded* (alternatively, Ω*-preserving*). These are embodied in the following theorem.

Theorem 3.5. *Let* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived and a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operators, respectively, in a strong* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*. Then:*

– I. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\varnothing) = \varnothing \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$

– II. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\Omega) = \Omega \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$

Proof. Let $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, respectively, in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

— I. Introduce $\mathbb{B} = \{0, 1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{P}(\delta) \quad \overset{\text{def}}{\longleftrightarrow} \quad \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\varnothing) = \varnothing \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$$

Then, to prove Item I., it only suffices to prove that,

$$\big(\forall \delta : 1 \preccurlyeq \delta \prec \lambda\big)\Big[\big(\mathrm{P}(1) = 1\big) \wedge \big(\mathrm{P}(\delta) = 1 \longrightarrow \mathrm{P}(\delta + 1) = 1\big)\Big]$$

– Case I. Let $1 = \delta$. Then, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(1)}(\varnothing) \longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\varnothing) = \varnothing$. Thus, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(1)}(\varnothing) = \varnothing$, implying $\mathrm{P}(1) = 1$. The base case therefore holds.

– CASE II. Let $1 \prec \delta \prec \lambda$ and assume that the inductive hypothesis $\mathrm{P}(\delta) = 1$ holds true. Then, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\varnothing) = \varnothing$ and consequently, it follows that

$$\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+1)}(\varnothing) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\varnothing) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\varnothing) = \varnothing$$

Hence, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+1)}(\varnothing) = \varnothing$, implying $\mathrm{P}(\delta+1) = 1$. The inductive case therefore holds.

Since $\mathrm{P}(\delta) = 1$ for all δ such that $1 \prec \delta \prec \lambda$, it follows that

$$\begin{aligned}\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\lambda+1)}(\varnothing) &\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\lambda)}(\varnothing)\\ &\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\Bigg(\bigcap_{\delta\prec\lambda} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\varnothing)\Bigg) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\varnothing) = \varnothing\end{aligned}$$

for all δ such that $1 \prec \delta \prec \lambda$, from which $\mathrm{P}(\lambda) = 1$ follows.

— II. Introduce $\mathbb{B} = \{0, 1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{Q}(\delta) \quad \overset{\text{def}}{\longleftrightarrow} \quad \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\Omega) = \Omega \qquad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$$

Then, to prove ITEM II., it only suffices to prove that,

$$\big(\forall \delta : 1 \preccurlyeq \delta \prec \lambda\big)\Big[\big(\mathrm{Q}(1) = 1\big) \wedge \big(\mathrm{Q}(\delta) = 1 \longrightarrow \mathrm{Q}(\delta+1) = 1\big)\Big]$$

– CASE I. Let $1 = \delta$. Then, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(1)}(\Omega) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega) = \Omega$. Thus, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(1)}(\Omega) = \Omega$, implying $\mathrm{Q}(1) = 1$. The base case therefore holds.

– CASE II. Let $1 \prec \delta \prec \lambda$ and assume that the inductive hypothesis $\mathrm{Q}(\delta) = 1$ holds true. Then, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\Omega) = \Omega$ and consequently, it follows that

$$\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+1)}(\Omega) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\Omega) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega) = \Omega$$

Hence, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+1)}(\Omega) = \Omega$, implying $\mathrm{Q}(\delta+1) = 1$. The inductive case therefore holds.

Since $\mathrm{Q}(\delta) = 1$ for all δ such that $1 \prec \delta \prec \lambda$, it follows that

$$\begin{aligned}\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\lambda+1)}(\Omega) &\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\lambda)}(\Omega)\\ &\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\Bigg(\bigcap_{\delta\prec\lambda} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\Omega)\Bigg) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega) = \Omega\end{aligned}$$

for all δ such that $1 \prec \delta \prec \lambda$, from which $\mathrm{Q}(\lambda) = 1$ follows. The proof of the theorem is complete. ☐

For any δ such that $1 \preccurlyeq \delta \prec \lambda$, the δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived set operator is $\cup$-*additive* (alternatively, $\cup$-*distributive*); the δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived set operator is $\cap$-*additive* (alternatively, $\cap$-*distributive*). The theorem follows.

Theorem 3.6. *Let* $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived and a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operators, respectively, and let* $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*. Then:*

– I. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) = \bigcup_{\mathscr{W}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$

– II. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}}) = \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \quad (\forall\delta:\ 1\preccurlyeq\delta\prec\lambda)$

Proof. Let $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, respectively, and let $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*}\mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega,\mathscr{T}_{\mathfrak{g}})$. Then:

— I. Introduce $\mathbb{B} = \{0,1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{P}(\delta) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}}) = \bigcup_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \quad (\forall\delta:1\preccurlyeq\delta\prec\lambda)$$

Then, to prove Item I., it only suffices to prove that,

$$\big(\forall\delta:\ 1\preccurlyeq\delta\prec\lambda\big)\Big[\big(\mathrm{P}(1)=1\big)\wedge\big(\mathrm{P}(\delta)=1 \longrightarrow \mathrm{P}(\delta+1)=1\big)\Big]$$

– Case I. Let $1=\delta$. Then, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}}) = \bigcup_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})$ holds true, implying $\mathrm{P}(1)=1$. The base case therefore holds.

– Case II. Let $1\prec\delta\prec\lambda$ and assume that the inductive hypothesis $\mathrm{P}(\delta)=1$ holds true. Then, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}}) = \bigcup_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}})$ and consequently, it follows that

$$\begin{aligned}
\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}}) \\
&= \mathfrak{g}\text{-Der}_{\mathfrak{g}}\Bigg(\bigcup_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}})\Bigg) \\
&= \bigcup_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \\
&\longleftrightarrow \bigcup_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{W}_{\mathfrak{g}})
\end{aligned}$$

Hence, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}}) = \bigcup_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{W}_{\mathfrak{g}})$, implying $\mathrm{P}(\delta+1)=1$. The inductive case therefore holds.

Since $\mathrm{P}(\delta)=1$ for all δ such that $1\prec\delta\prec\lambda$, it follows that $\mathrm{P}(\lambda)=1$ states that

$$\bigcap_{\delta\prec\lambda}\Bigg(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}})\Bigg) \longleftrightarrow \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\Bigg(\bigcap_{\delta\prec\lambda}\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}})\Bigg)$$

and it is evident that any element in $\bigcap_{\delta\prec\lambda}\Big(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}})\Big)$ is contained in $\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\Big(\bigcap_{\delta\prec\lambda}\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}})\Big)$. Thus, in order to prove that any element in $\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\Big(\bigcap_{\delta\prec\lambda}\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}})\Big)$ is also in $\bigcap_{\delta\prec\lambda}\Big(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}})\Big)$, let it be supposed that $\xi \in \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\Big(\bigcap_{\delta\prec\lambda}\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}})\Big)$ such that, for some $(\alpha,\beta)\prec(\lambda,\lambda)$

where $\alpha \leqslant \beta$, say, the statement $\xi \in \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \left(\bigcap_{\delta=\alpha,\beta} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \right)$ holds true. Then, $\xi \in \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\alpha)}(\mathscr{W}_{\mathfrak{g}})$ and therefore $\xi \in \bigcap_{\delta<\lambda} \left(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \right)$, implying $\mathrm{P}(\lambda) = 1$ holds.

— II. Introduce $\mathbb{B} = \{0,1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{Q}(\delta) \quad \overset{\text{def}}{\longleftrightarrow} \quad \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) = \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \quad (\forall \delta : 1 \leqslant \delta < \lambda)$$

Then, to prove Item ii., it only suffices to prove that,

$$(\forall \delta :\ 1 \leqslant \delta < \lambda)\big[(\mathrm{Q}(1) = 1) \wedge (\mathrm{Q}(\delta) = 1 \ \longrightarrow\ \mathrm{Q}(\delta+1) = 1)\big]$$

– Case i. Let $1 = \delta$. Then, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) = \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})$ holds true, implying $\mathrm{Q}(1) = 1$. The base case therefore holds.

– Case ii. Let $1 < \delta < \lambda$ and assume that the inductive hypothesis $\mathrm{Q}(\delta) = 1$ holds true. Then, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) = \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}})$ and consequently, it follows that

$$\begin{aligned}
\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) \quad &\longleftrightarrow \quad \mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) \\
&= \quad \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \right) \\
&= \quad \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \\
&\longleftrightarrow \quad \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{W}_{\mathfrak{g}})
\end{aligned}$$

Hence, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) = \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{W}_{\mathfrak{g}})$, implying $\mathrm{Q}(\delta+1) = 1$. The inductive case therefore holds.

Since $\mathrm{Q}(\delta) = 1$ for all δ such that $1 < \delta < \lambda$, it follows that $\mathrm{Q}(\lambda) = 1$ states that

$$\bigcap_{\delta<\lambda} \left(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \right) \quad \longleftrightarrow \quad \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \left(\bigcap_{\delta<\lambda} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \right)$$

and it is evident that any element in $\bigcap_{\delta<\lambda} \left(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \right)$ is contained in $\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \left(\bigcap_{\delta<\lambda} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \right)$. Thus, in order to prove that any element in $\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \left(\bigcap_{\delta<\lambda} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \right)$ is also in $\bigcap_{\delta<\lambda} \left(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \right)$, let it be supposed that $\zeta \in \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \left(\bigcap_{\delta<\lambda} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \right)$ such that, for some $(\alpha,\beta) < (\lambda,\lambda)$

where $\alpha \preccurlyeq \beta$, say, the statement $\zeta \in \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \left(\bigcap_{\delta=\alpha,\beta} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \right)$ holds true. Then, $\zeta \in \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\alpha)}(\mathscr{W}_{\mathfrak{g}})$ and therefore, it follows that the statement $\zeta \in \bigcap_{\delta \prec \lambda} \left(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \right)$ holds, implying $\mathrm{Q}(\lambda) = 1$ holds. The proof of the theorem is complete. □

The corollary stated below is an immediate consequence of the above theorem.

Corollary 3.7. *If* $\mathfrak{g}\text{-}\mathrm{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*operators* $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, $\mathbf{dc}_g \in \mathrm{DC}[T_g]$ *be a given pair of* $\mathfrak{T}_{\mathfrak{g}}$-*operators* $\mathrm{der}_{\mathfrak{g}}$, $\mathrm{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, *and* $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$-*space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, *then:*

- I. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) \subseteq \bigcup_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathrm{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$
- II. $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) \supseteq \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathrm{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$

For any (δ, η) such that $1 \preccurlyeq \delta \prec \eta \prec \lambda$, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\eta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\eta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$; $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\eta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\eta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$. Accordingly, the proposition follows.

Proposition 2. Let $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, respectively, and let $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

- I. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad \big(\forall (\delta, \eta) : 1 \preccurlyeq \delta \prec \eta \prec \lambda\big)$
- II. $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad \big(\forall (\delta, \eta) : 1 \preccurlyeq \delta \prec \eta \prec \lambda\big)$

Proof. Let $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, respectively, and let $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

— I. Set $\eta = \delta + \varepsilon$, where $1 \preccurlyeq \varepsilon$, introduce $\mathbb{B} = \{0, 1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{P}(\varepsilon) \quad \stackrel{\text{def}}{\longleftrightarrow} \quad \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+\varepsilon)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \varepsilon : 1 \preccurlyeq \varepsilon)$$

Then, to prove Item I., it only suffices to prove that,

$$\big(\forall \varepsilon : 1 \preccurlyeq \varepsilon\big)\Big[\big(\mathrm{P}(1) = 1\big) \wedge \big(\mathrm{P}(\varepsilon) = 1 \longrightarrow \mathrm{P}(\varepsilon + 1) = 1\big)\Big]$$

– Case I. Let $1 = \varepsilon$. Then, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{P}(1) = 1$. The base case therefore holds.

– CASE II. Let $1 \prec \varepsilon$ and assume that the inductive hypothesis $\mathrm{P}(\varepsilon) = 1$ holds true. Then, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+\varepsilon)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$ and consequently, it results that $\mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+\varepsilon)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$. But,

$$\begin{aligned}\mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+\varepsilon)}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+(\varepsilon+1))}(\mathscr{S}_{\mathfrak{g}})\\ &\subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\\ &\longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\end{aligned}$$

Hence, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+(\varepsilon+1))}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{P}(\varepsilon+1) = 1$. The inductive case therefore holds.

Since $\mathrm{P}(\delta) = 1$ for all δ such that $1 \prec \delta \prec \eta \prec \lambda$, it follows that

$$\begin{aligned}\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\lambda+1)}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\lambda)}(\mathscr{S}_{\mathfrak{g}})\\ &\longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}\left(\bigcap_{\eta \prec \lambda} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}})\right)\\ &\subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}})\\ &\longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\eta+1)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})\\ &\qquad\qquad \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\end{aligned}$$

for all δ such that $1 \prec \delta \prec \eta \prec \lambda$, from which $\mathrm{P}(\lambda) = 1$ follows.

— II. Set $\eta = \delta + \varepsilon$, where $1 \preccurlyeq \varepsilon$, introduce $\mathbb{B} = \{0, 1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{Q}(\varepsilon) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+\varepsilon)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \varepsilon : 1 \preccurlyeq \varepsilon)$$

Then, to prove ITEM II., it only suffices to prove that,

$$(\forall \varepsilon : 1 \preccurlyeq \varepsilon)\Big[\big(\mathrm{Q}(1) = 1\big) \wedge \big(\mathrm{Q}(\varepsilon) = 1 \longrightarrow \mathrm{Q}(\varepsilon+1) = 1\big)\Big]$$

– CASE I. Let $1 = \varepsilon$. Then, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{Q}(1) = 1$. The base case therefore holds.

– CASE II. Let $1 \prec \varepsilon$ and assume that the inductive hypothesis $\mathrm{Q}(\varepsilon) = 1$ holds true. Then, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+\varepsilon)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$ and consequently, it results that $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+\varepsilon)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$. But,

$$\begin{aligned}\mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+\varepsilon)}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+(\varepsilon+1))}(\mathscr{S}_{\mathfrak{g}})\\ &\supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\\ &\longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\end{aligned}$$

Hence, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+(\varepsilon+1))}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{Q}(\varepsilon+1) = 1$. The inductive case therefore holds.

Since $\mathrm{Q}(\delta) = 1$ for all δ such that $1 \prec \delta \prec \eta \prec \lambda$, it follows that

$$\begin{aligned} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\lambda+1)}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\lambda)}(\mathscr{S}_{\mathfrak{g}}) \\ &\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\Bigg(\bigcap_{\eta\prec\lambda} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}})\Bigg) \\ &\supseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \\ &\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\eta+1)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \\ &\qquad \supseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \end{aligned}$$

for all δ such that $1 \prec \delta \prec \eta \prec \lambda$, from which $\mathrm{Q}(\lambda) = 1$ follows. The proof of the proposition is complete. ☐

The corollary stated below is an immediate consequence of the above proposition.

Corollary 3.8. *If* $\mathfrak{g}\text{-}\mathrm{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *and* $\mathbf{dc}_g \in \mathrm{DC}[T_g]$ *be a given pair of* $\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathrm{der}_{\mathfrak{g}}$, $\mathrm{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, *and let* $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

- I. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad \big(\forall\,(\delta,\eta) :\ 1 \preccurlyeq \delta \prec \eta \prec \lambda\big)$
- II. $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad \big(\forall\,(\delta,\eta) :\ 1 \preccurlyeq \delta \prec \eta \prec \lambda\big)$

For any (δ,η) such that $1 \preccurlyeq \delta \prec \eta \prec \lambda$, the $(\delta+\eta)^{\mathrm{th}}$-order $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-derived set operator is equivalent to the composition of the δ^{th}-order and the η^{th}-order of the $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-derived set operator; likewise, the $(\delta+\eta)^{\mathrm{th}}$-order $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-coderived set operator is equivalent to the composition of the δ^{th}-order and the η^{th}-order of the $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-coderived set operator. The proposition follows.

Proposition 3. Let $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-coderived operators, respectively, and let $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

- I. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+\eta)}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \quad \big(\forall\,(\delta,\eta) :\ (1,1) \preccurlyeq (\delta,\eta)\big)$
- II. $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+\eta)}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \quad \big(\forall\,(\delta,\eta) :\ (1,1) \preccurlyeq (\delta,\eta)\big)$

where $(\delta,\eta) \prec (\lambda,\lambda)$.

Proof. Let $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-coderived operators, respectively, and let $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

— I. Introduce $\mathbb{B} = \{0,1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{P}(\delta,\eta) \overset{\mathrm{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+\eta)}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \\ \big(\forall\,(\delta,\eta) :\ (1,1) \preccurlyeq (\delta,\eta) \prec (\lambda,\lambda)\big)$$

Then, to prove ITEM I., it only suffices to prove that,

$$\big(\forall\,(\delta,\eta):\ (1,1)\preccurlyeq(\delta,\eta)\prec(\lambda,\lambda)\big)$$
$$\Big[\big(\mathrm{P}\,(1,1)=1\big)\wedge\big(\mathrm{P}\,(\delta,\eta)=1\ \longrightarrow\ \mathrm{P}\,(\delta+1,\eta+1)=1\big)\Big]$$

– CASE I. Let $(1,1)=(\delta,\eta)$. Then,

$$\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(2)}\,(\mathscr{S}_{\mathfrak{g}})\longleftrightarrow\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(1+1)}\,(\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(1)}\circ\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(1)}\,(\mathscr{S}_{\mathfrak{g}})\longleftrightarrow\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(2)}\,(\mathscr{S}_{\mathfrak{g}})$$

implying $\mathrm{P}\,(1,1)=1$. The base case therefore holds.

– CASE II. Let $(1,1)\prec(\delta,\eta)\prec(\lambda,\lambda)$ and assume that the inductive hypothesis $\mathrm{P}\,(\delta,\eta)=1$ holds true. Then, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+\eta)}\,(\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}\circ\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\eta)}\,(\mathscr{S}_{\mathfrak{g}})$ and consequently, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(2)}\circ\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+\eta)}\,(\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(2)}\circ\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}\circ\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\eta)}\,(\mathscr{S}_{\mathfrak{g}})$. But, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(2)}\circ\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+\eta)}\,(\mathscr{S}_{\mathfrak{g}})\longleftrightarrow\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{((\delta+1)+(\eta+1))}\,(\mathscr{S}_{\mathfrak{g}})$ and,

$$\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(2)}\circ\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}\circ\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\eta)}\,(\mathscr{S}_{\mathfrak{g}})$$
$$\updownarrow$$
$$\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(2)}\circ\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta-1)}\circ\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(1)}\circ\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\eta)}\,(\mathscr{S}_{\mathfrak{g}})$$
$$\updownarrow$$
$$\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+1)}\circ\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\eta+1)}\,(\mathscr{S}_{\mathfrak{g}})$$

Hence, it follows that $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{((\delta+1)+(\eta+1))}\,(\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+1)}\circ\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\eta+1)}\,(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{P}\,(\delta+1,\eta+1)=1$. The inductive case therefore holds.

Suppose $\mathrm{P}\,(\delta,\eta)=1$ holds for all (δ,η) such that $(1,1)\prec(\delta,\eta)\prec(\lambda,\lambda)$. Then,

$$\bigcap_{\delta+\eta\prec\lambda+\lambda}\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+\eta)}\,(\mathscr{S}_{\mathfrak{g}})\quad=\quad\bigcap_{\delta+\eta\prec\lambda+\lambda}\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}\circ\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\eta)}\,(\mathscr{S}_{\mathfrak{g}})$$
$$\updownarrow$$
$$\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\lambda+\lambda)}\,(\mathscr{S}_{\mathfrak{g}})\quad=\quad\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\lambda)}\circ\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\lambda)}\,(\mathscr{S}_{\mathfrak{g}})$$

from which $\mathrm{P}\,(\lambda,\lambda)=1$ follows.

— II. Introduce $\mathbb{B}=\{0,1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B}\ni\mathrm{Q}\,(\delta,\eta)\stackrel{\mathrm{def}}{\longleftrightarrow}\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+\eta)}\,(\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\circ\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\eta)}\,(\mathscr{S}_{\mathfrak{g}})$$
$$\big(\forall\,(\delta,\eta):\ (1,1)\preccurlyeq(\delta,\eta)\prec(\lambda,\lambda)\big)$$

Then, to prove ITEM II., it only suffices to prove that,

$$\big(\forall\,(\delta,\eta):\ (1,1)\preccurlyeq(\delta,\eta)\prec(\lambda,\lambda)\big)$$
$$\Big[\big(\mathrm{Q}\,(1,1)=1\big)\wedge\big(\mathrm{Q}\,(\delta,\eta)=1\ \longrightarrow\ \mathrm{Q}\,(\delta+1,\eta+1)=1\big)\Big]$$

– CASE I. Let $(1,1)=(\delta,\eta)$. Then,

$$\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(2)}\,(\mathscr{S}_{\mathfrak{g}})\longleftrightarrow\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(1+1)}\,(\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(1)}\circ\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(1)}\,(\mathscr{S}_{\mathfrak{g}})\longleftrightarrow\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(2)}\,(\mathscr{S}_{\mathfrak{g}})$$

implying $\mathrm{Q}\,(1,1)=1$. The base case therefore holds.

– Case ii. Let $(1,1) \prec (\delta,\eta) \prec (\lambda,\lambda)$ and assume that the inductive hypothesis $\mathrm{Q}(\delta,\eta) = 1$ holds true. Then, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+\eta)}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}})$ and consequently, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(2)} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+\eta)}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(2)} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}})$. But, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(2)} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+\eta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{((\delta+1)+(\eta+1))}(\mathscr{S}_{\mathfrak{g}})$ and,

$$\begin{gathered} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(2)} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \\ \updownarrow \\ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(2)} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta-1)} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(1)} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \\ \updownarrow \\ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+1)} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\eta+1)}(\mathscr{S}_{\mathfrak{g}}) \end{gathered}$$

Hence, it follows that $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{((\delta+1)+(\eta+1))}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+1)} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\eta+1)}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{Q}(\delta+1,\eta+1) = 1$. The inductive case therefore holds.

Suppose $\mathrm{Q}(\delta,\eta) = 1$ holds for all (δ,η) such that $(1,1) \prec (\delta,\eta) \prec (\lambda,\lambda)$. Then,

$$\begin{array}{ccc} \bigcap_{\delta+\eta\prec\lambda+\lambda} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+\eta)}(\mathscr{S}_{\mathfrak{g}}) & = & \bigcap_{\delta+\eta\prec\lambda+\lambda} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \\ & \updownarrow & \\ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\lambda+\lambda)}(\mathscr{S}_{\mathfrak{g}}) & = & \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\lambda)} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\lambda)}(\mathscr{S}_{\mathfrak{g}}) \end{array}$$

from which $\mathrm{Q}(\lambda,\lambda) = 1$ follows. The proof of the proposition is complete. □

The corollary stated below is an immediate consequence of the above proposition.

Corollary 3.9. *If* $\mathfrak{g}\text{-}\mathrm{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *and* $\mathbf{dc}_g \in \mathrm{DC}[T_g]$ *be a given pair of* $\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathrm{der}_{\mathfrak{g}}$, $\mathrm{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, *and let* $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, *then:*

– i. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+\eta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{der}_{\mathfrak{g}}^{(\delta)} \circ \mathrm{der}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \quad \big(\forall\,(\delta,\eta) :\ (1,1) \preccurlyeq (\delta,\eta)\big)$

– ii. $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+\eta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{cod}_{\mathfrak{g}}^{(\delta)} \circ \mathrm{cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \quad \big(\forall\,(\delta,\eta) :\ (1,1) \preccurlyeq (\delta,\eta)\big)$

where $(\delta,\eta) \prec (\lambda,\lambda)$.

For any (δ,η) such that $(1,1) \preccurlyeq (\delta,\eta) \prec (\lambda,\lambda)$, the $\delta\eta^{\text{th}}$-order $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-derived set operator is equivalent to the η^{th}-order of the δ^{th}-order of the $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-derived set operator; likewise, the $\delta\eta^{\text{th}}$-order $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-coderived set operator is equivalent to the η^{th}-order of the δ^{th}-order of the $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-coderived set operator. Accordingly, the following proposition presents itself.

Proposition 4. Let $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-coderived operators, respectively, and let $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

– i. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}}) = \big(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}\big)^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \ \big(\forall\,(\delta,\eta) :\ (1,1) \preccurlyeq (\delta,\eta) \prec (\lambda,\lambda)\big)$

– ii. $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}}) = \big(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\big)^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \ \big(\forall\,(\delta,\eta) :\ (1,1) \preccurlyeq (\delta,\eta) \prec (\lambda,\lambda)\big)$

Proof. Let $\mathfrak{g}\text{-Der}_{\mathfrak{g}},\ \mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, respectively, and let $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

— I. Introduce $\mathbb{B} = \{0,1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{P}(\eta) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}}) = \big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big)^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \eta :\ 1 \preccurlyeq \eta \prec \lambda)$$

Then, to prove Item i., it only suffices to prove that,

$$\big(\forall \eta :\ 1 \preccurlyeq \eta \prec \lambda\big)\big[\big(\mathrm{P}(1) = 1\big) \wedge \big(\mathrm{P}(\eta) = 1 \longrightarrow \mathrm{P}(\eta+1) = 1\big)\big]$$

– Case i. Let $1 = \eta$. Then,

$$\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta\times 1)}(\mathscr{S}_{\mathfrak{g}}) = \big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big)^{(1)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$$

implying $\mathrm{P}(1) = 1$. The base case therefore holds.

– Case ii. Let $1 \prec \eta \prec \lambda$ and assume that the inductive hypothesis $\mathrm{P}(\eta) = 1$ holds true. Then, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}}) = \big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big)^{(\eta)}(\mathscr{S}_{\mathfrak{g}})$ and consequently, it results that $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} \circ \big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big)^{(\eta)}(\mathscr{S}_{\mathfrak{g}})$. But, the relation $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\eta)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta(\eta+1))}(\mathscr{S}_{\mathfrak{g}})$ holds true and on the other hand, the relation $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} \circ \big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big)^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big)^{(\eta+1)}(\mathscr{S}_{\mathfrak{g}})$ also holds true. Hence, it follows that $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta(\eta+1))}(\mathscr{S}_{\mathfrak{g}}) = \big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big)^{(\eta+1)}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{P}(\eta+1) = 1$. The inductive case therefore holds.

Suppose $\mathrm{P}(\delta,\eta) = 1$ holds for all (δ,η) such that $(1,1) \prec (\delta,\eta) \prec (\lambda,\lambda)$. Then,

$$\begin{array}{ccc}
\bigcap_{\delta\prec\lambda}\Big(\bigcap_{\eta\prec\lambda} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}})\Big) & = & \bigcap_{\delta\prec\lambda}\Big(\bigcap_{\eta\prec\lambda} \big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big)^{(\eta)}(\mathscr{S}_{\mathfrak{g}})\Big) \\
 & \updownarrow & \\
\bigcap_{\delta\prec\lambda} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta\lambda)}(\mathscr{S}_{\mathfrak{g}}) & = & \bigcap_{\delta\prec\lambda} \big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big)^{(\lambda)}(\mathscr{S}_{\mathfrak{g}}) \\
 & \updownarrow & \\
\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\lambda\times\lambda)}(\mathscr{S}_{\mathfrak{g}}) & = & \big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\lambda)}\big)^{(\lambda)}(\mathscr{S}_{\mathfrak{g}})
\end{array}$$

from which $\mathrm{P}(\lambda,\lambda) = 1$ follows.

— II. Introduce $\mathbb{B} = \{0,1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{Q}(\eta) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}}) = \big(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\big)^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \eta :\ 1 \preccurlyeq \eta \prec \lambda)$$

Then, to prove Item ii., it only suffices to prove that,

$$\big(\forall \eta :\ 1 \preccurlyeq \eta \prec \lambda\big)\big[\big(\mathrm{Q}(1) = 1\big) \wedge \big(\mathrm{Q}(\eta) = 1 \longrightarrow \mathrm{Q}(\eta+1) = 1\big)\big]$$

– Case i. Let $1 = \eta$. Then,

$$\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta\times 1)}(\mathscr{S}_{\mathfrak{g}}) = \big(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\big)^{(1)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$$

implying $\mathrm{Q}(1) = 1$. The base case therefore holds.

– CASE II. Let $1 \prec \eta \prec \lambda$ and assume that the inductive hypothesis $\mathrm{Q}(\eta) = 1$ holds true. Then, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}}) = \big(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\big)^{(\eta)}(\mathscr{S}_{\mathfrak{g}})$ and consequently, it results that $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)} \circ \big(\big(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\big)^{(\eta)}(\mathscr{S}_{\mathfrak{g}})\big)$. But, the relation $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\eta)} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta(\eta+1))}(\mathscr{S}_{\mathfrak{g}})$ holds true and on the other hand, the relation $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)} \circ \big(\big(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\big)^{(\eta)}(\mathscr{S}_{\mathfrak{g}})\big) \longleftrightarrow \big(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\big)^{(\eta+1)}(\mathscr{S}_{\mathfrak{g}})$ also holds true. Hence, it follows that $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta(\eta+1))}(\mathscr{S}_{\mathfrak{g}}) = \big(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\big)^{(\eta+1)}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{Q}(\eta+1) = 1$. The inductive case therefore holds.

Suppose $\mathrm{Q}(\delta, \eta) = 1$ holds for all (δ, η) such that $(1,1) \prec (\delta, \eta) \prec (\lambda, \lambda)$. Then,

$$\begin{array}{ccc}
\bigcap_{\delta \prec \lambda}\Big(\bigcap_{\eta \prec \lambda} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}})\Big) & = & \bigcap_{\delta \prec \lambda}\Big(\bigcap_{\eta \prec \lambda}\big(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\big)^{(\eta)}(\mathscr{S}_{\mathfrak{g}})\Big) \\
 & \updownarrow & \\
\bigcap_{\delta \prec \lambda} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta\lambda)}(\mathscr{S}_{\mathfrak{g}}) & = & \bigcap_{\delta \prec \lambda}\big(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\big)^{(\lambda)}(\mathscr{S}_{\mathfrak{g}}) \\
 & \updownarrow & \\
\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\lambda\times\lambda)}(\mathscr{S}_{\mathfrak{g}}) & = & \big(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\lambda)}\big)^{(\lambda)}(\mathscr{S}_{\mathfrak{g}})
\end{array}$$

from which $\mathrm{Q}(\lambda, \lambda) = 1$ follows. The proof of the proposition is complete. □

An immediate consequence of the above proposition is the following corollary.

Corollary 3.10. *If* $\mathfrak{g}\text{-}\mathrm{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *and* $\mathbf{dc}_{g} \in \mathrm{DC}[T_{g}]$ *be a given pair of* $\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathrm{der}_{\mathfrak{g}}$, $\mathrm{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$*, and let* $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

- I. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \big(\mathrm{der}_{\mathfrak{g}}^{(\delta)}\big)^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \quad \big(\forall (\delta, \eta) : (1,1) \preccurlyeq (\delta, \eta) \prec (\lambda, \lambda)\big)$
- II. $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \big(\mathrm{cod}_{\mathfrak{g}}^{(\delta)}\big)^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \quad \big(\forall (\delta, \eta) : (1,1) \preccurlyeq (\delta, \eta) \prec (\lambda, \lambda)\big)$

For any δ such that $1 \preccurlyeq \delta \prec \lambda$, the union of a $\mathfrak{T}_{\mathfrak{g}}$-set and its $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-derived set includes the image of the $\mathfrak{T}_{\mathfrak{g}}$-set under the δ^{th}-order $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-derived set operator composition with itself; the intersection of a $\mathfrak{T}_{\mathfrak{g}}$-set and its $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-coderived set is included in the image of the $\mathfrak{T}_{\mathfrak{g}}$-set under the δ^{th}-order $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-coderived set operator composition with itself. These are embodied in the following theorem.

Theorem 3.11. *Let* $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *be a* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-derived and a* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-coderived operators, respectively, and let* $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*. Then:*

- I. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$
- II. $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$

Proof. Let $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-coderived operators, respectively, and let $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space

$\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

— I. Introduce $\mathbb{B} = \{0, 1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{P}(\delta) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \delta :\ 1 \preccurlyeq \delta \prec \lambda)$$

Then, to prove Item i., it only suffices to prove that,

$$\big(\forall \delta :\ 1 \preccurlyeq \delta \prec \lambda\big)\Big[\big(\mathrm{P}(1) = 1\big) \wedge \big(\mathrm{P}(\delta) = 1 \longrightarrow \mathrm{P}(\delta + 1) = 1\big)\Big]$$

– Case i. Let $1 = \delta$. Then, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ holds true, implying $\mathrm{P}(1) = 1$. The base case therefore holds.

– Case ii. Let $1 \prec \delta \prec \lambda$ and assume that the inductive hypothesis $\mathrm{P}(\delta) = 1$ holds true. Then, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and consequently, it follows that $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(2)} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(2)}\big(\mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)$. But, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(2)} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+1)} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})$ and, on the other hand, the relation $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(2)}\big(\mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \subseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ also holds true. Hence, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+1)} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{P}(\delta + 1) = 1$. The inductive case therefore holds.

Suppose $\mathrm{P}(\delta) = 1$ holds for all δ such that $1 \prec \delta \prec \lambda$. Then,

$$\begin{aligned}
\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\lambda+1)} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\lambda+1)}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(2)} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\lambda)} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\lambda)}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(2)} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\lambda+\lambda)}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(2)}\Bigg(\bigcap_{\delta+\delta \prec \lambda+\lambda} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+\delta)}(\mathscr{S}_{\mathfrak{g}})\Bigg) \\
&\subseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(2)} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+\delta)}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+1)} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \\
&\subseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})
\end{aligned}$$

from which $\mathrm{P}(\lambda) = 1$ follows.

— II. Introduce $\mathbb{B} = \{0, 1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{Q}(\delta) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \delta :\ 1 \preccurlyeq \delta \prec \lambda)$$

Then, to prove Item ii., it only suffices to prove that,

$$\big(\forall \delta :\ 1 \preccurlyeq \delta \prec \lambda\big)\Big[\big(\mathrm{Q}(1) = 1\big) \wedge \big(\mathrm{Q}(\delta) = 1 \longrightarrow \mathrm{Q}(\delta + 1) = 1\big)\Big]$$

– Case i. Let $1 = \delta$. Then, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ holds true, implying $\mathrm{Q}(1) = 1$. The base case therefore holds.

– Case ii. Let $1 \prec \delta \prec \lambda$ and assume that the inductive hypothesis $\mathrm{Q}(\delta) = 1$ holds true. Then, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and consequently, it follows that $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(2)} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(2)}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)$. But, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(2)} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+1)} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})$ and, on the other hand, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(2)}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \supseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \supseteq \mathscr{S}_{\mathfrak{g}} \cap$

$\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Hence, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{Q}(\delta+1) = 1$. The inductive case therefore holds.

Suppose $\mathrm{Q}(\delta) = 1$ holds for all δ such that $1 < \delta < \lambda$. Then,

$$\begin{aligned}
\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\lambda+1)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\lambda+1)}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(2)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\lambda)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\lambda)}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(2)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\lambda+\lambda)}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(2)}\Bigg(\bigcap_{\delta+\delta<\lambda+\lambda} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+\delta)}(\mathscr{S}_{\mathfrak{g}})\Bigg) \\
&\subseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(2)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+\delta)}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \\
&\supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \\
&\supseteq \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})
\end{aligned}$$

from which $\mathrm{Q}(\lambda) = 1$ follows. The proof of the theorem is complete. □

The following corollary is an immediate consequence of the above theorem.

Corollary 3.12. *If* $\mathfrak{g}\text{-Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *and* $\mathbf{dc}_g \in \mathrm{DC}[T_g]$ *be a given pair of* $\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathrm{der}_{\mathfrak{g}}$*,* $\mathrm{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$*, and let* $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

– I. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathrm{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \delta : 1 \leq \delta < \lambda)$

– II. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathscr{S}_{\mathfrak{g}} \cap \mathrm{cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \delta : 1 \leq \delta < \lambda)$

For any δ such $1 \leq \delta < \lambda$, the image of a $\mathfrak{T}_{\mathfrak{g}}$-set under the δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator is equivalent to the image of the relative complement of any $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived unit set in the $\mathfrak{T}_{\mathfrak{g}}$-set under the δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator; the image of the $\mathfrak{T}_{\mathfrak{g}}$-set under the δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator is equivalent to the image of the union of the $\mathfrak{T}_{\mathfrak{g}}$-set and any $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived unit set under the δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator. The theorem follows.

Theorem 3.13. *Let* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$*,* $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived and a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operators, respectively, and let* $(\{\xi\}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*. Then:*

– I. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big) \quad (\forall \delta : 1 \leq \delta < \lambda)$

– II. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}}\big) = \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\xi\}\big) \quad (\forall \delta : 1 \leq \delta < \lambda)$

Proof. Let $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, respectively, and let $(\{\xi\}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

— I. Introduce $\mathbb{B} = \{0, 1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{P}(\delta) \quad \overset{\text{def}}{\longleftrightarrow} \quad \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big) \quad (\forall \delta : 1 \leq \delta < \lambda)$$

Then, to prove ITEM I., it only suffices to prove that,

$$\big(\forall\delta:\ 1\preccurlyeq\delta\prec\lambda\big)\big[\big(\mathrm{P}\,(1)=1\big)\wedge\big(\mathrm{P}\,(\delta)=1\ \longrightarrow\ \mathrm{P}\,(\delta+1)=1\big)\big]$$

– CASE I. Let $1=\delta$. Then, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\,(\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\,(\{\xi\})\big)$ holds true, implying $\mathrm{P}\,(1)=1$. The base case therefore holds.

– CASE II. Let $1\prec\delta\prec\lambda$ and assume that the inductive hypothesis $\mathrm{P}\,(\delta)=1$ holds true. Then, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}\,(\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\,(\{\xi\})\big)$ and consequently, it follows that $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}\,(\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\,(\{\xi\})\big)$. But, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}\,(\mathscr{S}_{\mathfrak{g}})\longleftrightarrow\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+1)}\,(\mathscr{S}_{\mathfrak{g}})$ and, on the other hand, the relation $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\,(\{\xi\})\big)\longleftrightarrow\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+1)}\big(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\,(\{\xi\})\big)$ also holds true. Hence, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+1)}\,(\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+1)}\big(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\,(\{\xi\})\big)$, implying $\mathrm{P}\,(\delta+1)=1$. The inductive case therefore holds.

Suppose $\mathrm{P}\,(\delta)=1$ holds for all δ such that $1\prec\delta\prec\lambda$. Then,

$$\begin{array}{ccc}\bigcap_{\delta\prec\lambda}\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}\,(\mathscr{S}_{\mathfrak{g}}) & = & \bigcap_{\delta\prec\lambda}\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\,(\{\xi\})\big)\\ & \updownarrow & \\ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\lambda)}\,(\mathscr{S}_{\mathfrak{g}}) & = & \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\lambda)}\big(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\,(\{\xi\})\big)\end{array}$$

from which $\mathrm{P}\,(\lambda)=1$ follows.

— II. Introduce $\mathbb{B}=\{0,1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B}\ni\mathrm{Q}\,(\delta)\quad\stackrel{\text{def}}{\longleftrightarrow}\quad\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}}\big)=\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}}\cup\{\xi\}\big)\quad\big(\forall\delta:\ 1\preccurlyeq\delta\prec\lambda\big)$$

Then, to prove ITEM II., it only suffices to prove that,

$$\big(\forall\delta:\ 1\preccurlyeq\delta\prec\lambda\big)\big[\big(\mathrm{Q}\,(1)=1\big)\wedge\big(\mathrm{Q}\,(\delta)=1\ \longrightarrow\ \mathrm{Q}\,(\delta+1)=1\big)\big]$$

– CASE I. Let $1=\delta$. Then, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}\big)=\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}\cup\{\xi\}\big)$ holds true, implying $\mathrm{Q}\,(1)=1$. The base case therefore holds.

– CASE II. Let $1\prec\delta\prec\lambda$ and assume that the inductive hypothesis $\mathrm{Q}\,(\delta)=1$ holds true. Then, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}}\big)=\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}}\cup\{\xi\}\big)$ and consequently, it follows that $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}}\big)=\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}}\cup\{\xi\}\big)$. But on the one hand, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}}\big)\longleftrightarrow\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+1)}\big(\mathscr{S}_{\mathfrak{g}}\big)$ and, on the other hand, the relation $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}}\cup\{\xi\}\big)\longleftrightarrow\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+1)}\big(\mathscr{S}_{\mathfrak{g}}\cup\{\xi\}\big)$ also holds true. Hence, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+1)}\big(\mathscr{S}_{\mathfrak{g}}\big)=\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+1)}\big(\mathscr{S}_{\mathfrak{g}}\cup\{\xi\}\big)$, implying $\mathrm{Q}\,(\delta+1)=1$. The inductive case therefore holds.

Suppose $\mathrm{Q}\,(\delta)=1$ holds for all δ such that $1\prec\delta\prec\lambda$. Then,

$$\begin{array}{ccc}\bigcap_{\delta\prec\lambda}\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\,(\mathscr{S}_{\mathfrak{g}}) & = & \bigcap_{\delta\prec\lambda}\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}}\cup\{\xi\}\big)\\ & \updownarrow & \\ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\lambda)}\,(\mathscr{S}_{\mathfrak{g}}) & = & \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\lambda)}\big(\mathscr{S}_{\mathfrak{g}}\cup\{\xi\}\big)\end{array}$$

from which $\mathrm{Q}\,(\lambda)=1$ follows. The proof of the theorem is complete. $\square$

The corollary stated below is an immediate consequence of the above theorem.

Corollary 3.14. *If* $\mathfrak{g}\text{-}\mathrm{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *and* $\mathbf{dc}_g \in \mathrm{DC}[T_g]$ *be a given pair of* $\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathrm{der}_{\mathfrak{g}}$, $\mathrm{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, *and let* $(\{\xi\}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

- I. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\xi\})) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$
- II. $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}} \cup \{\xi\}) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$

Our research objective concerning the definitions and the essential properties of the concepts of δ^{th}-order derivative $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-coderived operators defined by transfinite recursion on the class of successor ordinals in $\mathscr{T}_{\mathfrak{g}}$-spaces is now complete. Of the notions of the δ^{th}-iterates of the $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-coderived operators in $\mathscr{T}_{\mathfrak{g}}$-spaces, we conclude the present section with two corollaries and two axiomatic definitions derived from these two corollaries.

The first corollary stated below gives the necessary and sufficient condition for a δ^{th}-order $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-derived operator to be a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-derived operator.

Corollary 3.15. *A necessary and sufficient condition for the* δ^{th}*-iterate* $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)} : \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega) \longmapsto \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$ *of* $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *to be a* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-derived operator in a strong* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *is that, for every* $(\{\xi\}, \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_3^*} \mathscr{P}(\Omega)$*, it satisfies:*

- I. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\varnothing) = \varnothing \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$
- II. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\xi\})) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$
- III. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$
- IV. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) = \bigcup_{\mathscr{U}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{U}_{\mathfrak{g}}) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$

The second corollary stated below gives the necessary and sufficient condition for a δ^{th}-order $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-coderived operator to be a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-coderived operator.

Corollary 3.16. *A necessary and sufficient condition for the* δ^{th}*-iterate* $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega) \longmapsto \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$ *of* $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *to be a* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-coderived operator in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *is that, for every* $(\{\xi\}, \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_3^*} \mathscr{P}(\Omega)$*, it satisfies:*

- I. $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\Omega) = \Omega \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$
- II. $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}) = \mathrm{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}} \cup \{\xi\}) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$
- III. $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}) \supseteq \mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$
- IV. $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) = \bigcap_{\mathscr{U}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{U}_{\mathfrak{g}}) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$

Hence, in a strong $\mathscr{T}_{\mathfrak{g}}$-space, for the δ^{th}-iterate of a set-valued map $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ to be characterized as a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-derived operator it must necessarily and sufficiently satisfy a list of *derived set* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-derived operator conditions* (ITEMS I.–IV. of COR. 3.15), and similarly, for the δ^{th}-iterate of a set-valued map $\mathfrak{g}\text{-Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ to be characterized as a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator it must necessarily and sufficiently satisfy a list of *derived set* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operator conditions* (Items v.–viii. of Cor. 3.16).

Evidently, Items i., ii., iii. and iv. of Cor. 3.15 state that the δ^{th}-iterate of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator $\mathfrak{g}\text{-Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is $\varnothing$*-grounded* (alternatively, $\varnothing$*-preserving*), ξ*-invariant* (alternatively, ξ*-unaffected*), $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}$*-intensive* and $\cup$*-additive* (alternatively, $\cup$*-distributive*), respectively. On the other hand, Items i., ii., iii. and iv. of Cor. 3.16 state that the δ^{th}-iterate of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is Ω*-grounded* (alternatively, Ω*-preserving*), ζ*-invariant* (alternatively, ζ*-unaffected*), $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$*-extensive* and $\cap$*-additive* (alternatively, $\cap$*-distributive*), respectively.

Viewing the δ^{th}-order derived set $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator conditions (Items i.–iv. of Cor. 3.15 above) as δ^{th}*-order* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived operator axioms*, the axiomatic definition of the concept of a δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator, then, can be defined as a δ^{th}-order set-valued map $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ satisfying a list of δ^{th}*-order* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived operator axioms.* The axiomatic definition of the concept of a δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator in strong $\mathscr{T}_{\mathfrak{g}}$-spaces follows.

Definition 3.17 (*Axiomatic Definition:* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-Derived Operator*)**.** The δ^{th}-iterate $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} : \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega) \longmapsto \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$ of $\mathfrak{g}\text{-Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is called a "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator of δ^{th} order" on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ for some ordinal δ such that $1 \leq \delta < \lambda$ if and only if, for any $\big(\{\xi\}, \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\big) \in \times_{\alpha \in I_3^*} \mathscr{P}(\Omega)$ such that $\{\xi\} \subset \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$, it satisfies each "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator axiom" in $\mathrm{AX}\big[\mathfrak{g}\text{-DE}^{(\delta)}[\mathfrak{T}_{\mathfrak{g}}]; \mathbb{B}\big] \stackrel{\text{def}}{=} \big\{\mathrm{Ax}_{\mathrm{DE},\nu}\big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big) : \nu \in I_4^*\big\}$, where the mapping $\mathrm{Ax}_{\mathrm{DE},\nu}$: $\mathfrak{g}\text{-DE}^{(\delta)}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathbb{B} \stackrel{\text{def}}{=} \{0,1\}$, $\nu \in I_4^*$, is defined as thus:

- $\mathrm{Ax}_{\mathrm{DE},1}\big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big) \stackrel{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\varnothing) = \varnothing$
- $\mathrm{Ax}_{\mathrm{DE},2}\big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big) \stackrel{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}) = \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)$
- $\mathrm{Ax}_{\mathrm{DE},3}\big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big) \stackrel{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$
- $\mathrm{Ax}_{\mathrm{DE},4}\big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big) \stackrel{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) = \bigcup_{\mathscr{U}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{U}_{\mathfrak{g}})$

Similarly, viewing the δ^{th}-order derived set $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator conditions (Items i.–iv. of Cor. 3.16 above) as δ^{th}*-order* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operator axioms*, the axiomatic definition of the concept of a δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator, then, can be defined as a δ^{th}-order set-valued map $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ satisfying a list of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operator axioms.* The axiomatic definition of the concept of a δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator in $\mathscr{T}_{\mathfrak{g}}$-spaces follows.

Definition 3.18 (*Axiomatic Definition:* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-Coderived Operator*)**.** The δ^{th}-iterate $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega) \longmapsto \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$ of $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is called a "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator of δ^{th} order" on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ for some ordinal δ such that $1 \leq \delta < \lambda$ if and only if, for any $\big(\{\xi\}, \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\big) \in \times_{\alpha \in I_3^*} \mathscr{P}(\Omega)$ such that $\{\xi\} \subset \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$, it satisfies each "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator axiom" in

$\mathrm{AX}\big[\mathfrak{g}\text{-}\mathrm{CD}^{(\delta)}\,[\mathfrak{T}_{\mathfrak{g}}]\,;\mathbb{B}\big] \overset{\text{def}}{=} \big\{\mathrm{Ax}_{\mathrm{CD},\nu}\big(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\big) :\ \nu \in I_4^*\big\}$, where the mapping $\mathrm{Ax}_{\mathrm{CD},\nu} : \mathfrak{g}\text{-}\mathrm{CD}^{(\delta)}\,[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathbb{B} \overset{\text{def}}{=} \{0,1\}$, $\nu \in I_4^*$, is defined as thus:

- $\mathrm{Ax}_{\mathrm{CD},1}\big(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\big) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\,(\Omega) = \Omega$
- $\mathrm{Ax}_{\mathrm{CD},2}\big(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\big) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\,(\mathscr{R}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{R}_{\mathfrak{g}} \cup \{\xi\}\big)$
- $\mathrm{Ax}_{\mathrm{CD},3}\big(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\big) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\,(\mathscr{R}_{\mathfrak{g}}) \supseteq \mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\,(\mathscr{R}_{\mathfrak{g}})$
- $\mathrm{Ax}_{\mathrm{CD},4}\big(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\big) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\,(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) = \bigcap_{\mathscr{U}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\,(\mathscr{U}_{\mathfrak{g}})$

On the essential properties of the δ^{th}-order derivative $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-coderived operators defined by transfinite recursion on the class of successor ordinals in $\mathscr{T}_{\mathfrak{g}}$-spaces, the discussion of the present section terminates here.

4. Discussion

4.1. Categorical and Ordinal Classifications. In the present section, based on the notions of *coarseness* (or, *smallness*, *weakness*), or alternatively, *finness* (or, *largeness*, *strongness*), the various relationships amongst the $\mathfrak{T}_{\mathfrak{a}}^{(\delta)}$, $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{a}}^{(\delta)}$-derived and $\mathfrak{T}_{\mathfrak{a}}^{(\delta)}$, $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{a}}^{(\delta)}$-coderived operators

$$\begin{cases}\mathrm{der}_{\mathfrak{a}}^{(\delta)},\ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a},\nu}^{(\delta)}\\ \mathrm{cod}_{\mathfrak{a}}^{(\delta)},\ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a},\nu}^{(\delta)}\end{cases} : \quad \begin{aligned} \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega)\\ \mathscr{S}_{\mathfrak{a}} &\longmapsto \begin{cases}\mathrm{der}_{\mathfrak{a}}^{(\delta)}\,(\mathscr{S}_{\mathfrak{a}}),\ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a},\nu}^{(\delta)}\,(\mathscr{S}_{\mathfrak{a}})\\ \mathrm{cod}_{\mathfrak{a}}^{(\delta)}\,(\mathscr{S}_{\mathfrak{a}}),\ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a},\nu}^{(\delta)}\,(\mathscr{S}_{\mathfrak{a}})\end{cases}\end{aligned} \tag{4.1}$$

are established in $\mathscr{T}_{\mathfrak{a}}$-spaces ($\mathfrak{a} \in \{\mathfrak{o}, \mathfrak{g}\}$) with respect to their category $\nu \in I_3^0$ and their ordinal $\delta \in [o] = \{\delta :\ 1 \preccurlyeq \delta \prec \lambda\}$, taking into account the required properties of the corresponding $\mathfrak{T}_{\mathfrak{a}}$, $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{a}}$-derived and $\mathfrak{T}_{\mathfrak{a}}$, $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{a}}$-coderived operators established in $\mathscr{T}_{\mathfrak{a}}$-spaces ($\mathfrak{a} \in \{\mathfrak{o}, \mathfrak{g}\}$) in a recent paper [1].

For illustrative purposes, the discussion will be furnished by $\big(\mathfrak{T}_{\mathfrak{a}}^{(\delta)}, \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{a}}^{(\delta)}\big)_{\mathfrak{a}=\mathfrak{o},\mathfrak{g}}$-*derived operators* and $\big(\mathfrak{T}_{\mathfrak{a}}^{(\delta)}, \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{a}}^{(\delta)}\big)_{\mathfrak{a}=\mathfrak{o},\mathfrak{g}}$-*coderived operators diagrams.* For clarity, the notations $\mathfrak{T} = (\Omega, \mathscr{T})$, der, $\mathfrak{g}$-Der, cod, $\mathfrak{g}$-Cod, …, $\mathrm{der}^{(\delta)}$, $\mathfrak{g}\text{-}\mathrm{Der}^{(\delta)}$, $\mathrm{cod}^{(\delta)}$, $\mathfrak{g}\text{-}\mathrm{Cod}^{(\delta)}$, … will be considered instead of $\mathfrak{T}_{\mathfrak{o}} = (\Omega, \mathscr{T}_{\mathfrak{o}})$, $\mathrm{der}_{\mathfrak{o}}$, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{o}}$, $\mathrm{cod}_{\mathfrak{o}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{o}}$, …, $\mathrm{der}_{\mathfrak{o}}^{(\delta)}$, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{o}}^{(\delta)}$, $\mathrm{cod}_{\mathfrak{o}}^{(\delta)}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{o}}^{(\delta)}$, …, respectively, or both will be considered interchangeably.

In a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}}) \supseteq (\Omega, \mathscr{T}_{\mathfrak{o}}) = \mathfrak{T}_{\mathfrak{o}}$, the so-called $\big(\mathfrak{T}_{\mathfrak{a}}, \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{a}}\big)_{\mathfrak{a}=\mathfrak{o},\mathfrak{g}}$-*coderived sets diagram* [*See* [1]: Diag. (4.1), §§ 4.1, p. 213.]

$$\begin{array}{ccccccc}
\mathrm{cod}\,(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \mathrm{cod}\,(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \mathrm{cod}\,(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \mathrm{cod}\,(\mathscr{S}_{\mathfrak{g}})\\
\cap\!\shortmid & & \cap\!\shortmid & & \cap\!\shortmid & & \cap\!\shortmid\\
\mathfrak{g}\text{-}\mathrm{Cod}_0\,(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \mathfrak{g}\text{-}\mathrm{Cod}_1\,(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \mathfrak{g}\text{-}\mathrm{Cod}_3\,(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \mathfrak{g}\text{-}\mathrm{Cod}_2\,(\mathscr{S}_{\mathfrak{g}})\\
\cap\!\shortmid & & \cap\!\shortmid & & \cap\!\shortmid & & \cap\!\shortmid\\
\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},0}\,(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},1}\,(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},3}\,(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},2}\,(\mathscr{S}_{\mathfrak{g}})\\
\cup\!\shortmid & & \cup\!\shortmid & & \cup\!\shortmid & & \cup\!\shortmid\\
\mathrm{cod}_{\mathfrak{g}}\,(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \mathrm{cod}_{\mathfrak{g}}\,(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \mathrm{cod}_{\mathfrak{g}}\,(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \mathrm{cod}_{\mathfrak{g}}\,(\mathscr{S}_{\mathfrak{g}})
\end{array}$$

as well as the so-called $\left(\mathfrak{T}_{\mathfrak{a}}, \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{a}}\right)_{\mathfrak{a}=\mathfrak{o},\mathfrak{g}}$-*derived sets diagram* [*See* [1]: DIAG. (4.2), §§ 4.1, p. 214.]

$$\begin{array}{ccccccc}
\operatorname{der}(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \operatorname{der}(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \operatorname{der}(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \operatorname{der}(\mathscr{S}_{\mathfrak{g}}) \\
\rotatebox[origin=c]{-90}{$\supseteq$} & & \rotatebox[origin=c]{-90}{$\supseteq$} & & \rotatebox[origin=c]{-90}{$\supseteq$} & & \rotatebox[origin=c]{-90}{$\supseteq$} \\
\mathfrak{g}\text{-}\mathrm{Der}_{0}(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \mathfrak{g}\text{-}\mathrm{Der}_{1}(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \mathfrak{g}\text{-}\mathrm{Der}_{3}(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \mathfrak{g}\text{-}\mathrm{Der}_{2}(\mathscr{S}_{\mathfrak{g}}) \\
\rotatebox[origin=c]{-90}{$\supseteq$} & & \rotatebox[origin=c]{-90}{$\supseteq$} & & \rotatebox[origin=c]{-90}{$\supseteq$} & & \rotatebox[origin=c]{-90}{$\supseteq$} \\
\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},0}(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},1}(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},3}(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},2}(\mathscr{S}_{\mathfrak{g}}) \\
\rotatebox[origin=c]{-90}{$\subseteq$} & & \rotatebox[origin=c]{-90}{$\subseteq$} & & \rotatebox[origin=c]{-90}{$\subseteq$} & & \rotatebox[origin=c]{-90}{$\subseteq$} \\
\operatorname{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \operatorname{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \operatorname{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \operatorname{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})
\end{array}$$

Let it be granted some pair $(\nu,\mu) \in I_3^0 \times I_3^0$ of categories and some pair of ordinals $(\delta,\eta) \in [o] \times [o]$. Suppose the relations $\begin{cases} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a},\nu}^{(\eta)} \lesssim \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a},\mu}^{(\delta)} \\ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a},\nu}^{(\eta)} \gtrsim \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a},\mu}^{(\delta)} \end{cases}$ stand for $\begin{cases} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a},\nu}^{(\eta)}(\mathscr{S}_{\mathfrak{a}}) \subseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a},\mu}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \\ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a},\nu}^{(\eta)}(\mathscr{S}_{\mathfrak{a}}) \supseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a},\mu}^{(\delta)}(\mathscr{S}_{\mathfrak{a}}) \end{cases}$ or equivalently, $\begin{cases} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a},\mu}^{(\delta)} \gtrsim \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a},\nu}^{(\eta)} \\ \mathfrak{g}\text{-}\mathrm{Der}_{\mu}^{(\delta)} \lesssim \mathfrak{g}\text{-}\mathrm{Der}_{\nu}^{(\eta)} \end{cases}$ stand for $\begin{cases} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a},\mu}^{(\delta)}(\mathscr{S}_{\mathfrak{a}}) \supseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a},\nu}^{(\eta)}(\mathscr{S}_{\mathfrak{a}}) \\ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a},\mu}^{(\delta)}(\mathscr{S}_{\mathfrak{a}}) \subseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a},\nu}^{(\eta)}(\mathscr{S}_{\mathfrak{a}}), \end{cases}$ respectively, in a $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$. Then, $\mathfrak{g}\text{-}\mathrm{Cod}_{\nu}^{(\eta)}$, $\mathfrak{g}\text{-}\mathrm{Der}_{\mu}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are *coarser* (or, *smaller*, *weaker*) than $\mathfrak{g}\text{-}\mathrm{Cod}_{\mu}^{(\delta)}$, $\mathfrak{g}\text{-}\mathrm{Der}_{\nu}^{(\eta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mu}^{(\delta)}$, $\mathfrak{g}\text{-}\mathrm{Der}_{\nu}^{(\eta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are *finer* (or, *larger*, *stronger*) than $\mathfrak{g}\text{-}\mathrm{Cod}_{\nu}^{(\eta)}$, $\mathfrak{g}\text{-}\mathrm{Der}_{\mu}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$; $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},\mu}^{(\delta)}$, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\nu}^{(\eta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are *finer* (or, *larger*, *stronger*) than $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},\nu}^{(\eta)}$,$\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\mu}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},\nu}^{(\eta)}$,$\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\mu}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are *coarser* (or, *smaller*, *weaker*) than $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},\mu}^{(\delta)}$, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\nu}^{(\eta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$.

In view of the above descriptions, for any pair $(\delta,\eta) \in [o] \times [o]$, the following $\left(\mathfrak{T}_{\mathfrak{a}}^{(\delta)}, \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{a}}^{(\delta)}\right)_{\mathfrak{a}=\mathfrak{o},\mathfrak{g}}$-*coderived operators diagram*, which is to be read horizontally, from left to right and vertically, from top to bottom, presents itself:

$$\begin{array}{ccccccc}
\mathrm{cod}^{(\eta)} & \lesssim & \mathrm{cod}^{(\eta)} & \lesssim & \mathrm{cod}^{(\eta)} & \gtrsim & \mathrm{cod}^{(\eta)} \\
\rotatebox[origin=c]{-90}{$\lesssim$} & & \rotatebox[origin=c]{-90}{$\lesssim$} & & \rotatebox[origin=c]{-90}{$\lesssim$} & & \rotatebox[origin=c]{-90}{$\lesssim$} \\
\mathfrak{g}\text{-}\mathrm{Cod}_{0}^{(\eta)} & \lesssim & \mathfrak{g}\text{-}\mathrm{Cod}_{1}^{(\eta)} & \lesssim & \mathfrak{g}\text{-}\mathrm{Cod}_{3}^{(\eta)} & \gtrsim & \mathfrak{g}\text{-}\mathrm{Cod}_{2}^{(\eta)} \\
\rotatebox[origin=c]{-90}{$\gtrsim$} & & \rotatebox[origin=c]{-90}{$\gtrsim$} & & \rotatebox[origin=c]{-90}{$\gtrsim$} & & \rotatebox[origin=c]{-90}{$\gtrsim$} \\
\mathfrak{g}\text{-}\mathrm{Cod}_{0}^{(\delta)} & \lesssim & \mathfrak{g}\text{-}\mathrm{Cod}_{1}^{(\delta)} & \lesssim & \mathfrak{g}\text{-}\mathrm{Cod}_{3}^{(\delta)} & \gtrsim & \mathfrak{g}\text{-}\mathrm{Cod}_{2}^{(\delta)} \\
\rotatebox[origin=c]{-90}{$\lesssim$} & & \rotatebox[origin=c]{-90}{$\lesssim$} & & \rotatebox[origin=c]{-90}{$\lesssim$} & & \rotatebox[origin=c]{-90}{$\lesssim$} \\
\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},0}^{(\delta)} & \lesssim & \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},1}^{(\delta)} & \lesssim & \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},3}^{(\delta)} & \gtrsim & \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},2}^{(\delta)} \\
\rotatebox[origin=c]{-90}{$\lesssim$} & & \rotatebox[origin=c]{-90}{$\lesssim$} & & \rotatebox[origin=c]{-90}{$\lesssim$} & & \rotatebox[origin=c]{-90}{$\lesssim$} \\
\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},0}^{(\eta)} & \lesssim & \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},1}^{(\eta)} & \lesssim & \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},3}^{(\eta)} & \gtrsim & \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},2}^{(\eta)} \\
\rotatebox[origin=c]{-90}{$\gtrsim$} & & \rotatebox[origin=c]{-90}{$\gtrsim$} & & \rotatebox[origin=c]{-90}{$\gtrsim$} & & \rotatebox[origin=c]{-90}{$\gtrsim$} \\
\mathrm{cod}_{\mathfrak{g}}^{(\eta)} & \lesssim & \mathrm{cod}_{\mathfrak{g}}^{(\eta)} & \lesssim & \mathrm{cod}_{\mathfrak{g}}^{(\eta)} & \gtrsim & \mathrm{cod}_{\mathfrak{g}}^{(\eta)}
\end{array} \tag{4.2}$$

On the other hand, for any pair $(\delta,\eta)\in[o]\times[o]$, the following $\left(\mathfrak{T}_{\mathfrak{a}}^{(\delta)},\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{a}}^{(\delta)}\right)_{\mathfrak{a}=\mathfrak{o},\mathfrak{g}}$-*derived operators diagram*, which is to be read horizontally, from left to right and vertically, from top to bottom, also presents itself:

$$
\begin{array}{ccccccc}
\mathrm{der}^{(\eta)} & \gtrsim & \mathrm{der}^{(\eta)} & \gtrsim & \mathrm{der}^{(\eta)} & \lesssim & \mathrm{der}^{(\eta)} \\
\wr\vee & & \wr\vee & & \wr\vee & & \wr\vee \\
\mathfrak{g}\text{-}\mathrm{Der}_{0}^{(\eta)} & \gtrsim & \mathfrak{g}\text{-}\mathrm{Der}_{1}^{(\eta)} & \gtrsim & \mathfrak{g}\text{-}\mathrm{Der}_{3}^{(\eta)} & \lesssim & \mathfrak{g}\text{-}\mathrm{Der}_{2}^{(\eta)} \\
\wr\wedge & & \wr\wedge & & \wr\wedge & & \wr\wedge \\
\mathfrak{g}\text{-}\mathrm{Der}_{0}^{(\delta)} & \gtrsim & \mathfrak{g}\text{-}\mathrm{Der}_{1}^{(\delta)} & \gtrsim & \mathfrak{g}\text{-}\mathrm{Der}_{3}^{(\delta)} & \lesssim & \mathfrak{g}\text{-}\mathrm{Der}_{2}^{(\delta)} \\
\wr\vee & & \wr\vee & & \wr\vee & & \wr\vee \\
\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},0}^{(\delta)} & \gtrsim & \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},1}^{(\delta)} & \gtrsim & \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},3}^{(\delta)} & \lesssim & \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},2}^{(\delta)} \\
\wr\vee & & \wr\vee & & \wr\vee & & \wr\vee \\
\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},0}^{(\eta)} & \gtrsim & \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},1}^{(\eta)} & \gtrsim & \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},3}^{(\eta)} & \lesssim & \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},2}^{(\eta)} \\
\wr\wedge & & \wr\wedge & & \wr\wedge & & \wr\wedge \\
\mathrm{der}_{\mathfrak{g}}^{(\eta)} & \gtrsim & \mathrm{der}_{\mathfrak{g}}^{(\eta)} & \gtrsim & \mathrm{der}_{\mathfrak{g}}^{(\eta)} & \lesssim & \mathrm{der}_{\mathfrak{g}}^{(\eta)}.
\end{array}
\tag{4.3}
$$

The relationships amongst the $\mathfrak{T}_{\mathfrak{a}}^{(\delta)}$, $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}^{(\delta)}$-derived operators $\mathrm{der}_{\mathfrak{a}}^{(\delta)}$, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a},\nu}^{(\delta)}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ and the $\mathfrak{T}_{\mathfrak{a}}^{(\delta)}$, $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}^{(\delta)}$-coderived operators $\mathrm{cod}_{\mathfrak{a}}^{(\delta)}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a},\nu}^{(\delta)}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$, respectively, are, therefore, established in $\mathscr{T}_{\mathfrak{a}}$-spaces ($\mathfrak{a}\in\{\mathfrak{o},\mathfrak{g}\}$) with respect to their category $\nu\in I_3^0$ and their ordinal $\delta\in[o]$.

4.2. A Nice Application.

It is the intent of the present section to present a nice application, highlighting some essential properties of the $\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$, $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-derived operators $\mathrm{der}_{\mathfrak{g}}^{(\delta)}$, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\nu}^{(\delta)}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ and $\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$, $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-coderived operators $\mathrm{cod}_{\mathfrak{g}}^{(\delta)}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},\nu}^{(\delta)}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$, respectively, in a $\mathscr{T}_{\mathfrak{g}}$-space with respect to their category $\nu\in I_3^0$ and their ordinal $\delta\in[o]$.

In considering the same $\mathscr{T}_{\mathfrak{g}}$-space upon which a nice application was presented in a recent paper [1, §§ 4.2], namely the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ based on the 7-point set $\Omega=\{\xi_\nu:\ \nu\in I_7^*\}$, and the latter topologized by the choice:

$$
\begin{aligned}
\mathscr{T}_{\mathfrak{g}}(\Omega) &= \left\{\varnothing,\ \{\xi_1\},\ \{\xi_1,\xi_3,\xi_5\},\ \{\xi_1,\xi_3,\xi_4,\xi_5,\xi_7\}\right\} \\
&= \left\{\mathscr{O}_{\mathfrak{g},1},\ \mathscr{O}_{\mathfrak{g},2},\ \mathscr{O}_{\mathfrak{g},3},\ \mathscr{O}_{\mathfrak{g},4}\right\} \\
\neg\mathscr{T}_{\mathfrak{g}}(\Omega) &= \left\{\Omega,\ \{\xi_2,\xi_3,\xi_4,\xi_5,\xi_6,\xi_7\},\ \{\xi_2,\xi_4,\xi_6,\xi_7\},\ \{\xi_2,\xi_6\}\right\} \\
&= \left\{\mathscr{K}_{\mathfrak{g},1},\ \mathscr{K}_{\mathfrak{g},2},\ \mathscr{K}_{\mathfrak{g},3},\ \mathscr{K}_{\mathfrak{g},4}\right\}
\end{aligned}
$$

with $\mathscr{R}_{\mathfrak{g}}=\{\xi_1,\xi_2,\xi_4\}$, $\mathscr{S}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}}\cup\{\xi_7\}$, $\mathscr{U}_{\mathfrak{g}}=\{\xi_3,\xi_5,\xi_6,\xi_7\}$, and $\mathscr{V}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}}\setminus\{\xi_3\}$, it was shown through calculations [*See* [1]: SYS. OF EQS (4.11), §§ 4.2, p. 216.] that the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operation of $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\nu}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ on the $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{R}_{\mathfrak{g}}$, $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$, and the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operation of $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},\nu}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ on the

$\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{U}_{\mathfrak{g}}$, $\mathscr{V}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, for all $\nu \in I_3^0$, result in:

$$\begin{cases} \mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}(\mathscr{W}_{\mathfrak{g}}) = \mathscr{K}_{\mathfrak{g},4} & \forall (\nu, \mathscr{W}_{\mathfrak{g}}) \in \{0,2\} \times \{\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\} \\ \mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}(\mathscr{W}_{\mathfrak{g}}) = \mathscr{O}_{\mathfrak{g},1} & \forall (\nu, \mathscr{W}_{\mathfrak{g}}) \in \{1,3\} \times \{\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\} \\ \mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu}(\mathscr{Y}_{\mathfrak{g}}) = \mathscr{O}_{\mathfrak{g},4} & \forall (\nu, \mathscr{Y}_{\mathfrak{g}}) \in \{0,2\} \times \{\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\} \\ \mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu}(\mathscr{Y}_{\mathfrak{g}}) = \mathscr{K}_{\mathfrak{g},1} & \forall (\nu, \mathscr{Y}_{\mathfrak{g}}) \in \{1,3\} \times \{\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\} \end{cases}$$

implying

$$\begin{cases} \mathfrak{g}\text{-Der}_{\mathfrak{g},0}(\mathscr{W}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Der}_{\mathfrak{g},1}(\mathscr{W}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Der}_{\mathfrak{g},3}(\mathscr{W}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g},2}(\mathscr{W}_{\mathfrak{g}}) \\ \mathfrak{g}\text{-Cod}_{\mathfrak{g},0}(\mathscr{Y}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g},1}(\mathscr{Y}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g},3}(\mathscr{Y}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g},2}(\mathscr{Y}_{\mathfrak{g}}) \end{cases}$$

for each $\mathscr{W}_{\mathfrak{g}} \in \{\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\}$ and $\mathscr{Y}_{\mathfrak{g}} \in \{\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\}$ [*See* [1]: Sys. of Eqs (4.13), §§ 4.2, p. 216.]. It was also shown through calculations [*See* [1]: Sys. of Eqs (4.12), §§ 4.2, p. 216.] that the $\mathfrak{T}_{\mathfrak{g}}$-derived operation of $\mathrm{der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on the $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{R}_{\mathfrak{g}}$, $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, and the $\mathfrak{T}_{\mathfrak{g}}$-coderived operation of $\mathrm{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on the $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{U}_{\mathfrak{g}}$, $\mathscr{V}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ result in:

$$\begin{cases} \mathrm{der}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}}) = \mathscr{K}_{\mathfrak{g},2} & \forall \mathscr{W}_{\mathfrak{g}} \in \{\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\} \\ \mathrm{cod}_{\mathfrak{g}}(\mathscr{Y}_{\mathfrak{g}}) = \mathscr{O}_{\mathfrak{g},2} & \forall \mathscr{Y}_{\mathfrak{g}} \in \{\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\} \end{cases}$$

implying

$$\begin{cases} \mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}(\mathscr{W}_{\mathfrak{g}}) \subseteq \mathrm{der}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}}) & \forall (\nu, \mathscr{W}_{\mathfrak{g}}) \in I_3^0 \times \{\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\} \\ \mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu}(\mathscr{Y}_{\mathfrak{g}}) \supseteq \mathrm{cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}}) & \forall (\nu, \mathscr{Y}_{\mathfrak{g}}) \in I_3^0 \times \{\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\} \end{cases}$$

in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ [*See* [1]: Sys. of Eqs (4.14), §§ 4.2, p. 216.].

Consider again the $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{R}_{\mathfrak{g}} = \{\xi_1, \xi_2, \xi_4\}$, $\mathscr{S}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}} \cup \{\xi_7\}$, $\mathscr{U}_{\mathfrak{g}} = \{\xi_3, \xi_5, \xi_6, \xi_7\}$, and $\mathscr{V}_{\mathfrak{g}} = \mathscr{U}_{\mathfrak{g}} \setminus \{\xi_3\}$. Then, for any $\delta \in [o]$, the $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-derived operation of $\mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on the $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{R}_{\mathfrak{g}}$, $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, and the $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-coderived operation of $\mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on the $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{U}_{\mathfrak{g}}$, $\mathscr{V}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, for all $\nu \in I_3^0$, produce the following results:

$$\text{(4.4)} \quad \begin{cases} \mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) = \mathscr{K}_{\mathfrak{g},4} & \forall (\nu, \mathscr{W}_{\mathfrak{g}}) \in \{0,2\} \times \{\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\} \\ \mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) = \mathscr{O}_{\mathfrak{g},1} & \forall (\nu, \mathscr{W}_{\mathfrak{g}}) \in \{1,3\} \times \{\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\} \\ \mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu}^{(\delta)}(\mathscr{Y}_{\mathfrak{g}}) = \mathscr{O}_{\mathfrak{g},4} & \forall (\nu, \mathscr{Y}_{\mathfrak{g}}) \in \{0,2\} \times \{\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\} \\ \mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu}^{(\delta)}(\mathscr{Y}_{\mathfrak{g}}) = \mathscr{K}_{\mathfrak{g},1} & \forall (\nu, \mathscr{Y}_{\mathfrak{g}}) \in \{1,3\} \times \{\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\} \end{cases}$$

Likewise, for any $\delta \in [o]$, the $\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-derived operation of $\mathrm{der}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on the $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{R}_{\mathfrak{g}}$, $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, and the $\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-coderived operation of $\mathrm{cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on the $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{U}_{\mathfrak{g}}$, $\mathscr{V}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, for all $\nu \in I_3^0$, also produce the following results:

$$\text{(4.5)} \quad \begin{cases} \mathrm{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) = \mathscr{K}_{\mathfrak{g},2} & \forall \mathscr{W}_{\mathfrak{g}} \in \{\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\} \\ \mathrm{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{Y}_{\mathfrak{g}}) = \mathscr{O}_{\mathfrak{g},2} & \forall \mathscr{Y}_{\mathfrak{g}} \in \{\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\} \end{cases}$$

By virtue of Sys. of Eqs (4.4), it follows that

$$\text{(4.6)} \quad \mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}^{(\lambda)}(\mathscr{W}_{\mathfrak{g}}) = \bigcap_{\delta < \lambda} \mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) = \varnothing \ \forall (\nu, \mathscr{W}_{\mathfrak{g}}) \in \{1,3\} \times \{\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\}$$

Again, by virtue of SYS. OF EQS (4.4), it also follows that

$$(4.7)\quad \begin{cases} \mathfrak{g}\text{-Der}^{(\lambda)}_{\mathfrak{g},\nu}(\mathscr{W}_{\mathfrak{g}}) & = \bigcap_{\delta<\lambda} \mathfrak{g}\text{-Der}^{(\delta)}_{\mathfrak{g},\nu}(\mathscr{W}_{\mathfrak{g}}) \neq \varnothing \quad \forall\,(\nu,\mathscr{W}_{\mathfrak{g}}) \in \{0,2\} \times \{\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}\} \\ \mathfrak{g}\text{-Cod}^{(\lambda)}_{\mathfrak{g},\nu}(\mathscr{Y}_{\mathfrak{g}}) & = \bigcap_{\delta<\lambda} \mathfrak{g}\text{-Cod}^{(\delta)}_{\mathfrak{g},\nu}(\mathscr{Y}_{\mathfrak{g}}) \neq \varnothing \quad \forall\,(\nu,\mathscr{Y}_{\mathfrak{g}}) \in I^0_3 \times \{\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}\} \\ \text{der}^{(\lambda)}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}}) & = \bigcap_{\delta<\lambda} \text{der}^{(\delta)}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}}) \neq \varnothing \quad \forall\,(\nu,\mathscr{W}_{\mathfrak{g}}) \in I^0_3 \times \{\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}\} \\ \text{cod}^{(\lambda)}_{\mathfrak{g}}(\mathscr{Y}_{\mathfrak{g}}) & = \bigcap_{\delta<\lambda} \text{cod}^{(\delta)}_{\mathfrak{g}}(\mathscr{Y}_{\mathfrak{g}}) \neq \varnothing \quad \forall\,(\nu,\mathscr{Y}_{\mathfrak{g}}) \in I^0_3 \times \{\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}\} \end{cases}$$

Hence, for any $\delta \in [o]$, it results that the following results hold true for each $\mathscr{W}_{\mathfrak{g}} \in \{\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}\}$ and $\mathscr{Y}_{\mathfrak{g}} \in \{\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}\}$:

$$(4.8)\quad \begin{cases} \mathfrak{g}\text{-Der}^{(\delta)}_{\mathfrak{g},0}(\mathscr{W}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Der}^{(\delta)}_{\mathfrak{g},1}(\mathscr{W}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Der}^{(\delta)}_{\mathfrak{g},3}(\mathscr{W}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}^{(\delta)}_{\mathfrak{g},2}(\mathscr{W}_{\mathfrak{g}}) \\ \mathfrak{g}\text{-Cod}^{(\delta)}_{\mathfrak{g},0}(\mathscr{Y}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Cod}^{(\delta)}_{\mathfrak{g},1}(\mathscr{Y}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Cod}^{(\delta)}_{\mathfrak{g},3}(\mathscr{Y}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}^{(\delta)}_{\mathfrak{g},2}(\mathscr{Y}_{\mathfrak{g}}) \end{cases}$$

For any $\delta \in [o]$, the $(\precsim,\succsim)$-relations $\mathfrak{g}\text{-Der}^{(\delta)}_{\mathfrak{g},0} \succsim \mathfrak{g}\text{-Der}^{(\delta)}_{\mathfrak{g},1} \succsim \mathfrak{g}\text{-Der}^{(\delta)}_{\mathfrak{g},3} \precsim \mathfrak{g}\text{-Der}^{(\delta)}_{\mathfrak{g},2}$ and $\mathfrak{g}\text{-Cod}^{(\delta)}_{\mathfrak{g},0} \precsim \mathfrak{g}\text{-Cod}^{(\delta)}_{\mathfrak{g},1} \precsim \mathfrak{g}\text{-Cod}^{(\delta)}_{\mathfrak{g},3} \succsim \mathfrak{g}\text{-Cod}^{(\delta)}_{\mathfrak{g},2}$ are thus verified. Clearly, for any $\delta \in [o]$, the following results also hold true:

$$(4.9)\quad \begin{cases} \mathfrak{g}\text{-Der}^{(\delta)}_{\mathfrak{g},\nu}(\mathscr{W}_{\mathfrak{g}}) \subseteq \text{der}^{(\delta)}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}}) \quad \forall\,(\nu,\mathscr{W}_{\mathfrak{g}}) \in I^0_3 \times \{\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}\} \\ \mathfrak{g}\text{-Cod}^{(\delta)}_{\mathfrak{g},\nu}(\mathscr{Y}_{\mathfrak{g}}) \supseteq \text{cod}^{(\delta)}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}}) \quad \forall\,(\nu,\mathscr{Y}_{\mathfrak{g}}) \in I^0_3 \times \{\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}\} \end{cases}$$

Thus, the $(\precsim,\succsim)$-relations $\mathfrak{g}\text{-Der}^{(\delta)}_{\mathfrak{g},\nu} \precsim \text{der}^{(\delta)}_{\mathfrak{g}}$ and $\mathfrak{g}\text{-Cod}^{(\delta)}_{\mathfrak{g},\nu} \succsim \text{cod}^{(\delta)}_{\mathfrak{g}}$, for all $\nu \in I^0_3$, are also verified.

The presentation of this nice application, highlighting some essential properties of the $\mathfrak{T}^{(\delta)}_{\mathfrak{g}}$, $\mathfrak{g}$-ν-$\mathfrak{T}^{(\delta)}_{\mathfrak{g}}$-derived operators $\text{der}^{(\delta)}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Der}^{(\delta)}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and $\mathfrak{T}^{(\delta)}_{\mathfrak{g}}$, $\mathfrak{g}$-ν-$\mathfrak{T}^{(\delta)}_{\mathfrak{g}}$-coderived operators $\text{cod}^{(\delta)}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}^{(\delta)}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, in a $\mathscr{T}_{\mathfrak{g}}$-space with respect to their category $\nu \in I^0_3$ and their ordinal $\delta \in [o]$ are, therefore, accomplished and ends here.

If the presentation be explored a step further, other interesting properties can be deduced from the study of other essential properties of $\mathfrak{T}^{(\delta)}_{\mathfrak{g}}$, $\mathfrak{g}$-ν-$\mathfrak{T}^{(\delta)}_{\mathfrak{g}}$-derived operators and $\mathfrak{T}^{(\delta)}_{\mathfrak{g}}$, $\mathfrak{g}$-ν-$\mathfrak{T}^{(\delta)}_{\mathfrak{g}}$-coderived operators in $\mathscr{T}_{\mathfrak{g}}$-spaces.

5. Conclusion

In a recent paper [1], we introduced the definitions and studied the essential properties of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, in $\mathscr{T}_{\mathfrak{g}}$-spaces. Mainly, we showed that $(\mathfrak{g}\text{-Der}_{\mathfrak{g}}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}) : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ is a pair of both *dual and monotone* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*operators* that is $(\varnothing,\Omega)$, $(\cup,\cap)$-*preserving*, and $(\subseteq,\supseteq)$-*preserving* relative to $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-(open, closed) sets [*See* [1]: CORS 3.15 & 3.16, §§ 2.2, p. 198.]. We also showed that $\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\right) : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ is a pair of *weaker* and *stronger* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*operators* [*See* [1]: THM. 3.1, §§ 2.2, p. 187.]. In the present paper, we have introduced by transfinite recursion on the class of successor ordinals the definitions and investigated the essential of the $\mathfrak{g}$-$\mathfrak{T}^{(\delta)}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}^{(\delta)}_{\mathfrak{g}}$-coderived operators $\mathfrak{g}\text{-Der}^{(\delta)}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}^{(\delta)}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, in $\mathscr{T}_{\mathfrak{g}}$-spaces [*See* § 3: THMS 3.1–3.13; CORS 3.2–3.16; PROPS 1–4; DEFS 3.17 & 3.18].

The following three statements sum up the outstanding facts resulting from the investigation of the essential of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-coderived operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, in $\mathscr{T}_{\mathfrak{g}}$-spaces:

– I. For any $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$, $\langle \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \rangle_{\delta \in [o]}$ is a monotone decreasing sequence of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-derived sets while $\langle \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \rangle_{\delta \in [o]}$ is a monotone increasing sequence of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-derived sets in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ [*See* § 3: THM. 3.1 & COR. 3.2].

– II. The $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-derived operator $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *weaker* than the $\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-derived operator $\text{der}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ while the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-coderived operator $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *stronger* than the $\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-coderived operator $\text{cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ [*See* § 3: PROP. 1; CORS 3.3 & 3.4].

– III. For any $(\{\xi\}, \mathfrak{g}\text{-Int}_{\mathfrak{g}}, \mathfrak{g}\text{-Cl}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{a}}]$, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-derived operator $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is $\varnothing$-*grounded* (alternatively, $\varnothing$-*preserving*), ξ-*invariant* (alternatively, ξ-*unaffected*), $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}$-*intensive* and $\cup$-*additive* (alternatively, $\cup$-*distributive*) [*See* § 3: COR. 3.15: ITEMS I.–IV.] while the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-coderived operator $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is Ω-*grounded* (alternatively, Ω-*preserving*), ξ-*invariant* (alternatively, ξ-*unaffected*), $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$-*extensive* and $\cap$-*additive* (alternatively, $\cap$-*distributive*) [*See* § 3: COR. 3.16: ITEMS I.–IV.] in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$.

Hence, it follows that the study of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-coderived operators $\begin{array}{rl} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) & \longrightarrow \mathscr{P}(\Omega) \\ \mathscr{S}_{\mathfrak{g}} & \longmapsto \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}), \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \end{array}$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ has resulted in several advantages. Indeed, it has resulted in axiomatic definitions of these $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-coderived operators in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ [*See* § 3: DEFS 3.17 & 3.18]. The $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-coderived structures $\mathfrak{D}_{\mathfrak{g}}^{(\delta)} \overset{\text{def}}{=} (\Omega, \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)})$ and $\mathfrak{C}_{\mathfrak{g}}^{(\delta)} \overset{\text{def}}{=} (\Omega, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)})$, then, are both themselves $\mathscr{T}_{\mathfrak{g}}$-spaces which may well be called $\mathscr{T}_{\mathfrak{g},\text{der}}^{(\delta)}$, $\mathscr{T}_{\mathfrak{g},\text{cod}}^{(\delta)}$-*spaces*, respectively. Accordingly, if Cantor [23, 24] had also considered the $\mathfrak{T}_{\mathfrak{o}|\mathbb{R}}^{(\delta)}$-derived operator $\begin{array}{rl} \text{der}_{\mathfrak{o}|\mathbb{R}}^{(\delta)} : \mathscr{P}(\mathbb{R}) & \longrightarrow \mathscr{P}(\mathbb{R}) \\ \mathscr{S}_{\mathfrak{o}} & \longmapsto \text{der}_{\mathfrak{o}|\mathbb{R}}^{(\delta)}(\mathscr{S}_{\mathfrak{o}}) \end{array}$ in his investigations of the convergence of Fourier series in $\mathbb{R}$, then the study of convergence in any of the $\mathscr{T}_{\mathfrak{g},\text{der}}^{(\delta)}$, $\mathscr{T}_{\mathfrak{g},\text{cod}}^{(\delta)}$-spaces $\mathfrak{D}_{\mathfrak{g}}^{(\delta)}$, $\mathfrak{C}_{\mathfrak{g}}^{(\delta)}$, respectively, might be made another subject of inquiry. The discovery of properties in this direction would definitely bring some benefits to the field of Mathematical Analysis, and the discussion of this paper ends here.

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