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RESEARCH ARTICLE

A New Link to Helices in Euclidean 3-Space

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ABSTRACT

In this paper, we introduce a novel approach for obtaining the parametric expression and description of a general helix, slant helix, and Darboux helix. The new method involves projecting α onto a plane passing through $\alpha(0)$ and orthogonal to the unit axis vector U in order to determine the position vector of the general helix α. The position vector of the helix with the plane projection γ and its axis U is then established. Additionally, a relation between the curvatures of α and γ is presented. The proposed technique is then applied to derive the parametric representation of a slant helix and Darboux helix, followed by the provision of various examples obtained through the application of this methodology.

Keywords: *General helix, slant helix, Darboux helix, position vector, Euclidean 3-space.*
AMS Subject Classification (2020): *Primary 14H50; Secondary 53A04, 53A55.*

1. Introduction

Let $\alpha = \alpha(s) : I \longrightarrow \mathbb{E}^3$ be an arbitrary curve in $\mathbb{E}^3$, where $I \subset \mathbb{R}$ is an open interval. Recall that the curve α is said to be a *unit speed curve* (or parameterized by arc-length parameter) if $\langle \alpha'(s), \alpha'(s) \rangle = 1$ for all $s \in I$, where $\langle , \rangle$ denotes the standard inner product of $\mathbb{E}^3$ given by

$$\langle , \rangle = dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system $\mathbb{E}^3$.In particular, the *norm* (or length) of a vector $v \in \mathbb{E}^n$ is given by $\|v\| = \sqrt{\langle v, v \rangle}$.

Given a unit speed curve α in Euclidean $3-$space $\mathbb{E}^3$ it is possible to define a Frenet frame $\{T, N, B\}$ along the curve α, where $T = \alpha'(s)$ is the unit tangent vector, $N = \frac{\alpha''(s)}{\|\alpha''(s)\|}$ is the principal normal vector, $B = T \times N$ is the binormal vector of α. Then the Frenet formulas are given by:

$$T' = \kappa N, \quad N' = -\kappa T + \tau B, \quad B' = -\tau N, \tag{1.1}$$

where $\kappa(s)$ and $\tau(s)$ are the curvature and the torsion of α, respectively.

When the Frenet frame of a regular curve makes an instantaneous helix motion in $\mathbb{E}^3$, there exists an axis of the frame's rotation. The direction of such axis is given by the vector

$$D = \tau T + \kappa B, \tag{1.2}$$

which is called a *Darboux vector field* of the curve α. The Darboux vector field enables us to rewrite the Frenet formulas as, [6],

$$T' = D \times T, \quad N' = D \times N, \quad B' = D \times B. \tag{1.3}$$

Monge initiated many important results in the theory of the curves in E^3 and Darboux pioneered the moving frame idea. After that, Frenet defined his moving frame and his special equations, which play important roles in mechanics, kinematics, and differential geometry ([3]).

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The geometry of the curve α can be described by the differentiation of the Frenet frame, which leads to the corresponding Frenet equations. One interesting topic in the theory of curves in $\mathbb{E}^3$ is to find new characterizations of important curves, especially for helices, slant helices, and Darboux helices. From elementary differential geometry it is well known that a curve whose unit tangent vector field T of α makes a constant angle θ with some fixed unit vector U, is called a *general* or *cylindrical helix* (or curve of the constant slope) in $\mathbb{E}^3$ ([8]). It is well-known that a regular curve α with the first curvature $\kappa \neq 0$ and second curvature τ in $\mathbb{E}^3$ is the general helix if and only if it has constant conical curvature $\frac{\tau}{\kappa}$. Scofield [7] has proved that the tangent indicatrix of a curve of precession is a spherical helix. A *curve of constant precession* is defined by the property that as the curve is traversed with unit speed, its centrode (Darboux vector field)

$$D = \tau T + \kappa B$$

revolves about a fixed axis with constant angle and constant speed. By using simple computations the author characterizes curves of constant precession by the fact that, up to reflection or phase shift of the arc-length s, $\kappa(s) = w\sin(\mu s)$ and $\tau(s) = w\cos(\mu s)$, for constant w and μ. In [5], Izumiya and Takeuchi have defined a new kind of helix called *slant helix* as a curve with non-vanishing curvature whose principal normal vector N makes a constant angle with a fixed direction. They proved that α is a slant helix if and only if the geodesic curvature of spherical image of principal normal indicatrix of α

$$\frac{\kappa^2}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\left(\frac{\tau}{\kappa}\right)'$$

is a constant function. In [4], Şenol defined special helices, called the *Darboux helices*, whose Darboux vector makes a constant angle with a fixed straight line. He proved that α is a Darboux helix if and only if

$$\frac{\left(\kappa^2+\tau^2\right)^{\frac{3}{2}}}{\kappa^2}\frac{1}{\left(\frac{\tau}{\kappa}\right)'}$$

is constant.

The determining of the position vector of some different curves is a one of important subjects. Recently, the parametric representation of general helices and slant helices as an important special curves in Euclidean space $\mathbb{E}^3$ is deduced by Ali in [1, 2].

In this paper, we give a new method which involves projecting a space curve α onto a plane passing through the starting point and orthogonal to the unit fixed vector in order to determine the position vector of the curve α. Using the method, we establish the position vector of the helix with a given projection curve γ and axis U along with a relation between the curvatures of the helix and the curvature of the projection curve. Then, we apply this methodology to derive the parametric representation of a slant helix and a Darboux helix. We also give some related examples and their figures by using *Mathematica*©.

2. Preliminaries

We now recall some basic concepts in classical differential geometry of curves in $\mathbb{R}^2$.

We will say that the curve $\alpha : I \longrightarrow \mathbb{R}^3$ is a plane curve if there exists a plane $P \subset \mathbb{R}^3$ such that $\alpha(I) \subset P$. Since we are considering only those properties of curves that are invariant under rigid motions, and we can always find a rigid motion of $\mathbb{R}^3$ taking the plane P to be the plane $z = 0$, we may restrict ourselves to the case $\alpha : I \longrightarrow \mathbb{R}^3$ where

$$\alpha(s) = (\alpha_1(s), \alpha_2(s), 0),$$

that is, to the case of differentiable maps $\alpha : I \longrightarrow \mathbb{R}^2$. Therefore, let us take a regular curve $\gamma : I \subset \mathbb{R} \longrightarrow \mathbb{R}^2$ lying in the plane $span\{e_1, e_2\}$. Then it has parameter equation

$$\gamma(\phi) = \int \frac{1}{k_\gamma(\phi)}\left(\cos\phi\, e_1 + \sin\phi\, e_2\right)\, d\phi, \tag{2.1}$$

where k_γ called the *curvature* of γ is defined by

$$k_\gamma = \frac{\|\gamma' \times \gamma''\|}{\|\gamma'\|^3} \tag{2.2}$$

and $\phi(s)$ called a *turning angle* of γ is given by

$$\phi(t) = \int k_\gamma(t)\ dt. \tag{2.3}$$

The *arc-length function* $s_\gamma = s_\gamma(t)$ of γ is defined by

$$s_\gamma = \int_0^t \|\dot{\gamma}(u)\|\, du. \tag{2.4}$$

It is known that there exists an oriented Frenet dihedron $\{T_\gamma, N_\gamma\}$ of the unit-speed curve γ, where T_γ and N_γ are the unit tangent vector and the principal normal vector of γ, respectively. Then the Frenet equations of the curve γ are given by

$$\begin{bmatrix} \dot{T}_\gamma \\ \dot{N}'_\gamma \end{bmatrix} = \begin{bmatrix} 0 & k_\gamma \\ -k_\gamma & 0 \end{bmatrix} \begin{bmatrix} T_\gamma \\ N_\gamma \end{bmatrix}, \tag{2.5}$$

where the curvature k_γ of the curve γ is defined by

$$k_\gamma = \langle \dot{T}_\gamma, N_\gamma \rangle.$$

In the differential geometry of curves, for a curve γ in $\mathbb{R}^2$ the following is well known ([6]):

Lemma 2.1. *Let $\gamma : (a, b) \longrightarrow \mathbb{R}^2$ be any regular plane curve in $\mathbb{R}^2$ whose signed curvature is k_γ. Then,*
(i) γ is a straight line if and only if $k_\gamma = 0$ for all $s \in (a, b)$;
(ii) γ is part of a circle if and only if $k_\gamma = constant$ for all $s \in (a, b)$.

3. Helices and their characterizations

Let $\alpha : I \longrightarrow \mathbb{R}^3$ be a unit speed curve with the curvatures κ and $\tau \neq 0$ in $\mathbb{R}^3$. Take the projection of the curve α onto the plane through the point $\alpha(0)$ and perpendicular to the fixed unit vector U.

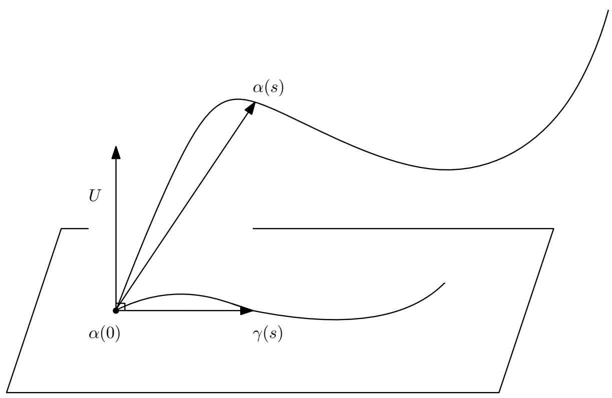

Figure 1. The curve α and its projection γ onto the plane P through the point $\alpha(0)$ and perpendicular to the unit vector U

From Figure 1, we see that the position vector of α, with respect to chosen origin $\alpha(0)$, can be written as

$$\alpha(s) = \gamma(s) + \delta_U(s)\, U \tag{3.1}$$

where γ is the projection of α, and δ_U is a differentiable function in s.
Differentiating the above relation with respect to s, we get

$$T = \gamma' + \frac{d\delta_U}{ds} U.$$

By taking the scalar product with U, we obtain

$$\langle T, U \rangle = \langle \gamma', U \rangle + \frac{d\delta_U}{ds}.$$

We have $\langle \gamma', U \rangle = 0$, since γ' lies in plane with normal vector U. The last equation gives

$$\frac{d\delta_U}{ds} = \langle T, U \rangle \,. \tag{3.2}$$

In what follows, we consider the following three cases: (A) α is a general helix; (B) α is a slant helix; (C) α is a Darboux helix.

(A) Let α be a general helix with the axis U. Then, for all $s \in I$ we get

$$\langle T, U \rangle = \cos\theta = constant, \tag{3.3}$$

and so the axis U is given by

$$U = \cos\theta T + \sin\theta B. \tag{3.4}$$

where $\theta \in (0, \pi)$ is the constant angle between T and U. Also, the curvatures κ and τ of the curve α satisfy the relation

$$\frac{\tau}{\kappa} = \cot\theta. \tag{3.5}$$

Now, from the relations (3.2) and (3.3) we get

$$\frac{d\delta_U}{ds} = \cos\theta,$$

is constant for all $s \in I$.
Integrating the above relation with the initial condition $\alpha(0)$ gives

$$\delta_U(s) = s\cos\theta, \tag{3.6}$$

as a real-valued function of s.
Substituting the relation (3.6) into the relation (3.1), we find

$$\alpha(s) = \gamma(s) + s\cos\theta U. \tag{3.7}$$

It means that α can be expressed by the given γ, U and θ.
Moreover, s is both the arc-length parameter of α and the arbitrary parameter of γ in (3.7). Then, differentiating the relation (3.7) with respect to s, we obtain

$$\frac{d\gamma}{ds} = \gamma'(s) = T - \cos\theta U,$$

with

$$\|\gamma'\| = \sin\theta.$$

So, from the relation (2.4), we can get the arc-length function s_γ of γ as

$$s_\gamma = s\sin\theta.$$

Then, we have the unit speed tangent vector T_γ of γ as

$$T_\gamma = \frac{\gamma'}{\|\gamma'\|} = \frac{1}{\sin\theta}T - \cot\theta U$$

Differentiating the last relation with respect to s_γ and using (1.1) and (3.4), we have

$$\frac{dT_\gamma}{ds_\gamma} = \frac{dT_\gamma}{ds}\frac{ds}{ds_\gamma} = \frac{\kappa}{\sin^2\theta}N.$$

and since $\dot{T}_\gamma = k_\gamma N_\gamma$,

$$\frac{\kappa}{\sin^2\theta}N = k_\gamma N_\gamma.$$

It is easily say that the normal vector N_γ of γ is parallel to the normal vector N of α, and so

$$\kappa = k_\gamma \sin^2\theta.$$

Then, from the relation (3.5) the torsion of α is given by

$$\tau = k_\gamma \sin\theta\cos\theta.$$

Therefore, we can give the following theorem and corollaries.

Theorem 3.1. *Let γ be a curve parameterized by the arc-length s_γ lies in a plane with the normal vector U, and let $\theta \in (0, \pi)$ is a constant angle. Then, there is a general helix α by the arc-length s with the curvatures κ and $\tau \neq 0$ in $\mathbb{R}^3$ such that $\langle T, U\rangle = \cos\theta$ for all s, and its position vector given by*

$$\alpha(s) = \gamma(s) + s\cos\theta U,$$

and its arc-length parameter s is given by

$$s = \frac{s_\gamma}{\sin\theta}.$$

Corollary 3.1. *The curvatures κ and $\tau \neq 0$ of the general helix α with in $\mathbb{R}^3$ are given by*

$$\kappa(s) = k_\gamma \sin^2\theta,$$

and

$$\tau(s) = k_\gamma \sin\theta\cos\theta,$$

where k_γ is the curvature of γ.

Corollary 3.2. *The general helix α is a circular helix if and only if k_γ is constant.*

Moreover, from the relation (2.3) and Corollary 3.1 we get $\phi(s) = \int \frac{\kappa}{\sin^2\theta}\, ds$. Therefore, the following corollary can be given using the relation (2.1) and Theorem 3.1.

Corollary 3.3. *Every general helix α with the curvatures $\kappa > 0$ in $\mathbb{R}^3$ can be parameterized by*

$$\alpha(s) = \int \frac{\sin^2\theta}{\kappa}\left(\cos\phi\, e_1 + \sin\phi\, e_2\right)\, d\phi + s\cos\theta U,$$

where $\phi(s) = \int \frac{\kappa}{\sin^2\theta}\, ds$.

(B) Let α be a slant helix with the axis U. Then, for all $s \in I$ we have

$$\langle N, U\rangle = \cos\varphi = constant. \tag{3.8}$$

Define a curve β as the integral curve of the principal normal N of α, i.e.,

$$\beta = \int N\, ds.$$

Since $\beta' = N$, we get $\langle \beta', \beta'\rangle = 1$, and so $\beta' = T_\beta$. Then, the arc-length of β is equal to the arc-length s of α. Also, from the relation (3.8), $\langle T_\beta, U\rangle = \cos\varphi$, and then the curve β is a general helix with the axis U given by

$$U = \cos\varphi T_\beta + \sin\varphi B_\beta. \tag{3.9}$$

Moreover, the Frenet apparatus of β can be given

$$\begin{array}{l} T_\beta = \beta' = N, \\ N_\beta = \frac{\beta''}{\|\beta''\|} = \frac{1}{\sqrt{\kappa^2+\tau^2}}\left(-\kappa T + \tau B\right), \\ B_\beta = T_\beta \times N_\beta = \frac{1}{\sqrt{\kappa^2+\tau^2}}\left(\tau T + \kappa B\right), \\ \kappa_\beta = \sqrt{\kappa^2+\tau^2}, \\ \tau_\beta = \frac{\kappa^2}{\kappa^2+\tau^2}\left(\frac{\tau}{\kappa}\right)'. \end{array} \tag{3.10}$$

Now, let $\bar{\gamma}$ be the projection of the curve β onto the plane P with the fixed unit normal vector U. Then, from Theorem (3.1) the position vector of β can be written as

$$\beta(s) = \bar{\gamma}(s) + s\cos\varphi U, \tag{3.11}$$

and from Corollary 3.1 the relations between the curvatures of β and the curvature $k_{\bar{\gamma}}$ of $\bar{\gamma}$ are given by

$$\begin{array}{l} \kappa_\beta(s) = k_{\bar{\gamma}} \sin^2\varphi, \\ \tau_\beta(s) = k_{\bar{\gamma}} \sin\varphi\cos\varphi. \end{array} \tag{3.12}$$

From the relations (3.10) and (3.12) we can write the curvatures κ and τ of α as

$$\begin{aligned}\kappa &= k_{\bar{\gamma}} \sin^2 \varphi \sin w, \\ \tau &= k_{\bar{\gamma}} \sin^2 \varphi \cos w,\end{aligned} \tag{3.13}$$

where $w = w(s)$ is some differentiable function of s.
Since β is a general helix, its conical curvature yields

$$\frac{\tau_\beta}{\kappa_\beta} = \frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}} \left(\frac{\tau}{\kappa}\right)' = \cot\varphi. \tag{3.14}$$

Putting the relation (3.13) into the relation (3.14), we get

$$w' = -k_{\bar{\gamma}} \sin\varphi \cos\varphi.$$

Integrating the above relation gives

$$w = -\sin\varphi\cos\varphi \int k_{\bar{\gamma}} ds, \tag{3.15}$$

where $\varphi \in (0,\pi)$ is the constant angle between N and U.

Remark 3.1. It is easily seen that the curvatures κ and τ are expressed in terms of the curvature $k_{\bar{\gamma}}$ and the angle φ using the relations (3.13) and (3.15).

Furthermore, from the relation (3.10) we have

$$T = \frac{1}{\sqrt{\kappa^2+\tau^2}} \left(-\kappa N_\beta + \tau B_\beta\right),$$

or from the relation (3.13)

$$T = -\sin w N_\beta + \cos w B_\beta, \tag{3.16}$$

Then we can get the position vector of α as $\alpha(s) = \int T(s)\ ds$.

Example 3.1. Let us choose a plane curve

$$\bar{\gamma} = \left(\cos s_{\bar{\gamma}}, \sin s_{\bar{\gamma}}, 0\right), \tag{3.17}$$

which lies in the plane $z = 0$ with the normal $U = (0,0,1)$, and the angle between N and U is $\varphi = \frac{\pi}{6}$. It is assumed that the curve $\bar{\gamma}$ is the projection of the integral curve β of N onto the plane $z = 0$. Since $\beta' = N$ is a unit vector, β is a helix which has the same arc-length as the curve α. Then from Theorem (3.1) the arc-length function $s_{\bar{\gamma}}$ of $\bar{\gamma}$ is

$$s_{\bar{\gamma}} = s\sin\varphi = \frac{s}{2}. \tag{3.18}$$

Therefore, from the relation (3.11) the position vector of β is found as

$$\beta(s) = \left(\cos\left(\frac{s}{2}\right), \sin\left(\frac{s}{2}\right), \frac{\sqrt{3}s}{2}\right),$$

and the Frenet apparatus of β is obtained as

$$\begin{aligned}T_\beta &= \left(-\tfrac{1}{2}\sin\left(\tfrac{s}{2}\right), \tfrac{1}{2}\cos\left(\tfrac{s}{2}\right), \tfrac{\sqrt{3}}{2}\right), \\ N_\beta &= \tfrac{\beta''}{\|\beta''\|} = \left(-\cos\left(\tfrac{s}{2}\right), -\sin\left(\tfrac{s}{2}\right), 0\right), \\ B_\beta &= T_\beta \times N_\beta = \left(\tfrac{\sqrt{3}}{2}\sin\left(\tfrac{s}{2}\right), -\tfrac{\sqrt{3}}{2}\cos\left(\tfrac{s}{2}\right), \tfrac{1}{2}\right), \\ \kappa_\beta &= \tfrac{1}{4}, \\ \tau_\beta &= \tfrac{\sqrt{3}}{4}.\end{aligned}$$

Differentiating two times the relation (3.17) with respect to the arc-length s and using (3.18), we obtain

$$\begin{aligned}\bar{\gamma}' &= \left(-\tfrac{1}{2}\sin\left(\tfrac{s}{2}\right), \tfrac{1}{2}\cos\left(\tfrac{s}{2}\right), 0\right), \\ \bar{\gamma}'' &= \left(-\tfrac{1}{4}\cos\left(\tfrac{s}{2}\right), -\tfrac{1}{4}\sin\left(\tfrac{s}{2}\right), 0\right),\end{aligned}$$

and

$$\bar{\gamma}' \times \bar{\gamma}'' = \left(0, 0, \frac{1}{8}\right).$$

Then, from the relations (2.2) and (3.15) the curvature of $\bar{\gamma}$ is $k_{\bar{\gamma}} = 1$, and $w(s) = -\frac{\sqrt{3}s}{4}$. Using the relation (3.16) we can obtain the unit tangent vector T of α as follows:

$$\begin{aligned} T \;&= \Big(\tfrac{\sqrt{3}}{2}\sin\left(\tfrac{s}{2}\right)\cos\left(\tfrac{\sqrt{3}s}{4}\right) - \sin\left(\tfrac{\sqrt{3}s}{4}\right)\cos\left(\tfrac{s}{2}\right), \\ &-\sin\left(\tfrac{\sqrt{3}s}{4}\right)\sin\left(\tfrac{s}{2}\right) - \tfrac{\sqrt{3}}{2}\cos\left(\tfrac{\sqrt{3}s}{4}\right)\cos\left(\tfrac{s}{2}\right), \tfrac{1}{2}\cos\left(\tfrac{\sqrt{3}s}{4}\right)\Big) \end{aligned}$$

and since $\beta' = N$, the unit normal vector N of α can be given as follows:

$$N = \left(-\frac{1}{2}\sin\left(\frac{s}{2}\right), \frac{1}{2}\cos\left(\frac{s}{2}\right), \frac{\sqrt{3}}{2}\right).$$

Finally, using the above relations, we can obtain the unit binormal vector B of α as follows:

$$\begin{aligned} B \;&= T \times N \\ &= \Big(-\tfrac{\sqrt{3}}{2}\sin\left(\tfrac{s}{2}\right)\sin\left(\tfrac{\sqrt{3}s}{4}\right) - \cos\left(\tfrac{\sqrt{3}s}{4}\right)\cos\left(\tfrac{s}{2}\right), \\ &-\tfrac{\sqrt{3}}{2}\sin\left(\tfrac{\sqrt{3}s}{4}\right)\cos\left(\tfrac{s}{2}\right) - \cos\left(\tfrac{\sqrt{3}s}{4}\right)\sin\left(\tfrac{s}{2}\right), -\tfrac{1}{2}\sin\left(\tfrac{\sqrt{3}s}{4}\right)\Big). \end{aligned}$$

Moreover, from the relations (3.13) and (3.15) the curvatures of α is

$$\begin{aligned} \kappa &= -\tfrac{1}{4}\sin\left(\tfrac{\sqrt{3}s}{2}\right), \\ \tau &= \tfrac{1}{4}\cos\left(\tfrac{\sqrt{3}s}{2}\right). \end{aligned}$$

By using integration of Wolfram Mathematica, we obtain the parameter equation of α (see Figure 2) as follows

$$\begin{aligned} \alpha(s) \;&= \textstyle\int T(s)\, ds \\ &= \begin{pmatrix} -14\sin\left(\tfrac{s}{2}\right)\sin\left(\tfrac{\sqrt{3}s}{4}\right) - 8\sqrt{3}\cos\left(\tfrac{s}{2}\right)\cos\left(\tfrac{\sqrt{3}s}{4}\right), \\ \tfrac{3\sqrt{3}}{2}\sin\left(\tfrac{s}{2}\right)\cos\left(\tfrac{\sqrt{3}s}{2}\right) - \tfrac{5}{2}\cos\left(\tfrac{s}{2}\right)\sin\left(\tfrac{\sqrt{3}s}{2}\right), \tfrac{2\sqrt{3}}{3}\sin\left(\tfrac{\sqrt{3}}{4}s\right) \end{pmatrix} \end{aligned}$$

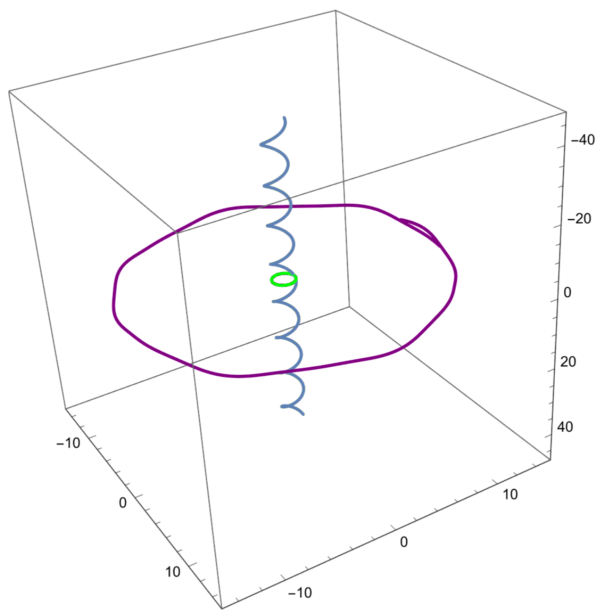

Figure 2. The slant curve α is purple, the helix curve β is blue, and the plane curve $\bar{\gamma}$ is green

(C) Let α be a Darboux helix with the axis U. Then, for all $s \in I$ we have

$$\langle W, U\rangle = \cos\phi,$$

is constant. Here, W is the unit Darboux vector of α, i.e. $W = \frac{D}{\|D\|} = \frac{1}{\sqrt{\kappa^2+\tau^2}}(\tau T + \kappa B)$. Now, define a curve $\bar{\beta}$ as the integral curve of the principal normal W of α, i.e.,

$$\bar{\beta} = \int W\, ds. \tag{3.19}$$

Since $\bar{\beta}' = W$, we get $\langle \bar{\beta}', \bar{\beta}'\rangle = 1$, and so $\bar{\beta}' = T_{\bar{\beta}}$. Then, the arc-length of $\bar{\beta}$ is equal to the arc-length s of α. Also, from the relation (3.8), $\langle T_{\bar{\beta}}, U\rangle = \cos\phi$, and then the curve β is a general helix with the axis U given by

$$U = \cos\phi T_{\bar{\beta}} + \sin\phi B_{\bar{\beta}}.$$

By using the relations (1.1), (1.3) and (3.19), the Frenet apparatus of $\bar{\beta}$ can be given

$$\begin{aligned}
&T_{\bar{\beta}} = \bar{\beta}' = W,\\
&N_{\bar{\beta}} = \frac{\bar{\beta}''}{\|\bar{\beta}''\|} = \frac{1}{\sqrt{\kappa^2+\tau^2}}\left(\kappa T - \tau B\right),\\
&B_{\bar{\beta}} = T_{\bar{\beta}} \times N_{\bar{\beta}} = -N,\\
&\kappa_{\bar{\beta}} = \frac{\kappa^2}{\kappa^2+\tau^2}\left(\frac{\tau}{\kappa}\right)',\\
&\tau_{\bar{\beta}} = \sqrt{\kappa^2+\tau^2},
\end{aligned} \tag{3.20}$$

Now, let $\bar{\bar{\gamma}}$ be the projection of the curve $\bar{\beta}$ onto the plane P with the fixed unit normal vector U. Then, from Theorem (3.1) the position vector of $\bar{\beta}$ can be written as

$$\bar{\beta}(s) = \bar{\bar{\gamma}}(s) + s\cos\phi U, \tag{3.21}$$

and from Corollary 3.1 the relations between the curvatures of $\bar{\beta}$ and the curvature $k_{\bar{\bar{\gamma}}}$ of $\bar{\bar{\gamma}}$ are given by

$$\begin{aligned}
\kappa_{\bar{\beta}}(s) &= k_{\bar{\bar{\gamma}}}\sin^2\phi,\\
\tau_{\bar{\beta}}(s) &= k_{\bar{\bar{\gamma}}}\sin\phi\cos\phi.
\end{aligned} \tag{3.22}$$

From the relations (3.20) and (3.22) we can write the curvatures κ and τ of α as

$$\begin{aligned}
\kappa &= k_{\bar{\bar{\gamma}}}\sin\phi\cos\phi\sin\psi,\\
\tau &= k_{\bar{\bar{\gamma}}}\sin\phi\cos\phi\cos\psi,
\end{aligned} \tag{3.23}$$

where $\psi = \psi(s)$ is some differentiable function of s.
Since β is a general helix, its conical curvature yields

$$\frac{\tau_\beta}{\kappa_\beta} = \frac{\left(\kappa^2+\tau^2\right)^{3/2}}{\kappa^2}\frac{1}{\left(\frac{\tau}{\kappa}\right)'} = \cot\phi. \tag{3.24}$$

Putting the relation (3.23) into the relation (3.24), we get

$$\psi' = -k_{\bar{\bar{\gamma}}}\sin^2\phi.$$

Integrating the above relation gives

$$\psi = -\sin^2\phi\int k_{\bar{\bar{\gamma}}}ds, \tag{3.25}$$

where $\phi \in (0, \pi)$ is the constant angle between W and U.
Furthermore, from the relation (3.20) we have

$$T = \frac{1}{\sqrt{\kappa^2+\tau^2}}\left(\tau T_{\bar{\beta}} + \kappa N_{\bar{\beta}}\right),$$

or from the relation (3.13)

$$T = \cos\psi T_{\bar{\beta}} + \sin\psi N_{\bar{\beta}}, \tag{3.26}$$

Then we can get the position vector of α as $\alpha(s) = \int T(s)\ ds$.

Example 3.2. Let us derive the parameter equation of a Darboux helix α, which is defined by the arc-length s, using the plane curve lying in the plane $z = 0$ with the normal $U = (0, 0, 1)$, and an angle $\phi = \frac{\pi}{6}$ as described in Example 3.1. It is assumed that the curve $\bar{\bar{\gamma}}$ is the projection of the integral curve $\bar{\beta}$ of W onto the plane $z = 0$. Since $\bar{\beta}' = W$ is a unit vector, $\bar{\beta}$ is a helix which has the same arc-length as the curve α. Then, from Example 3.1, the position vector of $\bar{\beta}$ is

$$\bar{\beta}(s) = \left(\cos\left(\frac{s}{2}\right), \sin\left(\frac{s}{2}\right), \frac{\sqrt{3}s}{2}\right)$$

with the Frenet apparatus

$$\begin{aligned}
T_{\bar{\beta}} &= \left(-\tfrac{1}{2}\sin\left(\tfrac{s}{2}\right), \tfrac{1}{2}\cos\left(\tfrac{s}{2}\right), \tfrac{\sqrt{3}}{2}\right),\\
N_{\bar{\beta}} &= \frac{\bar{\beta}''}{\|\bar{\beta}''\|} = \left(-\cos\left(\tfrac{s}{2}\right), -\sin\left(\tfrac{s}{2}\right), 0\right),\\
B_{\bar{\beta}} &= T_{\bar{\beta}} \times N_{\bar{\beta}} = \left(\tfrac{\sqrt{3}}{2}\sin\left(\tfrac{s}{2}\right), -\tfrac{\sqrt{3}}{2}\cos\left(\tfrac{s}{2}\right), \tfrac{1}{2}\right),\\
\kappa_{\bar{\beta}} &= \tfrac{1}{4},\\
\tau_{\bar{\beta}} &= \tfrac{\sqrt{3}}{4}.
\end{aligned}$$

Since the curvature of $\bar{\bar{\gamma}}$ is $k_{\bar{\bar{\gamma}}} = 1$, the relation yields $\psi(s) = -\frac{s}{4}$.
Using the relation (3.26) we can obtain the unit tangent vector T of α as follows:

$$T = \left(\frac{1}{4}\sin\left(\frac{3s}{4}\right) - \frac{3}{4}\sin\left(\frac{s}{4}\right), \frac{3}{4}\cos\left(\frac{s}{4}\right) - \frac{1}{4}\cos\left(\frac{3s}{4}\right), \frac{\sqrt{3}}{2}\cos\left(\frac{s}{4}\right)\right)$$

By using integration of Wolfram Mathematica, we obtain the parameter equation of α (see Figure 3) as follows

$$\begin{aligned}
\alpha(s) &= \int T(s)\, ds\\
&= \left(3\cos\left(\frac{s}{4}\right) - \frac{1}{3}\cos\left(\frac{3s}{4}\right), 3\sin\left(\frac{s}{4}\right) - \frac{1}{3}\sin\left(\frac{3s}{4}\right), 2\sqrt{3}\sin\left(\frac{s}{4}\right)\right)
\end{aligned}$$

Moreover,

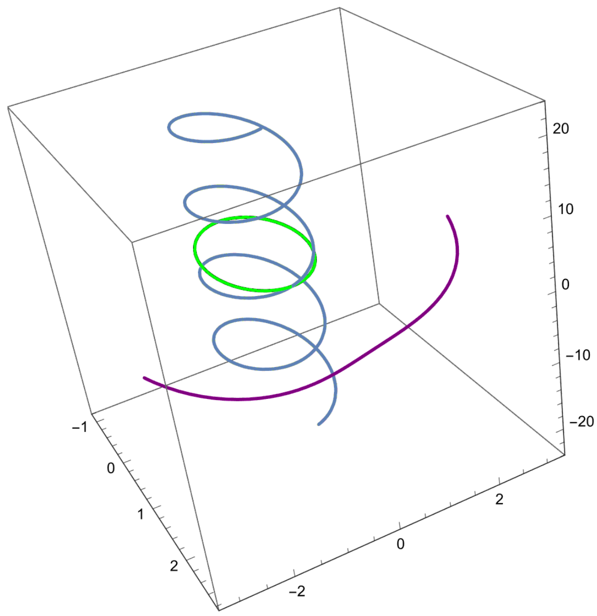

Figure 3. The Darboux curve α is purple, the helix curve β is blue, and the plane curve $\bar{\bar{\gamma}}$ is green

Example 3.3. Let us choose the clothoid (also called the Cornu spiral) defined by

$$\bar{\bar{\gamma}} = \left(0, \int_0^{s_{\bar{\bar{\gamma}}}} \cos\frac{s_{\bar{\bar{\gamma}}}^2}{2} ds_{\bar{\bar{\gamma}}}, \int_0^{s_{\bar{\bar{\gamma}}}} \sin\frac{s_{\bar{\bar{\gamma}}}^2}{2} ds_{\bar{\bar{\gamma}}}\right) \tag{3.27}$$

that lies in the plane $x = 0$ with the normal $U = (1, 0, 0)$, and the angle $\phi = \frac{\pi}{3}$. Clothoids have been used in highway design for many years. The clothoid has curvature $k_{\bar{\bar{\gamma}}} = s_{\bar{\bar{\gamma}}}$. We will obtain the parameter equation of a Darboux helix α given by the arc-length s, such that $\langle W, U\rangle = \cos\frac{\pi}{3} = \frac{1}{2}$.
Now, we assume that the curve $\bar{\bar{\gamma}}$ is the projection of the integral curve $\bar{\beta}$ of W onto the plane $x = 0$. Here, since $\bar{\beta}' = W$ is a unit vector, $\bar{\beta}$ is a helix which has the same arc length as the curve α. Then from Theorem (3.1) the arc-length function $s_{\bar{\bar{\gamma}}}$ of $\bar{\bar{\gamma}}$ is

$$s_{\bar{\bar{\gamma}}} = s\sin\varphi = \frac{\sqrt{3}s}{2}. \tag{3.28}$$

Differentiating two times the relation (3.27) with respect to s and using (3.28) we have

$$\begin{aligned}\bar{\bar{\gamma}}' &= \frac{\sqrt{3}}{2}\left(0, \cos\frac{3s^2}{8}, \sin\frac{3s^2}{8}\right),\\ \bar{\bar{\gamma}}'' &= \frac{3\sqrt{3}s}{8}\left(0, -\sin\frac{3s^2}{8}, \cos\frac{3s^2}{8}\right),\end{aligned}$$

Then, from the relations (3.23) and (3.25) the curvatures of α is

$$\begin{aligned}\kappa &= -\tfrac{\sqrt{3}s}{4}\sin\left(\tfrac{3s^2}{8}\right),\\ \tau &= \tfrac{\sqrt{3}s}{4}\cos\left(\tfrac{3s^2}{8}\right),\end{aligned}$$

Therefore, from the relation (3.21) the position vector of $\bar{\beta}$ is

$$\begin{aligned}\bar{\beta}(s) &= \bar{\bar{\gamma}}(s) + s\cos\phi U\\ &= \left(0, \int_0^{\frac{\sqrt{3}s}{2}}\cos\frac{3s^2}{8}ds, \int_0^{\frac{\sqrt{3}s}{2}}\sin\frac{3s^2}{8}ds\right) + \frac{s}{2}(1,0,0)\\ &= \left(\frac{s}{2}, \int_0^{\frac{\sqrt{3}s}{2}}\cos\frac{3s^2}{8}ds, \int_0^{\frac{\sqrt{3}s}{2}}\sin\frac{3s^2}{8}ds\right)\end{aligned}$$

with the Frenet apparatus

$$\begin{aligned}&T_\beta = \left(\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\cos\tfrac{3s^2}{8}, \tfrac{\sqrt{3}}{2}\sin\tfrac{3s^2}{8}\right),\\ &N_\beta = \tfrac{\beta''}{\|\beta''\|} = \left(0, -\sin\tfrac{3s^2}{8}, \cos\tfrac{3s^2}{8}\right),\\ &B_\beta = T_\beta \times N_\beta = \left(\tfrac{\sqrt{3}}{2}, -\tfrac{1}{2}\cos\tfrac{3s^2}{8}, -\tfrac{1}{2}\sin\tfrac{3s^2}{8}\right),\\ &\kappa_{\bar{\beta}} = \tfrac{3s}{4},\\ &\tau_{\bar{\beta}} = \tfrac{\sqrt{3}s}{4}.\end{aligned}$$

Then, from the relation (3.26) we get the unit tangent vector T of α as follows

$$T = \left(\frac{1}{2}\cos\frac{3s^2}{8}, \left(\frac{\sqrt{3}-2}{2}\right)\cos^2\frac{3s^2}{8}, \left(\frac{\sqrt{3}-2}{4}\right)\sin\frac{3s^2}{4}\right).$$

By taking integration of the above relation, we obtain the parameter equation of α (see Figure 4) as follows

$$\begin{aligned}\alpha(s) &= \int T(s)\ ds\\ &= \left(\frac{1}{2}\int_0^s\cos\frac{3s^2}{8}ds, \left(\frac{\sqrt{3}-2}{2}\right)\int_0^s\cos^2\frac{3s^2}{8}ds, \left(\frac{\sqrt{3}-2}{4}\right)\int_0^s\sin\frac{3s^2}{4}ds\right)\end{aligned}$$

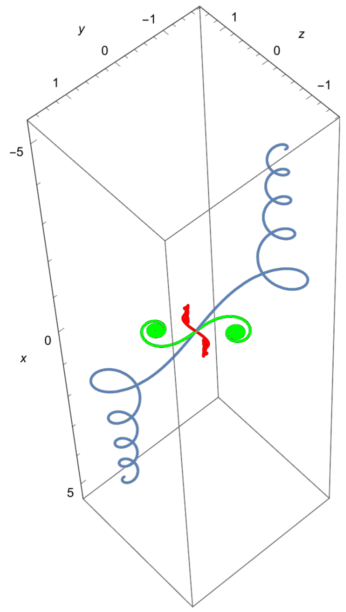

Figure 4. The Darboux curve α is red, the helix curve β is blue, and the plane curve $\bar{\bar{\gamma}}$ is green

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Author's contributions

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References

[1] Ali, A.T.: *Position vectors of general helices in Euclidean 3-space*. Bull. Math. Anal. Appl. **3**(2), 198–205 (2011).
[2] Ali, A.T.: *Position vectors of slant helices in Euclidean 3-space*. J. Egyptian Math. Soc. **20**(1), 1–6 (2012).
[3] Boyer, C.B.: A history of mathematics. John Wiley & Sons, Inc., New York (1991).
[4] Şenol, A.: *General helices in space forms*. Ph.D. Thesis, Ankara University (2008).
[5] Izumiya, S., Takeuchi, N.: *New special curves and developable surfaces*. Turkish J. Math. **28**(2), 153–163 (2004).
[6] O'Neill, B.: Elementary differential geometry. (2nd edition) Elsevier/Academic Press, Amsterdam (2006).
[7] Scofield, P.D.: *Curves of constant precession*. Amer. Math. Monthly **102**(6), 531–537 (1995).
[8] Struik, D.J.: Lectures on Classical Differential Geometry. Addison-Wesley Press, Inc., Cambridge, MA (1950).

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