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An Alternative Approach to the Axiomatic Characterization of the Interval Shapley Value

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Abstract — This study presents a new approach to the axiomatic characterization of the interval Shapley value. This approach aims to improve our comprehension of the particular characteristics of the interval Shapley value in a provided context. Furthermore, the research contributes to the related literature by expanding and applying the fundamental axiomatic principles used to define the interval Shapley value. The proposed axioms encompass symmetry, gain-loss, and differential marginality, offering a distinctive framework for understanding and characterizing the interval Shapley value. Through these axioms, the paper examines and interprets the intrinsic properties of the value objectively, presenting a new perspective on the interval Shapley value. The characterization highlights the importance and distinctiveness of the interval Shapley value.

Keywords *Cooperative interval games, uncertainty, interval Shapley value, axiomatic characterization*

Mathematics Subject Classification (2020) 91A12, 90B70

1. Introduction

The Shapley value, first introduced by Shapley [1] in 1953, is a significant solution concept in cooperative game theory. Over the years, the Shapley value has developed substantially and become a captivating concept. Originally devised for cooperative games with transferable utility (TU games), which involve a finite set of players and real-numbered coalition values, the Shapley value has not only sustained its significance but has also undergone substantial development over the years. This concept is a foundational framework for equitable reward or cost allocation. It transcends its initial mathematical underpinnings and finds applications across diverse disciplines, such as operations research (OR), economics, sociology, and computer science [2]. It delves into the intricacies of complex problems related to reward and cost-sharing, offering a nuanced approach to evaluating contributions within coalitions.

In many real-world situations, the intricacies that arise from interactions between individuals and organizations require modeling. Game theory is valuable for comprehending and analyzing complex situations within a rigorous mathematical framework. Consider the dynamics between two companies operating in a competitive market. Each company aims to increase its market share and maximize profitability. However, each company's decisions are inevitably influenced by the strategies employed

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by the others. If a company chooses to reduce its prices, the other may follow suit, potentially affecting the profitability of both businesses. Game theory can be utilized to model and analyze competitive market dynamics, an instance of resource sharing within a group. When a group must distribute a limited resource, everyone naturally tries to protect their interests. For example, when a team decides on project roles, each member aims to maximize their skills and contributions. However, game theory can facilitate finding an optimal solution and achieving equilibrium among competing interests. These examples highlight the broad range of situations where game theory can be applied. Nevertheless, individuals frequently encounter interval uncertainty, providing a new perspective on cooperative interval games. Particularly, it addresses scenarios where individuals or companies consider collaboration and need to formalize a contract. In such cases, it is difficult to determine exact coalition payoffs, and only the minimum and maximum values can be clearly defined with certainty.

Each cooperative interval game represents an interval payoff, the interval Shapley value. This value holds significant influence as an interval solution concept in cooperative interval game theory, particularly in real-world applications and OR situations. It is characterized by the special subclass of cooperative interval games. This paper aims to present a novel axiomatic approach to characterizing the interval Shapley value, which does not rely on additivity or marginality but instead incorporates interval data. This paper explores the interval Shapley value and its axiomatic characterizations within cooperative interval games, drawing inspiration from [3]. Several characterizations of the interval Shapley value and grey Shapley value can be found in the literature, as documented in [4–7]. Numerous studies have been conducted on the Shapley value. For example, [1] uses the axioms of additivity (ADD), efficiency (EFF), symmetry (SYM), and the null player property (NULL). [8] characterizes the Shapley value by using EFF, SYM, and strong monotonicity property (SMON). As characterized by [3], the Shapley value uses a new axiom called coalitional strategic equivalence (CSE). Moreover, numerous characterizations of the Shapley value can be found in the literature [9–12].

The manuscript aims to present an innovative axiomatic characterization of the interval Shapley value. Departing from the conventional reliance on additivity and marginality, this characterization introduces a novel approach using (a specific concept) to establish a new perspective. The research deals with Shapley value and its axiomatic characterizations, inspired by the scientific contributions of [3]. The motivation behind characterization is to redefine a value using different axioms. By using a specific set of principles based on Gain-Loss, differential marginality, and symmetry axioms, we can redefine the Shapley value in a way that is different from existing characterizations. These selected principles enhance our approach's originality and provide a unique perspective for understanding cooperative game theory. The selection of these axioms strengthens the innovative nature of our work, deviating from traditional frameworks and presenting a novel conceptualization of the Shapley value. The conscious choice of Gain-Loss, differential marginality, and symmetry as guiding principles set our characterization apart from conventional approaches, contributing a new and distinctive viewpoint to the ongoing discourse surrounding the Shapley value. In essence, axiomatic characterization aims to provide an interval value by introducing a different point of view through specific axioms. These axioms serve as tools to analyze the characterization. As a result, we derive new interval properties and define this value with particular characteristics. In this study, we propose a new alternative characterization. The rest of the paper is organized as follows. Section 2 provides basic information and materials on cooperative and interval game theory. In Section 3, the interval Shapley value is characterized axiomatically with the axioms of gain-loss, differential marginality, and symmetry axioms with interval data. We conclude our paper by offering a comprehensive evaluation with potential perspectives for future studies.

2. Preliminaries

A coalition game in coalition form is represented by an ordered pair $\langle N, v\rangle$ where N is the set of players and $v : 2^N \to \mathbb{R}$ is the characteristic function. The set 2^N denotes the set of all the subsets of N, each element of which is referred to as a coalition. A coalition game in coalition form is often employed as a TU game. G^N denotes the cooperative players in coalition form. The characteristic function of a game, denoted as $v \in G^N$, assigns the payoff $v(S)$ to each coalition $S \in 2^N$. Throughout this study, the notation 's' represents the cardinality of the coalition S instead of the notation $|S|$ for the number of elements in S.

Example 2.1. Let $N = \{1, 2, 3\}$ denote the set of players. Players 1 and 2 want to produce left gloves, while Player 3 wants to produce right gloves. The game's contribution is zero when producing only left or only right gloves, and it is 30 units when producing gloves together. This situation can be represented by the game $\langle N, v\rangle$. Here, the characteristic functions can be formulated as follows:

$$v(\emptyset) = 0$$

$$v(1) = v(2) = v(3) = v(12) = 0$$

$$v(13) = v(23) = v(N) = 30$$

Shapley value, one of the key concepts in cooperative game theory, will be discussed. Single-point solutions are represented through the transformation $f : G^N \to \mathbb{R}$ [13, 14].

Definition 2.2. The Shapley value of a cooperative game, denoted as $v \in G^N$, is articulated through the mapping $f : G^N \to \mathbb{R}$. Specifically, the Shapley value for player i is expressed as:

$$f_i(v) = \sum_{i\in S} \frac{\Delta_v(s)}{s}$$

In this context, the term $\Delta_v(s) = \sum_{T\subseteq S}(-1)^{s-t}v(T)$ embodies the concept of marginal contribution, a measure originally delineated by [15]. The Shapley value provides a fair allocation of the total payoff among players by considering all possible permutations of players and their contributions within coalitions. The set of all the games are $(2^{|N|} - 1)$- dimensional linear space where unanimity games form a basis. The unanimity game with the coalition of S, $u_S : 2^N \to \mathbb{R}$ is defined by

$$u_S(T) = \begin{cases} 1, \ S \subseteq T \\ 0, \text{ otherwise} \end{cases}$$

for $S \in 2^N \setminus \{\emptyset\}$. For more details, see [16].

This section provides an overview of the historical background for cooperative interval games [17–19]. An interval game is defined by $< N, w >$. Here, $N = \{1, 2, \cdots, n\}$ is the set of players, and the characteristic function is $w : 2^N \to I(\mathbb{R})$ where $I(\mathbb{R})$ is the set of all the closed intervals in $\mathbb{R}$. The interval set $w(S)$ has form $[\underline{w}(S), \overline{w}(S)]$ for each coalition $S \in 2^N$ where $\underline{w}(S)$ is the lower value and $\overline{w}(S)$ is the upper value. We denote the set of all the interval games with the player set N by IG^N.

We use another subtraction operator different from Moore's subtraction operator for this study [20]. We define $I - J$, only when $|I| \geq |J|$, as $I - J = \left[\underline{I} - \underline{J}, \overline{I} - \overline{J}\right]$ where $\underline{I} - \underline{J} \leq \overline{I} - \overline{J}$. It is noted that I is weakly superior to J, denoted by $I \succcurlyeq J$, if and only if $\underline{I} \geq \underline{J}$ and $\overline{I} \geq \overline{J}$. For $w_1, w_2 \in IG^N$, we state that $w_1 \preccurlyeq w_2$ if $w_1(S) \preccurlyeq w_2(S)$, for all $S \in 2^N$, and we define $< N, w_1 + w_2 >$ and $< N, \lambda w >$ by $(w_1 + w_2)(S) = w_1(S) + w_2(S)$ and $(\lambda w)(S) = \lambda \cdot w(S)$, for all $S \in 2^N$, such that $\lambda \in \mathbb{R}^+$. Additionally, for $w_1, w_2 \in IG^N$ with $|w_1(S)| \geq |w_2(S)|$, for all $S \in 2^N$, $< N, w_1 - w_2 >$ is defined by

$(w_1 - w_2)(S) = w_1(S) - w_2(S)$. Interval solutions are interval payoff vectors in $I(\mathbb{R})$. We denote the set of all the interval payoff vectors by $I(\mathbb{R})^N$. We designate a game $< N, w >$ as size monotonic if $< N, |w| >$ is monotonic such that $|w|(S) \leq |w|(T)$, for all $S, T \in 2^N$ with $S \subseteq T$. For further use, the class of size monotonic interval games with the player set N is denoted by $SMIG^N$ (for more details, see [16]).

Definition 2.3. The interval Shapley value, denoted by $\Phi : SMIG^N \to I(\mathbb{R})^N$, is as follows:

$$\Phi(w) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^{\sigma}(w)$$

An interval game $\langle N, Iu_S \rangle$ is defined by

$$(Iu_S)(T) = u_S(T)I$$

in which $I \in I(\mathbb{R})$ and u_S is the unanimity game in there $S \in 2^N \setminus \{\emptyset\}$, for all $T \in 2^N \setminus \{\emptyset\}$. The interval Shapley value of the interval game Iu_S is defined by

$$\Phi_i(Iu_S) = \begin{cases} I/|S|, & i \in S \\ [0,0], & i \notin S \end{cases}$$

The set of the additive cone generated by the set

$$K = \left\{ I_S u_S \mid S \in 2^N \setminus \{\emptyset\}, I_S \in I(\mathbb{R}) \right\}$$

is denoted by KIG^N. Therefore, each element in the cone is a finite sum of elements in K. We note that KIG^N is a subset of $SMIG^N$, and in the specific subclass of cooperative interval games, the interval Shapley value is axiomatically characterized within KIG^N.

3. Axiomatization

This section presents a novel characterization of the Shapley value defined for cooperative interval games. Furthermore, we recommend using precise axioms and the main theorem in this characterization. The interval solution, denoted by a function $f : IG^N \to I(\mathbb{R})^N$ is characterized by assigning a $|N|$-dimensional real vector to each interval game within the set N. The vector represents the distribution of interval payoffs that can be achieved through collaborative efforts among individual players in the game. Initially, we articulate the established axioms governing solutions $f : IG^N \to I(\mathbb{R})^N$.

The efficiency axiom is a fundamental principle of game theory, emphasizing the imperative of efficient resource utilization and maximization of total welfare. This axiom states that game outcomes must be economically efficient. Consequently, resources must be distributed optimally, and the condition of any player should not be improved through a more effective allocation of the existing resources. The efficiency axiom is an important feature in game theory, frequently used to balance payoffs and strategies. The efficiency axiom is extended by defining it in the context of the interval concept.

Axiom 3.1 (I-EFF): For all $w \in IG^N$, it holds that

$$\sum_{i \in N} f_i(w) = w(N)$$

Player $i \in N$ is a null player in $v \in G^N$ if $v(S) = v(S \setminus \{i\})$, for all $S \subseteq N$. The concept of a "null player" refers to a player who has no impact on the strategies of other players and exerts no influence on the outcomes of the game. This condition is employed in analyses to denote situations where the participation or influence of a specific player can be considered negligible. The null player axiom is recognized as a significant tool in modeling equilibrium and outcome analyses within game theory. We expand the null axiom by defining it in the context of the interval concept.

Axiom 3.2 (I-NULL): If $i \in N$ is a null player in-game $w \in IG^N$, then $f_i(w) = [0, 0]$.

If $v(S \cup \{i\}) = v(S \cup \{j\})$, for all $S \subseteq N \backslash \{i, j\}$, then two players $i, j \in N$ are called symmetric in $v \in G^N$.

The symmetry axiom states that a player or situation is symmetrical with others. This axiom contributes to the understanding of equality and equilibrium in analyses. In game theory, symmetry typically describes situations where players share similar roles or strategies. The symmetry axiom is widely used to determine game equilibrium points and optimal game strategies. We extend the symmetry axiom by defining within the framework of the interval concept.

Axiom 3.3 (I-SYM): If $i,\, j \in N$ are symmetric in $w \in IG^N$, then $f_i(w) = f_j\,(w)$.

The additivity axiom asserts that an individual player's payoffs contribute to the total payoff of the game. This axiom emphasizes the idea that the combined contributions of individuals are reflected in the aggregate of game outcomes. As players aim to maximize their individual payoffs by making strategic decisions, the additivity axiom is used to determine the game's overall success. In game theory, the additivity axiom is a foundational principle essential for analyzing games' aggregate outcomes and reaching optimal strategies. We define the additivity axiom within the framework of the interval concept, extending its scope.

Axiom 3.4 (I-ADD): For all $w, w' \in IG^N$,

$$f\,(w + w') = f(w) + f(w')$$

where $(w + w') \in IG^N$ is provided by

$$(w + w')(S) = w(S) + w'(S)$$

for all $S \subseteq N$.

In game theory, the gain-loss axiom is pivotal in explaining how players assess their situations regarding acquired gains and incurred losses. Players begin their strategic journeys from a designated starting point, analyzing this point against the gains and losses accumulated as the game progresses. However, the evaluation process depends on the magnitude of gains and losses and their ability to improve the current circumstances. In essence, players carefully observe whether there is an equal amount of gain or loss, highlighting the importance of this dynamic. The gain-loss axiom is extended by defining within the framework of the interval concept.

Axiom 3.5 (I-GL): For all $w, w' \in IG^N$ and $i \in N$ such that

$$w(N) = w'(N) \quad \text{ and } \quad f_i(w) \succcurlyeq f_i(w')$$

there is some $j \in N$ such that $f_j(w) \preccurlyeq f_j(w')$.

The marginality axiom is a fundamental principle in game theory that directs attention to the impact of a player's marginal contribution on the total payoff. This axiom is foundational in scrutinizing players' strategic choices by emphasizing the decisive effect of marginal changes in evaluating a player's decisions and contributions. Pursuing increased returns through marginal contributions is a prominent tenet of this axiom. We define the marginality axiom within the framework of the interval concept.

Axiom 3.6 (I-M): For all $w, w' \in IG^N$ and $i \in N$ such that

$$w(S \cup \{i\}) - w(S) = w'(S \cup \{i\}) - w'(S)$$

for all $S \subseteq N\backslash\{i\}, f_i(w) = f_i(w')$.

According to this axiom, the variation in the game's payoffs is determined by the marginal change in a player's contribution. Following this principle, if a new game is introduced to symmetric players, any equal change in their marginal contributions should have an equal impact on their payoffs. The principle states that any equal adjustment in players' marginal contributions should be fairly reflected in the corresponding payoffs. The definition of the differential marginality axiom is extended by providing it within the framework of the interval concept.

Axiom 3.7 (**I-DM**): For all $w, w' \in IG^N$ and $i,\, j \in N$ such that

$$w(S \cup \{i\}) - w(S \cup \{j\}) = w'(S \cup \{i\}) - w'(S \cup \{j\})$$

for all $S \subseteq N\backslash\{i,j\},\; f_i(w) - f_j(w) = f_i(w') - f_j(w')$.

Axiomatic characterization involves redefining a concept by specifying its properties through axioms. Therefore, we have presented the axioms required to define the interval Shapley value. Three of these axioms serve as the foundational elements for the main theorem. We axiomatically characterize the interval Shapley value by utilizing these three properties. The relevant theorem will be presented, and its proof will be provided.

Theorem 3.1. The value that satisfies the Axioms I-GL, I-DM, and I-SYM is referred to as the interval Shapley value on KIG^N.

PROOF. Interval Shapley value obeys I-GL and I-SYM axioms by the definition of this value. Moreover, [21] demonstrates that the Shapley value satisfies DM. Therefore, the interval Shapley value satisfies I-DM. We aim to establish the converse. Interval Shapley value obeys I-GL and I-SYM. Let w and w' belong to KIG^N. Consider the symmetric game $w' \in KIG^N$ where w' is uniformly zero across all coalitions, i.e., $w'_i(S) = [0,0]$, for all $i \in S$. According to I-SYM, $f_i(w') = f_j(w')$, for all $i \neq j$, and

$$\sum_{i=1}^{n} f_i(w') = [0,0]$$

follows from I-GL. Consequently, $f_i(w') = [0,0]$, for all $i \in N$. By I-DM, it can be deduced that for any interval game $w \in KIG^N$ and any player $i \in N$,

$$w_i(S) = [0,0]\,, \text{ for all } S \subseteq N, \text{ implies } f_i(w) = [0,0] \tag{3.1}$$

That is, null players get nothing. In other words, players with null contributions receive no payoff. We use Shapley's insight that any game v can be explained as the sum of primitive games, allowing for a detailed analysis of its fundamental components and strategic foundations. This concept can be redefined by extending it into the domain of interval games, as explained below:

$$w = \sum_{S \subseteq N : S \neq \emptyset} \lambda_S u_S \tag{3.2}$$

where

$$\lambda_S u_S(T) = \begin{cases} \lambda_S, & \text{if } S \subseteq T \\ [0,0]\,, & \text{otherwise} \end{cases}$$

The interval Shapley value finds its formulation in the following manner:

$$f_i(w) = \sum_{S \subseteq N : S \neq \emptyset} f_i(\lambda_S u_S) = \sum_{S : i \in S} \frac{\lambda_S}{|S|}$$

Define the index I of w as the minimal quantity of non-zero terms requisite in an expression delineating

w in the form specified by (3.2). The theorem is established through induction on the set I. If $I = 0$, then every is a null and hence $f_i(w) = [0,0]$ by (3.1). For all $i, j \in S$, I-SYM implies that $f_i(w) = f_j(w)$; supplemented by the prerequisite that $\sum_{i=1}^{n} f_i(w) = w(N)$. Consequently, we deduce that $f_i(w) = \frac{\lambda_S}{|S|}$, for all $i \in S$. Thus, $f(w)$ constitutes the interval Shapley value for the case of $I \in \{0,1\}$. We assume that $f(w)$ represents the interval Shapley value for any index up to I. Consider w with an index of $I+1$, expressed as

$$w = \sum_{l=1}^{I+1} \lambda_{S_l} u_{S_l}$$

where $\lambda_{S_l} \neq [0,0]$, for all I. Let $S = \bigcap_{l=1}^{I+1} S_l$ and suppose that $i \notin S$. Define the game

$$w' = \sum_{l:i\in S_l} \lambda_{S_l} u_{S_l}$$

The index of w is at most I and $w'_i(T) = w_i(T)$, for all S. Consequently, by induction and I-DM, it follows that

$$f_i(w) = f_i(w') = \sum_{l=i\in S_l} \frac{\lambda_S}{|S|}$$

which represents the interval Shapley value of i. We still need to demonstrate that $f_i(w)$ is the interval Shapley value when $i \in S = \bigcap_{l=1}^{I+1} S_l$. According to I-SYM, $f_i(w)$ is a constant c, for all members of S; similarly, the interval Shapley value is some constant c', for all members of S. By I-GL, it follows that $c = c'$. □

The following example illustrates how to construct a model and compute the interval Shapley value in a real-life operational research scenario, as presented by [14]. Consider an inventory situation characterized by interval data and formulate an associated interval game. Player 3 owns a storage facility with a capacity for a single container, while Players 1 and 2 each possess one container. If Player 1 is permitted to store their container, they will receive a benefit between 20 and 40. If Player 2 is allowed to store their container, the corresponding benefit falls within the range of [60, 80].

Example 3.2. The situation described above corresponds to the interval game $\langle N, w\rangle$ with $N = \{1,2,3\}$ and $w(S) = [0,0]$ if $3 \notin S, w(\emptyset) = [0,0], w(1,3) = [20,40], w(2,3) = [60,80]$, and $w(N) = [80,100]$, i.e., a big boss interval game with Player 3 as a big boss. Then, the interval marginal vectors are provided in the following table where $\sigma : N \to N$ is identified with $(\sigma(1), \sigma(2), \sigma(3))$. Firstly, for $\sigma_1 = (1,3,2)$, we calculate the interval marginal vectors. Then,

$$m_1^{\sigma_1}(w) = w(1) = [0,0]$$

$$m_2^{\sigma_1}(w) = w(N) - w(13) = [80,100] - [20,40] = [60,60]$$

and

$$m_3^{\sigma_1}(w) = w(13) - w(1) = [20,40] - [0,0] = [20,40]$$

The others can be calculated similarly, which are shown in Table 1.

Table 1. Interval marginal vectors

σ	$m_1^\sigma(w)$	$m_2^\sigma(w)$	$m_3^\sigma(w)$
$\sigma_1 = (1,2,3)$	$m_1^{\sigma_1}(w) = [0,0]$	$m_2^{\sigma_1}(w) = [0,0]$	$m_3^{\sigma_1}(w) = [80,100]$
$\sigma_2 = (1,3,2)$	$m_1^{\sigma_2}(w) = [0,0]$	$m_2^{\sigma_2}(w) = [60,60]$	$m_3^{\sigma_2}(w) = [20,40]$
$\sigma_3 = (2,1,3)$	$m_1^{\sigma_3}(w) = [0,0]$	$m_2^{\sigma_3}(w) = [0,0]$	$m_3^{\sigma_3}(w) = [80,100]$
$\sigma_4 = (2,3,1)$	$m_1^{\sigma_4}(w) = [20,20]$	$m_2^{\sigma_4}(w) = [0,0]$	$m_3^{\sigma_4}(w) = [60,80]$
$\sigma_5 = (3,1,2)$	$m_1^{\sigma_5}(w) = [20,40]$	$m_2^{\sigma_5}(w) = [60,60]$	$m_3^{\sigma_5}(w) = [0,0]$
$\sigma_6 = (3,2,1)$	$m_1^{\sigma_6}(w) = [20,20]$	$m_2^{\sigma_6}(w) = [60,80]$	$m_3^{\sigma_6}(w) = [0,0]$

Table 1 illustrates the interval marginal vectors of the cooperative interval game in Example 3.2. The average of the six interval marginal vectors is the interval Shapley value of this game, which can be observed as:

$$\Phi(w) = \left(\left[10, \frac{40}{3}\right], \left[30, \frac{100}{3}\right], \left[40, \frac{160}{3}\right]\right)$$

4. Conclusion

This study aims to provide an axiomatic characterization of the Shapley value using the axioms above. It is argued that these axioms uniquely define the Shapley value. The paper surveys cooperative game theory in the literature, focusing on two specific subtraction operators: Moore's subtraction operator and the special subtraction operator. In the last decade, several axiomatic characterizations of the Shapley value have been using the special subtraction operator. Shortly, we plan to introduce new axiomatic characterizations for the Shapley value using Moore's subtraction operator. The classical game's dividends, initially introduced by Hars [15], play a pivotal role in characterizing the classical Shapley value. The utilization of dividends enables the characterization of the interval Shapley value in cooperative interval games with compact real-valued coalitional interval values.

In conclusion, further exploration of this idea shows promise as a potential avenue for future research. This approach offers a valuable perspective for comprehending and assessing the Shapley value. Regarding future research, further exploration of this concept presents an exciting and fruitful area for characterizing the Shapley value. Furthermore, a more comprehensive examination of Grey Game Theory, which considers uncertainty and incomplete information, enhances the accuracy of modeling cooperative games. Within this framework, characterization methods and the Shapley value enable a better understanding of collaboration dynamics among players, especially in situations involving uncertainty. This characterization has the potential to provide new insights for further characterizations and is amenable to extension within the domain of grey games.

Furthermore, it acts as a guiding framework to facilitate characterizations of other relevant values, such as the Banzhaf and T-value. In this respect, it offers a nuanced understanding of Shapley value and guides researchers seeking to characterize different values. As a result, this characterization emerges as a comprehensive framework that directs research toward the Shapley value and guides research into related values within the cooperative game theory. Therefore, future investigations could deepen understanding and knowledge in this field by conducting a comprehensive analysis incorporating both cooperative games and Grey Game Theory.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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